The Riemann problem for nonlinear polarization waves in two-component Bose-Einstein condensates

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1 Introduction

The experimental realization of Bose-Einstein condensates consisting of two species offers the possibility of studying rich dynamical behaviors associated to the relative motion of the components. It has been recently noticed \cite{1} that this “polarization dynamics” (out-of-phase motion of the components) can be separated from the “density dynamics” (on-phase motion of the components), even in the case of large amplitude soliton waves, provided that the difference between intra- and inter-species interaction constants is small. In Ref. \cite{2} we have derived the general equations for the polarization dynamics, have presented the associated rich variety of periodic nonlinear polarization waves and described the solution of the Riemann problem for a specific initial discontinuous polarization profile. The equations obtained in \cite{2} were found to be equivalent to the Landau-Lifshitz equations for the easy plane magnetics \cite{3}. It is also worth noticing that they provide the dispersive generalization of the two-layer fluid dynamics \cite{4}. The resulting solution of the Riemann problem consisted of a wave structure formed by a rarefaction wave, a plateau, and a dispersive shock. In the present work we analyze in detail a few model cases for which the Riemann problem admits clean cut solutions: either a pure rarefaction wave (Sec. 3.1) or a pure dispersive shock wave (cnoidal, Sec. 3.2 or trigonometric, Sec. 3.3). The fact that the polarization dynamics is described by an exactly integrable equation makes it possible for the first time to very accurately describe the region of the dispersive shock by using the Whitham averaging technique.

2 Polarization dynamics

We consider a two-component one-dimensional Bose-Einstein condensate described by order parameters $\psi_{\uparrow}(x, t)$ and $\psi_{\downarrow}(x, t)$ which obey the following coupled Gross-Pitaevskii equations

\begin{equation}
\frac{i\hbar}{2m} \partial_t \psi_{\uparrow, \downarrow} + \frac{\hbar^2}{2m} \partial_x^2 \psi_{\uparrow, \downarrow} - \left( g |\psi_{\uparrow, \downarrow}|^2 + (g - \delta g) |\psi_{\downarrow, \uparrow}|^2 \right) \psi_{\uparrow, \downarrow} = 0.
\end{equation}

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We assume in Eqs. (1) that the two intra-species non linear coefficients \( g_{↑↑} \) and \( g_{↓↓} \) have the same value, denoted as \( g \). This situation is exactly realized in the mixture of the two hyperfine states \( |F = 1, m_F = ±1⟩ \) of \(^{23}\text{Na} \) [5]. The inter-species coefficient \( g_{↑↓} \) is written as \( g - \delta g \), and we assume that

\[
0 < \delta g \ll g .
\]  

Conditions (2) are realized in the case of \(^{23}\text{Na} \) (\( \delta g / g ≃ 0.07 \)) and the left condition ensures the mean-field miscibility of the two species (see Refs. [6,7]). At equilibrium both components are at rest with the same uniform density \( |\psi_↑|^2 = |\psi_↑|^2 \equiv \rho_0 / 2 \). It can be shown that this system exhibits two types of waves that can be called “density waves” – corresponding to global motion of the condensate – and “polarization waves” – corresponding to relative motion between the two components. In the small amplitude and long wavelength limit, the velocity of density waves and the velocity of polarization waves write respectively

\[
c_{d} = \sqrt{\frac{\rho_0 (2g - \delta g)}{2m}}, \quad c_{p} = \sqrt{\frac{\rho_0 \delta g}{2m}} .
\]  

In the limit (2), \( c_{p} \) is very small compared to \( c_{d} \). As a result, the typical length scale \( \xi_{p} \) (also called “polarization healing length”) and characteristic time scale \( T_{p} \) of the polarization waves defined by

\[
\xi_{p} = \frac{\hbar}{2mc_{p}}, \quad T_{p} = \frac{\xi_{p}}{c_{p}} = \frac{\hbar}{\rho_0 \delta g} ,
\]  

are much larger than the corresponding characteristic length and characteristic time associated with density waves. In order to study the polarization nonlinear waves, it is thus appropriate to pass to the non-dimensional variables (see Ref. [1])

\[
x / \xi_{p} \to x, \quad t / T_{p} \to t .
\]  

