A kinetic approach to blast waves in one-dimensional conservative and dissipative fluids

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Self-similar blast waves originating from a punctual release of energy in a gas have become a staple of hydrodynamics since the seminal work of Taylor, von Neumann and Sedov. Yet they have received very little attention at the kinetic level, which can provide insights beyond the limits of hydrodynamics— including transient regimes and a microscopic understanding of the shock front (which must be reduced to a singularity at the macroscopic level). As a first step toward such kinetic description, we study the blast in a one-dimensional gas of hard particles. It is amenable to analysis even in the presence of dissipation (leading to "snowplow" dynamics) or inhomogeneous mass repartition, as found in granular materials and astrophysical systems. Furthermore, even the conservative case proves to be of remarkable pedagogical interest, in demonstrating subtle aspects of dimensional analysis and their resolution through kinetic insights. We show that it can effectively behave like a zero-dimensional system, reduced to the shock front, depending on whether a length scale appears in the initial mass distribution.

I. INTRODUCTION

Consequences of the punctual release of a very large quantity of energy in the atmosphere became, as one might expect, a rather active yet secretive research topic during the 1940s, so that numerous physicists— among which Taylor [1, 2], Sedov [3] and Von Neumann [4], under the scrutiny of their respective governments— independently discovered the basic characteristics of the self-similar blast wave that follows such an explosion. Since then, this problem has become a textbook example of dimensional analysis and hydrodynamics, with thousands of articles following up on the latter aspect in such diverse applications as plasmas and astrophysical systems, and adding several refinements such as heterogeneous or non-conservative media [5]. Yet, to the best of our knowledge, only two experiments [7, 8] and a few numerical or analytical investigations [6, 9, 10] have tackled aspects of this problem from the microscopic point of view, at the scale of particle collisions, and only recently have related problems in shock fronts been investigated at the same level of description (mostly in the context of granular materials, see for instance [11–14]).

We focus here on a supersonic blast wave in strong shock conditions: a hydrodynamical solution that presents a discontinuity on the shock front, due to the speed of matter displacement being much larger than the speed of sound or even heat transfer in the ambient gas, so that no perturbation can propagate downstream into the fluid and result in a smoothing of the boundary. The speed of sound is proportional to the thermal velocity \( \sqrt{\Theta/2m} \) for temperature \( \Theta \) and particle mass \( m \), and heat conductivity also depends on \( \Theta \). The most representative case is therefore that of the "cold gas" with \( \Theta = 0 \) outside of the blast region, which will be considered throughout this paper. As self-similar phenomena must not depend on microscopic particulars, we further simplify the problem by considering hard-core interactions and modelling energy dissipation by inelastic collisions with fixed restitution coefficient in Sec.IV, although our results should apply to any type of dissipation whose rate increases with temperature.

Previous works derived the scaling laws in the system, such as the time-dependence of the radius of the blast \( R(t) \), from the conservation of energy in the elastic case and momentum in the inelastic case [18]. We recall these derivations in Sec.II and suggest that conservation laws must be supplemented by kinetic insights to give rise to the proper scaling properties. We first consider the 1D elastic gas in Sec.III, which is highly singular if the initial mass repartition is uniform or scale-free. This singularity demonstrates how intricacies of dimensional analysis at the macroscopic level are in fact easily solved using microscopic insight. It also connects to the specific role of the foremost moving particle (or the shock front in \( d > 1 \)), which is again relevant in inelastic systems. Indeed, we show in Sec.IV that this role proves crucial during the transient phase of dissipative blasts. Our characterisation of this intermediate regime agrees with analyses from experiments [7, 8], and we conclude on the question of how momentum conservation applies in dissipative, standard conservative and accelerated conservative blasts.

II. SCALING LAWS

This short section recalls general results on scaling laws in the blast in arbitrary dimension, to which we will later confront our results. As we expect the solution to be self-similar, macroscopic quantities such as the number of
particles in the blast \( N(t) \), its radius \( R(t) \) or its total energy \( E(t) \) (in case it is not conserved) will follow simple power-laws that may be determined by dimensional analysis. Note the applicability of dimensional analysis is in fact rigorously expressed in group-theoretic terms and expounded upon in the classic work by Barenblatt [15]. See the Appendix for a brief summary of relevant basic theorems and an explanation of terminology such as similarity.

