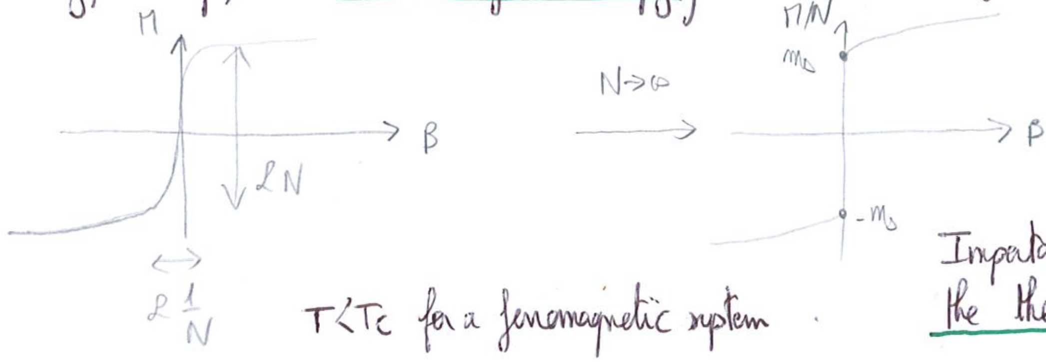
I) Introduction

The difficulties

(*) abundance / complexity → identify an order parameter

It is usually the derivative of some free energy (Helmholtz, Gibbs ...) wrt an external field: $M = -\frac{\partial F}{\partial B}\Big|_T$ for a magnet

(*) We know since Onsager that a phase transition ⟺ some singularity in the free energy (meaning, always, the relevant free energy). But $\ln Z$ analyticity problem...



$T<T_c$ for a ferromagnetic system.

Important to consider the thermodynamic limit

Classification of phase transitions

We distinguish

→ First order transitions: at least one first order derivative of the free energy is discontinuous

→ Continuous " : all first-order " are continuous; a higher order derivative is singular: hence 2nd order transitions, 3rd order (less usual)

Characterization of 2nd order p.t. by critical exponents ... reveal universality.

Hence the interest of...

The Ising model

→ Definition

→ Transition results from a competition between $\boxed{\text{energy} \Leftrightarrow \text{entropy}}$

Quite generic paradigm... but not universal! See entropy driven phase transitions

→ Back to Ising: energy / entropy competition strongly affected by [space dimension d ; order param dimension]

Discuss $O(n)$ models: $n=1$ (Ising); $n=2$ (XY); $n=3$ (Heisenberg); $\sum_{\alpha=1}^{n} (S_\alpha)^2 = 1$, $\vec{S} \in \mathbb{R}^n$

We start by discussing the effect of space dimension.

d=1 No phase transition with short range interactions. Use transfer matrix formalism

Start with Ising ($n=1$) with periodic boundary conditions

$$\beta H = -K \sum_{i=1}^{N} S_i S_{i+1} + -\tilde{B} \underbrace{\sum_{i=1}^{N} S_i}_{= \frac{1}{2} \sum_{i=1}^{N} (S_i + S_{i+1})} \quad ; \quad S_{N+1} = S_1$$

$\tilde{B} = \beta B$

$K = \beta J$

$$Z = \sum_{S_1 \dots S_N = \pm 1} \prod_{i=1}^{N} \underbrace{e^{K S_i S_{i+1} + \frac{\tilde{B}}{2}(S_i + S_{i+1})}}$$

$$T(S_i, S_{i+1}) = \langle S_i | \mathbb{T} | S_{i+1} \rangle$$

$$\mathbb{T} = \begin{pmatrix} e^{K+\tilde{B}} & e^{-K} \\ e^{-K} & e^{K-\tilde{B}} \end{pmatrix}$$ transfer matrix

$$= \sum_{S_1, \dots S_N} \langle S_1 | \mathbb{T} | S_2 \rangle \langle S_2 | \mathbb{T} | S_3 \rangle \cdots \langle S_N | \mathbb{T} | \underbrace{S_1}_{\equiv S_{N+1}} \rangle$$

$$= \mathrm{Tr}\left(\mathbb{T}^N \right)$$

$$= t_+^N + t_-^N$$

; t_+ and t_- are the 2 eigenvalues of $\mathbb{T}$

$t_+ > t_-$ by convention

$t_\pm = e^K \cosh \tilde{B} \pm \sqrt{e^{2K} \sinh^2 \tilde{B} + e^{-2K}}$

In the thermodynamic limit, t_+ dominates and the free energy per spin reads

$$\beta f = \frac{\beta F}{N} \xrightarrow[N \to \infty]{} -\ln t_+$$

We can also get the magnetization and correlation function:

$$\langle S_k \rangle = \frac{1}{Z} \sum_{S_1, \dots S_N} \langle S_1 | \mathbb{T} | S_2 \rangle \dots \langle S_{k-1} | \mathbb{T} | S_k \rangle S_k \langle S_k | \mathbb{T} | S_{k+1} \rangle \dots \langle S_{N-1} | \mathbb{T} | S_N \rangle$$

and we introduce the Pauli matrix $\hat{S}_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

$$\langle S_k \rangle = \frac{1}{Z} \mathrm{Tr}\left[\mathbb{T}^N \hat{S}_z \right]$$ since $\hat{S}_z | S_k \rangle = S_k | S_k \rangle$