In these new variables, the density and the polarization degrees of freedom decouple. More precisely, the total density modulations propagating at large velocity, leave, after a short setup time, a large region where the dynamics is governed by polarization effects, in which the total density \( |\psi_↑|^2 + |\psi_↓|^2 \) is constant as well as the total current. In order to describe the polarization dynamics, we use the following parametrization:

\[
\begin{pmatrix}
\psi_↑ \\
\psi_↓
\end{pmatrix} = \sqrt{\rho_0} e^{i\Phi / 2} \begin{pmatrix}
\cos \phi \\
\sin \phi e^{i\phi / 2}
\end{pmatrix} ,
\]  

where \( \rho_0 \cos \theta = |\psi_↑|^2 - |\psi_↓|^2 \) represents the relative density and \( \phi = \text{arg}(\psi_↓) - \text{arg}(\psi_↑) \) represents the potential of the relative velocity between the two components. In the following, we work in a reference frame where there is no total density flux: this imposes \( \partial_x \Phi = \cos \theta \partial_x \phi \). The two dynamical fields describing the polarization dynamics can be chosen as the relative density described by \( w \equiv \cos \theta \) and the relative velocity \( v \equiv \partial_x \phi \). The polarization dynamics corresponding to (1) is governed by the equations

\[
\partial_t w - \partial_x [(1 - w^2)v] = 0 , \quad \partial_t v - \partial_x [(1 - v^2)w] + \partial_x \left[ \frac{1}{\sqrt{1 - w^2}} \partial_x \left( \frac{\partial_x w}{\sqrt{1 - w^2}} \right) \right] = 0 .
\]  

that can be shown to be equivalent to the dissipationless form of the Landau-Lifshitz equation governing the dynamics of the effective magnetization \( S \) in an asymmetric ferromagnet with “easy-plane magnetization”:

\[
S = \begin{pmatrix}
\sin \theta \cos \phi \\
\sin \theta \sin \phi \\
\cos \theta
\end{pmatrix} , \quad \partial_t S = S \times \left( \partial^2_x S - S \hat{z} \right) .
\]
3 The Riemann problem: Evolution of an initial step-like profile

As a typical application of the theory, we consider an initial step-like distribution of the form

\[ w(x, t = 0) = \begin{cases} w_L & \text{when } x < 0, \\ w_R & \text{when } x > 0, \end{cases} \quad \text{and} \quad v(x, t = 0) = \begin{cases} v_L & \text{when } x < 0, \\ v_R & \text{when } x > 0. \end{cases} \tag{9} \]

Since the step-like initial distribution (9) does not include any parameter having the dimension of a length, the subsequent solution will, for all times, depend only on the self-similar variable \( z = x/t \). In the following we will not compute the most general time evolved profile resulting from the initial condition (9), but rather choose specific initial conditions illustrating different typical results. The general case should then be a mixture of these “pure solutions”.

3.1 Dispersionless limit: the pure rarefaction wave

If \( w \) and \( v \) slowly vary over one polarization healing length, then we can neglect the dispersion effects described by the terms proportional to \( \partial_x^2 \partial_t \) in Eq. (8). Following the method presented in Ref. [4] one can cast the resulting equations in a symmetric form by introducing the following “Riemann invariants”

\[ \lambda_\pm(w, v) = wv \pm \sqrt{(1 - w^2)(1 - v^2)}, \tag{10} \]

which satisfy the nonlinear dispersionless equations

\[ \partial_t \lambda_\pm + V_\pm(\lambda_-, \lambda_+) \partial_x \lambda_\pm = 0, \quad \text{where} \quad V_\pm = \frac{3}{2} \lambda_\pm + \frac{1}{2} \lambda_\mp = 2wv \pm \sqrt{(1 - w^2)(1 - v^2)}. \tag{11} \]

Eqs. (11) have the familiar form of equations describing the dynamics of a compressible gas in terms of the Riemann invariants; however the relationships between the Riemann invariants (\( \lambda_-, \lambda_+ \)) and the physical variables (\( w = \cos \theta, v = \partial_x \phi \)) are more complicated in the present case than for a gaseous system. As explained above, the initial profile (9) yields solutions for which \( \lambda_\pm \) only depend on the self-similar variable \( z = x/t \). In this case, Eqs. (11) solve very easily

\[ \lambda_+ = \text{const.}, \quad \text{and} \quad V_-(\lambda-(z), \lambda_+) = z, \quad \text{or} \quad \lambda_- = \text{const.}, \quad \text{and} \quad V_+(\lambda-, \lambda_+(z)) = z. \tag{12} \]