We first recall briefly the standard argument most famously given by Taylor [1, 2]. At the macroscopic level, the dynamics are completely controlled by the laws of conservation of energy and mass, as the energy liberated by the detonation is distributed within the growing radius \( R \) of the blast wave (over an increasing number of particles \( N \) with total mass \( M \)), so that the only parameters describing the evolution are time \( t \), radius \( R \), total energy \( E \), and mass density \( \rho \) in the initial state of the gas from which we can deduce \( M \). By dimensional analysis, the evolution of the radius must then obey

\[
R(t, E, \rho) = R_0 \left( \frac{E t^2}{\rho_i} \right)^{2/(d+2)}
\]

as this is the only combination of the three other parameters that has the dimensions of a length. In more descriptive terms, Taylor expressed this as

\[
\frac{M \dot{R}^2}{E} = \text{constant}
\]

which represents the fact that a finite fraction of the energy is always invested in coherent motion (i.e. as kinetic energy of the flow) so that particles propagate at the velocity of the shock front \( \dot{R} \) (or a velocity proportional to it). The rest of the energy goes into random motion, that is thermal energy, but as long as the blast expands the ratio of kinetic over thermal energy should tend to a finite limit. In general terms, we will put it as

\[
E(t) \sim N(t) \dot{R}^2(t) \sim R^d(t) \dot{R}^2(t)
\]

\[
\sim \frac{R^{d+2}(t)}{t^2} \sim t^{(d+2)\delta-2} \quad \text{assuming } R(t) \sim t^\delta.
\]

If energy is conserved, \( E(t) = E_i \), we find

\[
R(t) \sim t^{2/(d+2)} \quad \text{(conservative case)}.
\]

hence Taylor’s famous result \( R \sim t^{2/5} \) in \( d = 3 \). However all previous relations hold even if \( E(t) \) is not constant, as in the case of inelastic collisions between the particles (or in radiative gases). Then another conservation law must be considered: that of total momentum

\[
\Pi(t) \sim M(t) \dot{R}(t) \sim \frac{R(t)^{d+1}}{t} \sim t^{\delta(d+1)-1}
\]

so that

\[
R(t) \sim t^{1/(d+1)} \quad \text{(dissipative case)}.
\]

This behaviour, recently suggested in a kinetic theoretical context [9], is also well known in astrophysical blasts as a proposition by Oort [17]. It is verified by molecular dynamics simulations in Fig.1

A puzzle immediately arises: why should we consider energy conservation for elastic particles, and momentum conservation for inelastic particles, when the second law also applies to first case? This conundrum is most easily resolved in the one-dimensional case, which is the main focus of this paper (for higher dimensions, see [18]). On the other hand, a system of elastic particles in one dimension exhibits an important singularity, which we first investigate in Sec.III so as to clarify the conditions for the generality of our analysis.

### III. ELASTIC PARTICLES

**Singularity of the one-dimensional case**

The kinetics of a blast in an homogeneous fluid of elastic hard spheres was first considered a few years ago [6], and we recall it to introduce the inelastic and inhomogeneous cases, but also because it exhibits a strong peculiarity in dimension \( d = 1 \) that was not noted before, and can be understood through dimensional analysis.
FIG. 1. Scaling laws for a dissipative blast in dimension \(d = 2\) (left) and \(d = 3\) (right). Scaling laws for particle number \(N\), total energy \(E\) and radius \(R\) from top to bottom, rescaled by their initial value. Solid lines are theoretical asymptotes, and dashes represent initial ballistic (linear) growth.

Indeed, if all particles are identical, every collision simply transfers the velocity of the moving particle to its partner (see the collision rule (22) in Sec.IV), and thus, there is a single particle in motion with constant velocity at any given time. The corresponding time dependence disagrees with the law in general dimension given in the previous section:

\[
R(t) \sim t \neq t^{2/(d+2)} = t^{2/3}.
\]  

(7)

However, exponent \(2/(d + 2)\) had been derived from the principle that the proper dimensional parameters in the scaling law for \(R\) were \(t\), \(E_i\) and \(\rho_i\), where \(\rho_i\) is the linear mass of the homogeneous system

\[
[\rho_i] = M/L^d = M/L.
\]  