$\Rightarrow \langle S_k | \hat{S}_z = \langle S_k | S_k$

$\mathbb{T} |+\rangle = 2\cos$

$\mathbb{T} |-\rangle = 2\sin$

when $B=0$, $\mathbb{T} = \begin{pmatrix} e^K & e^{-K} \\ e^{-K} & e^K \end{pmatrix}$

and the spectrum simplifies: $2\cosh K$ with eigenvector $\frac{1}{\sqrt{2}} \begin{vmatrix} 1 \\ 1 \end{vmatrix} \equiv |+\rangle$

$2\sinh K$ " " $\frac{1}{\sqrt{2}} \begin{vmatrix} 1 \\ -1 \end{vmatrix} \equiv |-\rangle$;

$\hat{S}_z |+\rangle = |-\rangle$

$\hat{S}_z |-\rangle = |+\rangle$

We compute the trace in the eigenbasis:

$$\langle S_k \rangle = \frac{1}{Z}\Big[\langle + | \mathbb{T}^N \hat{S}_z | + \rangle + \langle - | \mathbb{T}^N \hat{S}_z | - \rangle \Big]$$

$$= \frac{1}{Z}\Big[\underbrace{\langle + | \mathbb{T}^N | - \rangle}_{t_-^N \langle + | - \rangle} + \underbrace{\langle - | \mathbb{T}^N | + \rangle}_{t_+^N \langle - | + \rangle} \Big]$$

$= 0$, as a consequence of symmetry ($\neq 0$ if $B \neq 0$)

The correlation function is interesting:

$$\underbrace{\langle S_i \, S_{i+k} \rangle}_{= \langle S_1 S_{1+k} \rangle} = \frac{1}{Z} \sum_{S_1 \dots S_N} S_1 \, S_{1+k} \prod_{j=1}^{N} \langle S_j | \mathbb{T} | S_{j+1} \rangle$$

$$= \frac{1}{Z} \operatorname{Tr}\Big[\hat{S}_z \, \mathbb{T}^k \, \hat{S}_z \, \mathbb{T}^{N-k} \Big]$$ that we may again compute in the eigenbasis

$$= \frac{1}{Z}\Big[\underbrace{\langle + | \hat{S}_z \mathbb{T}^k \hat{S}_z | + \rangle}_{t_-^k} t_+^{N-k} + t_-^{N-k} \underbrace{\langle + | \hat{S}_z \mathbb{T}^k | \hat{S}_z | - \rangle}_{t_+^k}$$

$$= \frac{t_+^{N-k} \, t_-^k + t_-^{N-k} \, t_+^k}{t_+^N + t_-^N}$$

$$\xrightarrow[N \to \infty]{k \text{ fixed}} \left(\frac{t_-}{t_+} \right)^k \equiv e^{-k/\xi} \quad ; \quad \xi = -\frac{1}{\log(t_-/t_+)} = -\frac{1}{\log \tanh K}$$

Note the generality of the transfer matrix method: with nearest neighbours interactions in $d=1$, for an arbitrary interaction $\langle S_i | \mathbb{T} | S_j \rangle \equiv e^{-\beta u(S_i, S_j)} > 0$

For a q-states Potts model, $\mathbb{T}$ would be a $q \times q$ matrix.

Calling t_+ and t_- the two largest eigenvalues, we still have

$$Z = \operatorname{Tr}[\mathbb{T}^N] = t_+^N + t_-^N + \dots$$

and $$\xi = -\frac{1}{\ln(t_-/t_+)}$$

$\pi|+\rangle$

Perron-Frobenius theorem states that for a finite matrix with >0 entries

$\pi|-\rangle$

the largest eigenvalue is always non degenerate $\Rightarrow$ ξ is always finite and $\log Z$ (and also Z) are analytic functions of the parameters involved in βu.