For a given boundary condition, one can compute the constant Riemann invariant \( \lambda_+ \) (or \( \lambda_- \)) and then obtain the physical solutions \((w, v)\) by inverting one of the two solutions of Eqs. (12). For instance, the left solution of (12) yields two implicit formulae for the relative density \( w(z) \) and the relative velocity \( v(z) \):

\[ w(z)v(z) + \sqrt{1 - w(z)^2} \sqrt{1 - v(z)^2} = \lambda_+, \quad \text{and} \quad 2w(z)v(z) - \sqrt{1 - w(z)^2} \sqrt{1 - v(z)^2} = z. \tag{13} \]

In the general case, the initial discontinuity (9) evolves into a complex structure [2]. However, for some specific choices of initial distribution the solution consists in a single rarefaction wave connecting the left and right boundary flows. This case is depicted in Fig. 1 where it is compared with the result of numerical simulations. In the numerics the initial profile cannot be perfectly sharp as (9), and is taken of the form

\[ w(x, t = 0) = (w_R + w_L)/2 + (w_R - w_L)/2 \tanh(x/\zeta_0) \quad \text{with} \quad \zeta_0 = 1, \tag{14} \]

and a similar initial distribution for \( v(x, t = 0) \) (change \( w_{L/R} \) by \( v_{L/R} \)).

3.2 Dispersive shock wave: the pure cnoidal case

As shown in Ref. [2], the case where the initial profile (9) evolves into a pure rarefaction wave is not typical: one often observes the concomitant formation of a dispersive shock wave. Such a structure can be described as a nonlinear periodic solution of the polarization equations (8) whose parameters (amplitude,
velocity, period, etc.) are slowly modulated over one wavelength and one period of oscillation. This means that we can apply the Whitham averaging method for its description (see, e.g., Refs. [8,9]). It has been shown that Eq. (8) is integrable by the inverse scattering method (see, e.g., Ref. [10]), and its single phase periodic solutions have been derived in Ref. [2]; the relative density $w(x,t) = \cos \theta$ can be expressed in terms of Jacobi elliptic functions $cn$ and $sn$ (see Ref. [11]) and parametrized by a set of four integration constants $-1 \leq w_1 \leq w_2 \leq w_3 \leq w_4 \leq 1$:

$$w(x,t) = w_3 + \frac{(w_4 - w_3) cn^2(W,m)}{1 + \frac{w_4 - w_3}{w_3 - w_1} sn^2(W,m)}, \quad \text{where} \quad \begin{cases} W = \frac{1}{\lambda} \sqrt{(w_3 - w_1)(w_4 - w_2)} (x - Ut), \\ m = \frac{(w_4 - w_3)(w_2 - w_1)}{(w_4 - w_2)(w_3 - w_1)}. \end{cases}$$

(15)

The velocity of the wave $U$ can be expressed in terms of the parameters $w_i$ and the relative velocity $v(x,t)$ is a simple function of the relative density $w(x,t)$ and of the parameters $w_i$ (see Ref. [2]). Periodic nonlinear waves of type (15) are denoted as “cnoidal waves”.

In order to describe the slow modulation of the periodic wave parameters $w_i$, we adapted the Whitham theory derived in Ref. [10] for the “easy-axis ferromagnet”, to the case of the “easy-plane ferromagnet” corresponding to Eq. (8). A modulated cnoidal wave is described by four Riemann invariants (different from (10)) \( \{\lambda_i(w_1, w_2, w_3, w_4)\}_{1 \leq i \leq 4} \) obeying hydrodynamics equations similar to (11):

$$\partial_t \lambda_i + V_i(\lambda_1, \lambda_2, \lambda_3, \lambda_4) \partial_x \lambda_i = 0.$$  

(16)