(8)

If there is a single particle in motion, this linear mass is not relevant: only mass \(m\) of the particle is involved, and we thus naturally find that velocity should be constant

\[
R(t, E_i, m) = \left(\frac{2E_i t^2}{m}\right)^{1/2} \sim t.
\]  

(9)

In fact, having combined microscopic properties of the particles such as mass \(m\), size \(\sigma\) and mean free path \(l\) into a single parameter \(\rho_i\) already implied a similarity hypothesis, as we will demonstrate below. We may reintroduce these parameters in our dimensional analysis, so that there are now 5 parameters for 3 dimensions, hence 2 dimensionless parameters. Let us first consider velocity \(\dot{R}\) since, unlike \(R\), it still has only one obvious typical scale that we take as prefactor

\[
\dot{R}(t, E_i, m, l, \sigma) = \sqrt{\frac{E_i}{m}} \Psi\left(\frac{t}{\tau}, \frac{\sigma}{l}\right) = \sqrt{\frac{E_i}{m}} \Psi(\psi_1, \psi_2)
\]  

(10)

with \(\tau\) the typical collision time for a particle holding energy \(E_i\)

\[
\tau = \sqrt{\frac{ml^2}{E_i}}.
\]  

(11)

Furthermore, space on the line that lies inside a particle is irrelevant for the dynamics and we may let it vanish while keeping all inter-particle distances constant (which is exceptional to the 1D case), so that

\[
\lim_{\psi_2 \to 0} \Psi(\psi_1, \psi_2) = \hat{\Psi}(\psi_1).
\]  

(12)

Finally, if particles are identical, their collisions have no effect on velocity, so \(\tau\) may go to infinity without changing the dynamics

\[
\lim_{\psi_1 \to 0} \hat{\Psi}(\psi_1) = \text{constant}
\]  

(13)
TABLE I. Exponents observed in molecular dynamics simulations on a line for elastic particles initially at rest except one at \( r = 0 \). The scale-free row corresponds to particles whose masses are distributed according to a power law \( m(r) = m_0 |r|^{-\omega} \), thus without typical length scale. On the scale-bound row, the masses are similarly chosen, then the mass of every second particle is doubled, so that a typical scale equal to the mean free path is imposed on the system (however convergence to the asymptotic expression gets very slow for large exponents).

Hence \( \hat{R} \) is immediately expressed by its prefactor, and we find the ballistic behaviour

\[
R(t, E_i, m, l, \sigma) \propto t \sqrt{\frac{E_i}{m}}
\]  

with full similarity (of the first kind, see the Appendix for precisions on this vocabulary) in variables \( l \) and \( \sigma \). In higher dimensions, the same ballistic limit is found if \( \psi_1 \to 0 \) or \( \psi_2 \to 0 \) (meaning that the collision probability vanishes through either particles being infinitely small or infinitely far apart, or \( E_i = 0 \)) but this limit is singular: any non-zero value of these parameters gives rise to the standard behaviour detailed in section II.

An alternate way of understanding this problem comes from rewriting the equation so as to involve the number \( N \) of particles in motion, which itself is some (as yet unknown) function of the parameters:

\[
\hat{R}(t, E_i, m, l, \sigma) = \sqrt{\frac{E_i}{m}} \Phi\left( N, \frac{\sigma}{l} \right) = \sqrt{\frac{E_i}{Nm}} \Phi\left( \frac{\sigma}{l} \right)
\]  

where the second equality comes from conservation of energy over particles with total mass \( Nm \). Then as we know that any non singular value of the remaining parameter \( \sigma/l \) should not affect scaling exponents, we just have to solve

\begin{align*}
\text{If } N(t) &\sim R^d(t), \quad \hat{R}(t) \sim R^{-d/2}(t) \quad \iff \quad R(t) \sim t^{2/(d+2)}. \\
\text{If } N(t) &\sim 1, \quad \hat{R}(t) \sim 1 \quad \iff \quad R(t) \sim t.
\end{align*}

Hence, expression (15) exhibits similarity of the second kind in parameter \( N \) (as we needed to solve a differential equation to obtain the exponent) while equation (10) was fully self-similar of the first kind and dimensional analysis was sufficient. This demonstrates that self-similarity may be arduous to detect under the wrong set of variables, and how this is decided by conservation laws: using velocity \( \hat{R} \) helps us understand why one should consider parameter \( m \) rather than \( \rho \) in the singular case, because momentum \( m\hat{R} \) is conserved in the latter. Still, this subtlety can only be fully understood by considering a more general way to relate masses to the conservation laws.

**Spatial mass repartition**

There seems to be a clear way of removing the singularity of the one-dimensional case: setting particles with different masses, so that collisions distribute energy instead of transferring it entirely to the lead particle. At least two choices may prove interesting:

- use a spatially homogeneous mass distribution, for instance a binary alternation of masses 1 and 2,
- have particle masses decay as a power-law with distance to the origin: \( m(r) = m_0 r^{-\omega} \).

The second option is inspired by a classic example of self-similar blast in astrophysics: a supernova in a region of space where, due to gravitational effects, the density of matter varies with the distance to the center of the star. Setting \( \omega = 1 - d_{\text{eff}} \) we may also use it as a one-dimensional analog to angular sections in higher dimensions \( d_{\text{eff}} > 1 \), where instead of the mass density, it is the number of particles at a given distance from the center that increases. In any case, the usual dimensional analysis is easily adapted to find

\[
R(t) \sim t^{2/(d+2-\omega)}
\]
Results of a molecular dynamics simulation, shown in Table I, allow for a better understanding of the singularity discussed above: binary alternating masses give as expected per Taylor’s reasoning (for the standard case $\omega = 0$)

$$R(t) \sim t^{2/d+2} = t^{2/3} \quad (d = 1)$$

and the peculiarity of dimension 1 disappears, since the proper dimensional parameter in $R(t)$ is indeed $\rho_i$. However, a power-law mass distribution $m(r) = m_0 r^{-\omega}$ retains the singularity of the 1D case

$$N(t) \sim R(t) \sim t^{2/(2-\omega)}$$

as deduced from dimensional analysis with mass parameter $m$

$$R(t, E_i, m_0, \omega) = \left( \frac{E_i t^2}{m_0} \right)^{1/(2-\omega)} \tilde{R}(\omega).$$

Thus, the determining factor for the observed regime is in fact the absence of a typical length scale in the singular 1D system, which is not solved by adding an inherently scale-free spatial mass repartition such as a power-law. However the binary alternation adds a length scale which is the separation between two different particles, and the singularity disappears. The microscopic explanation for these different behaviours is that in the scale-free (“zero-dimensional”) case, for any choice of $\omega$, only the mass of the leading particle determines the shock propagation speed $\tilde{R}(t)$, so that interparticle distances are never relevant.

Interestingly enough, this peculiarity of the foremost particle is magnified if one adds inelasticity to the parameters: we can then find an intermediate asymptotic regime in which the system still exhibits similarity in certain microscopic variables, but also depends on this additional dimensionless parameter, and exhibits exponential and logarithmic rather than power-law behaviours.

IV. INELASTIC PARTICLES

To investigate the conundrum of scaling and conservation laws in the inelastic case (see Sec.II), we benefit from the fact that the dynamics of a blast in a one-dimensional system at zero initial temperature can be analysed exactly. For clarity and ease of calculation, let us further impose that all particles are initially equidistant with separation $l = 1$ (although this constraint will be relaxed in simulations).

Let us recall the collision rule for two inelastic particles with restitution coefficient $\alpha \in [0, 1]$ (for a discussion, see for instance [20])

$$v_1^* = av_1 + (1 - a)v_2$$

$$a = \frac{(m_1 - \alpha m_2)}{(m_1 + m_2)}$$

where $v_1^*$ is the velocity of particle 1 after collision with particle 2, and $v_2^*$ is similarly computed. For equal masses, $a = (1 - \alpha)/2$ and elastic particles correspond to $\alpha = 1$, therefore if particles are both elastic and identical, $v_1^* = v_2^*$ and velocities are simply exchanged upon collision.

A. First collisions and leader behaviour

Figure 2 represents the logarithmic growth of the number of moving particles in the blast. This behaviour is easily understood by the following argument: all particles in the blast are set in motion by the leading particle, whose velocity decreases geometrically (and whose identity is switched) with each collision. As long as none of its followers can catch up with it, the velocity of the shock wave thus decreases exponentially.