For the sake of brevity, we do not write here the formulae linking the wave parameters $w_i$ and the Riemann invariants $\lambda_i$, neither the formulae for the Riemann velocities $V_i$. In the purely cnoidal situation considered in this subsection, the self-similar solutions of the Whitham equations (16) are determined by a set of equations analogous to (12), where now three Riemann invariants $\lambda_4$ are constant, and the only $z$-dependent term is $\lambda_3(z)$. Then, the $z$-dependence of the $w_i$’s which parametrize the cnoidal wave (15) is given by $w_i = w_i[\lambda_1, \lambda_2, \lambda_3(z), \lambda_4]$, where $\lambda_3(z)$ is determined by the implicit relation

$$V_3(\lambda_1, \lambda_2, \lambda_3(z), \lambda_4) = z.$$  

(17)

The constants $\lambda_1, \lambda_2$ and $\lambda_4$ are defined by the necessary matching between the end points of the cnoidal wave (15) and the left and right boundary conditions fixed by (9). This procedure can be simplified by noticing that these matching conditions correspond to the matching between the Riemann invariants of the Whitham modulation theory (the $\lambda_i$’s) and the Riemann invariants computed in the dispersionless
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The λ±'s which characterize the left and right boundary conditions. Since only λ3 can vary, these matching conditions read

\[ \lambda_1 = \lambda_-(w_L, v_L) = \lambda_-(w_R, v_R), \quad \lambda_2 = \lambda_+(w_R, v_R), \quad \text{and} \quad \lambda_4 = \lambda_+(w_L, v_L). \]  

(18)
The left-hand-side relations in (18) are only verified for specific boundary conditions leading to a pure dispersive shock. The comparison of the analytic predictions (17) and (18) with the numerical solution is displayed in Fig. 2. We choose to display the slowly varying envelop of the dispersive shock wave \((w_3, w_4)\) (15) as well as its first oscillation, which corresponds in this case to a bright soliton (limit \(w_3 \rightarrow w_2\) of (15), see Ref. [2]).

3.3 Dispersive shock wave: the pure trigonometric case

Eq. (8) admits periodic nonlinear solutions which do not consist in a cnoidal wave, but which are purely trigonometric [2], and which can be considered as limiting cases of (15) obtained for \(w_2 = w_3\):

\[ w = w_3 + \frac{(w_4 - w_3) \cos^2 W}{1 + \frac{w_4 - w_3}{w_4 - w_1} \sin^2 W}, \quad \text{where} \quad W = \sqrt{(w_3 - w_1)(w_4 - w_1)} (x - Ut)/2. \]  

(19)

It is possible to describe a dispersive shock wave consisting uniquely in a modulated periodic wave of type (19). This is achieved thanks to a Whitham averaging technique alternative to the so-called Gurevitch-Pitaevskii scheme [8] (which has been used in Sec. 3.2 for describing a purely cnoidal dispersive shock) in which \(\lambda_4(z) = \lambda_3(z)\) whereas the other Riemann invariants are \(z\)-independent. The final result and its comparison with our numerical simulations is presented in Fig. 3.

4 Conclusion

We have presented in this work some model solutions of the Riemann problem for the polarization dynamics of a two-component Bose-Einstein condensate, in the experimental relevant situation where the difference between intra- and inter-species interaction constants is small. These model solutions correspond to clean cut cases for which an initially discontinuous profile leads to a single rarefaction wave
Figure 3. Left: Riemann invariants used in the Whitham averaging technique plotted as a function of $z = x/t$ for an initial profile (9) with boundary values $(w_L, v_L) = (-0.4, 0.2)$ and $(w_R, v_R) = (0.2, -0.4)$. Right: Comparison between the results of the Whitham method for $w(z)$ (envelope $(w_3, w_4)$ in red and first oscillation in green) and the numerical solution at $t = 500$ (dark blue curve) of Eqs. (8) for the initial condition (14).

(Sec. 3.1) or a single dispersive shock wave (Secs. 3.2 and 3.3). In particular, the pure shocks have been shown to be very accurately described by using the Whitham averaging method. This has been made possible thanks to the exact integrability of the nonlinear equation describing the polarization dynamics.

Much more remains to be done. In particular, one has to understand how the solution of the general Riemann problem can be described by using the building blocks presented in this report. Work in this direction is in progress.

References