In the initial phase where every particle has gone through a single collision, the $N - 1$th gives

$$v_N^* = \frac{1 + \alpha}{2} v_{N-1}$$

$$v_{N-1}^* = \frac{1 - \alpha}{2} v_{N-1}$$
FIG. 2. Number of moving particles every 100 collisions up to 30000 in a simulation with $\alpha = 0.99$. Dashes correspond to the asymptotic solution (29) up to an additive constant. The recollision phase is seen for equal masses and spacing as the region where squares cluster.

FIG. 3. Velocity of moving particles as a function of their position. The leader is isolated with much higher velocity than its direct followers (as $\alpha = 0.99$). The first particles on the left have entered the recollision phase, but the middle section exhibits the exponential decay predicted for the initial phase.

so that if $\alpha > 0$, the velocity of the leading particle is always higher than its followers (for $\alpha = 0$ immediate aggregation occurs), hence

\[ v_N = \left( \frac{1 + \alpha}{2} \right)^{N-1} \]  
(25)

\[ v_k = \frac{1 - \alpha}{2} \left( \frac{1 + \alpha}{2} \right)^{k-1} \quad \forall k < N \]  
(26)

representing a trail of particles with velocity exponentially decreasing with their index, see Fig.3. If $\alpha$ is close to 1, the leader retains a large fraction of the initial momentum for a long time. In that phase, the time needed to reach particle $N$ thus grows exponentially:

\[ t_N = \sum_{k=1}^{N-1} \left( \frac{2}{1+\alpha} \right)^k = \left( \frac{2}{1+\alpha} \right)^N \frac{1+\alpha}{1-\alpha} - \frac{2}{1-\alpha} \]  
(27)

(hence $v_N \propto 1/t_N$). As $N \to \infty$

\[ \ln(t_N) \approx \ln \left( \frac{1+\alpha}{1-\alpha} \right) + N \ln \left( \frac{2}{1+\alpha} \right) \]  
(28)
which leads to the asymptote observed in Fig.2:

\[ N \approx \frac{\ln \left( \frac{1 - \alpha}{1 + \alpha} t_N \right)}{\ln \left( \frac{2}{1 + \alpha} \right)} \quad (29) \]

This phase is a case of non-self-similar intermediate asymptotics [15]: energy is no longer a constant parameter, but no other conservation law is applicable instead because none concerns solely the leading particle (contrary to the elastic case), which is still entirely responsible for the dynamics; instead, the behaviour depends on a new dimensionless parameter \( \alpha \).

**B. Recollision and scaling law**

Eventually, some trail particles become faster than the leader, since the latter’s velocity decreases exponentially with \( N \) (which at this point is both the number of collisions and the number of particles in motion) while the typical velocity in the trail evolves as \( 1/N \) and the length of the trail is bounded by \( N \).

Once the leader starts being re-accelerated by collisions coming from behind, and hence receives some of their momentum, the system evolves towards the expected behaviour \( N \propto t^{1/2} \). This evolution is slowed down by recollision cascades happening in the trail, but the asymptotic power law is reached after the transitory logarithmic phase whose length diverges as \( \alpha \to 1 \), cf. Figure 4. This intermediate regime is the one exhibited in both experimental works on blasts in granular systems [7, 8].

![FIG. 4. Scaling laws for the number of moving particles \( N \). Left : for \( \alpha = 0.99 \), a ballistic linear growth is followed by the logarithmic regime (29) (note the switch to lin-log scale while the other regimes are plotted in log-log), until the leading particle is accelerated by recollisions coming from the trail and the asymptotic law \( N \sim \sqrt{t} \) emerges. Right : for \( \alpha = 0.5 \), the asymptotic law is immediately observed.](image)

Thus, it seems that the scaling law emerges from the competition between cooling occurring at the level of the leader, and pressure induced by particles coming from behind. Scaling laws directed by momentum conservation thus correspond to microscopic situations where particles on the inside of the shock wave are pushing those in front of them and new particles constantly accumulate at the front, in agreement with Oort’s idea of “snowplow” dynamics [17] which finds here a confirmation at the microscopic level.

Let us finally note that the conservation of momentum is generally a subtle matter, whether particles are elastic or not. In the conservative case,

\[ E(t) \sim N(t)\dot{R}^2(t) = E(0) \quad (30) \]

and one expects

\[ \Pi(t) \sim N(t)\dot{R}(t) \sim 1/\dot{R}(t). \quad (31) \]

However momentum must be conserved at the level of the whole system:

\[ \Pi(t) = \Pi(0). \quad (32) \]
If one separates particles into those to the left and to the right of the mass centre (assuming that it moves toward the right), we may define two quantities $\Pi_-$ (to the left) and $\Pi_+$ (to the right) which can take opposite signs and evolve as $1/R(t)$, for instance as

$$\Pi(t) \sim t^{-\omega/(1-\omega)}$$

(33)

for a mass distribution $m(r) = m_0 r^{-\omega}$, provided this exponent is positive: as noted in [6], the 1D elastic blast tends to symmetrize like its higher-dimensional counterpart, and it quickly occurs that $\Pi_+(t) \approx -\Pi_-(t)$. Thus an important momentum transfer occurs between opposite sectors of the blast, going through the centre.

On the other hand, if that exponent is negative (corresponding to an *accelerated* shock as $R(t)$ increases with time), the previous equation predicts a decreasing momentum. It cannot occur on both sides due to the global conservation law, hence this leads to an asymmetry

$$\Pi_+(t) \sim 1/R(t)$$

(34)

$$\Pi_-(t) = \Pi(0) - \Pi_+(t)$$

(35)

where momentum concentrates behind the centre of mass. This situation for elastic particles is in fact very similar to the dissipative blast in one respect: here masses are decreasing with $r$ in such way that collisions between two successive particles do not lead to a reversal of the velocity, so that all particles are moving in the same direction and there is no symmetrization of the blast, just like for inelastic particles. In higher dimensions, dissipative and accelerated-conservative blasts are both known to exhibit an empty core [5, 21].

V. CONCLUSION

We have first considered the case of a one-dimensional system of elastic hard particles in which a large quantity of energy is released punctually, and shown that, unless the repartition of particle masses contains an intrinsic length scale, this system is effectively zero-dimensional as the blast reduces to its front (a single particle in motion). Adding energy dissipation to the collisions removes this singularity, yet its trace lingers in a transient phase, during which the front still is solely responsible for the dynamics of the system, but is constrained by no conservation law, so that self-similarity disappears until the emergence of a later regime. This transient phase displays the logarithmic behaviour noted by the authors of experimental studies in granular systems [7, 8], while the asymptotic scaling regime is the momentum-conserving snowplow made famous in astrophysical systems [17].

Both the conservative and dissipative systems analysed in this article demonstrate the usefulness of kinetic insight to properly deduce scaling behaviours from global symmetries such as conservation laws. Due to the singular nature of the shock front, the correct reasoning may often be obscured if we remain at the hydrodynamic level of description, either in analytical and numerical studies. Further evidence of this idea will arise in the study of blasts in higher dimensions [18].

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APPENDIX: CONDITIONS AND KINDS OF SELF-SIMILARITY

As Section III involves some subtler points of dimensional analysis, it seems useful to recall briefly a few well-known results. Given a dimensional quantity \( a \) defined as a function of \( n \) variables with \( k \) independent dimensions:

\[
a = \Psi(x_1 \ldots x_n)
\]

we may rewrite it as a different function of \( k \) of the initial variables, containing all independent dimensions, and \( n-k \) dimensionless ratios

\[
a = \hat{\Psi}(x_1 \ldots x_k, \pi_1 \ldots \pi_{n-k})
\]

where \( \pi_i \) are constructed by making variables \( x_{k+1} \ldots x_n \) dimensionless with combinations of \( x_1 \ldots x_k \) raised to relevant powers:

\[
\pi_i = \frac{x_{k+1}^{a_{i1}} \cdots x_k^{a_{ik}}}{x_1^{a_{i1}} \cdots x_k^{a_{ik}}} \quad \text{such that } [\pi_i] = 1.
\]

The physical law connecting \( a \) to parameters cannot depend explicitly on the dimensional quantities \( x_1 \ldots x_k \) (else it would vary with our choice of units), hence the Buckingham \( \Pi \) theorem:

\[
a = \Psi(x_1 \ldots x_n) = x_1^{a_{11}} \cdots x_k^{a_{k}} \Pi(\pi_1 \ldots \pi_{n-k}) \quad \text{such that } [\Pi] = [\pi_i] = 1.
\]

A famous consequence is that if all variables have independent dimensions, then no ratios \( \pi_i \) may be constructed and \( a \) must have a power-law dependence in the variables \( x_i \), with exponents uniquely determined by the requirement 

\[
[a] = [\prod_i x_i^{a_i}].
\]

These relationships are usually termed "scaling laws", and following Barenblatt [15], the observable \( a \) is said to be fully similar in any parameter that is absent from \( \Pi \), so that its variation only affects \( a \) as a simple power-law rescaling through the dimensional prefactor in (39).

Self-similarity in the usual sense is a direct extension of this situation. Let us assume that among the parameters \( x_i \) are some free variables such as coordinates. For instance, \( a \) can be a field (instead of a simple quantity) with coordinates in space \( r \) and time \( t \) in the case of dynamical systems, or \( r \) and a measurement scale \( l \) for fractals. Such a system exhibits similarity if those variables always appear together in ratios \( \pi_i \), so that for any change in position \( r \to r' \) there is a corresponding change in scale \( l \to l' \) or time \( t \to t' \) that restores the initial values for all dimensionless quantities \( \pi_i \) and \( \Pi \). The analogy with the previous paragraph is clear: for free variables, they needn’t be absent from \( \Pi \) to ensure similarity, but they must always be transformable in ways that leave \( \Pi \) invariant. Barenblatt argued that travelling waves are in fact a limiting case of self-similarity, as can be intuited from this definition.

We may now present the most important aspect of this theory as pertains to our work. In general, systems are not self-similar because their laws involve too many parameters, so that, for instance, \( t \) occurs in multiple dimensionless ratios \( \pi_i \), some of which contain no other free variable. Therefore, a given translation \( t \to t' \) will produce irreducible changes in the system, that cannot be compensated by a simple rescaling of other coordinates. As there are generally many microscopic parameters that may intervene in a phenomenon, this would probably prevent any real system from
being self-similar, were it not for the notion of intermediate asymptotics: let us say (for the sake of concreteness) that the field $a$ depends on $r$ and $t$ as

$$ a(r, t) = x_1^{\alpha_1} \cdots x_k^{\alpha_k} \Pi(\pi_1(r, t), \pi_2(t), \ldots, \pi_{n-k}(t)) \quad (40) $$

where $\pi_1$ contains multiple free variables while the $\pi_{i>1}$ are the irreducible part of the dependency in one variable. One possibility is that some ratios $\pi_i$ be constant (or change slowly enough that we can ignore their dependence in $t$) in the range of interest. Thus, every conservation law in the system is a step toward similarity, as it allows to remove some combination of the variables from the dimensionless relation $\Pi$.

Another possibility is for the system to be *asymptotically* self-similar, if the following limit is well defined

$$ \lim_{t \to 0 \text{ or } \infty} \Pi(\pi_1 \cdots \pi_{n-k}) = \bar{\Pi}(\pi_1) \quad (41) $$

meaning that similarity occurs in a range of $t$ where the excess parameters are either 0 or infinite, yet the function $\Pi$ has a finite limit. This is a strong assumption, and most of the time $\Pi$ does not converge to a proper function. Yet in some cases there is *self-similarity of the second kind*: if $\Pi$ diverges or vanishes in the limit that we consider, but as a power-law of the dimensionless ratios

$$ \lim_{t \to 0 \text{ or } \infty} \Pi(\pi_1, \pi_2 \cdots \pi_{n-k}) = \prod_{i=2}^{n-k} \pi_i^{\beta_i} \bar{\Pi}(\pi_1) \quad (42) $$

it retains all the scaling properties associated with self-similarity, with the subtlety that exponents $\beta_i$ cannot be determined by dimensional analysis, as they apply to dimensionless parameters. Hence they must be computed from the full equations describing the system; in general, they will depend continuously on parameter values (for instance they may evolve in time). This case is called *partial similarity* since $\Pi$ is not invariant under easily determined transformations of the dimensional parameters (as in the case of full similarity), but $\bar{\Pi}$ obviously is.