~~NB If $t_+ = t_-$ for some parameter (say Temperature), then $\xi = \infty$ clearly, but note also that $\frac{1}{N} \log Z$ becomes singular:~~

$$\cancel{\frac{1}{N} \log Z = \log t_+ + \frac{1}{N} \log\left(1 + \left(\frac{t_-}{t_+}\right)^N\right)}$$

Thus, a one-dimensional model can exhibit a singularity only at $T=0$ when some matrix elements become infinite $\Rightarrow$ phase transition

↳ no phase transition at $T \neq 0 \Rightarrow$ Ising model is paramagnetic $\forall T > 0$

($d=2$) → Onsager's exact solution (see Kardar), 1944. Onsager used a $2^L \times 2^L$ transfer matrix to study a lattice of width L, in a row by row description. For any finite L, Perron-Frobenius thm applies and the largest eigenvalue of $\mathbb{T}$ is non degenerate. But for $L \to \infty$, the top 2 eigenvalues merge, and ξ can diverge. This gives

$$\frac{\ln Z}{N} = \ln 2 + \frac{1}{2} \int_{-\pi}^{\pi} \frac{dq_x \, dq_y}{(2\pi)^2} \ln\left[\cosh^2(2K) - \sinh(2K)(\cos q_x + \cos q_y)\right]$$

Onsager paper long & complicated; approach was simplified in following years.

↳ a mathematical tour de force

↳ the first exact solution for a phase transition, showing that critical behaviour was quite different from mean-field prediction, of Landau type. It took about 30 years to reconcile the 2 results, by the renormalization group.

Onsager presented without proof $\quad m = \left(1 - \sinh^{-4}(2K)\right)^{1/8}$ for magnetization

$\Rightarrow \beta = 1/8$

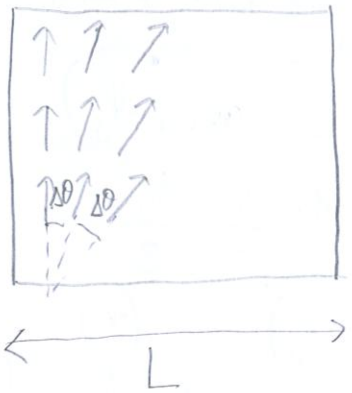
($d=3$) Wilson's joke

$d=1$, and $d>1$: transition or not..., where does the difference come from?

↳ notion of lower critical dimension

So far, we discussed the effect of space dimension d on the nature of fluctuations, and consequence for $\exists$ phase transition. The symmetry of the spin in the $O(n)$ model is also essential and affects the lower critical dimension $\Rightarrow$ importance of n ($n=1 \Leftrightarrow$ discrete spin $\Leftrightarrow$ standard Ising ; $n=2 \Leftrightarrow$ continuous, XY model ; $n=3$, also continuous $\Leftrightarrow$ Heisenberg model).

To get a flavor of physics involved, consider an XY / Heisenberg model. It is possible to form macroscopic excitations at very little cost $\Rightarrow$ detrimental to order, and we should expect the lower critical dimension to increase).



Angle $\Delta\theta \sim \frac{1}{L}$ between a layer and the next

Compared to ground state:

$$\Delta E \propto L^{d-1} \left(\underset{\substack{\downarrow \\ \text{in ground state}}}{1} - \underset{\substack{\downarrow \\ \text{with the } \Delta\theta \text{ tilt}}}{\cos\Delta\theta} \right) J \times L$$

$$\propto L^{d-1} (\Delta\theta)^2 \, L \propto L^{d-2}$$

For $d \leq 2$, we thus found essentially "costless" excitations (spin waves - magnons) which lead us to anticipate/expect that for $d \leq 2$, order cannot be sustained, entropic destabilization will prevail. This is correct, as rationalized by the

$\boxed{\text{Mermin-Wagner theorem}}$

There is no spontaneous breaking of $\overset{\text{continuous}}{\text{symmetry}}$ (no phase transition) with short-range interactions in dimension $d \leq 2$: $d_{lower} = 2$ for $n \geq 2$. For a discrete symmetry, i.e. $n=1$, we have seen $d_{lower} = 1$.

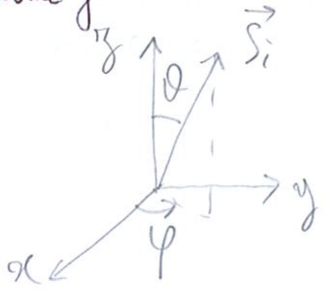
Mermin-Wagner: sketch of a proof

Take $O(n)$ model with $n \geq 2$, eg the Heisenberg model ($n=3$). We are going to show that fluctuations destroy long range order, for space dim $d \leq 2$.

$$\mathcal{H} = -J \sum_{\langle ij \rangle} \vec{S}_i \cdot \vec{S}_j \quad , \quad \text{on a cubic lattice, lattice spacing } a$$
$$|\vec{S}_i| = 1 \quad \text{for all spins.}$$

The ground state is degenerate. Assume all spins // z axis.

Assume fluctuations small, ie $|\theta|$ small ; $\vec{S}_i \begin{vmatrix} \sin\theta_i \cos\varphi_i \\ \sin\theta_i \sin\varphi_i \\ \cos\theta_i \end{vmatrix}$



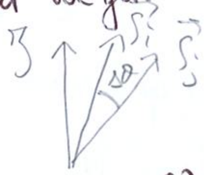
$\theta \equiv$ co-latitude

$$\vec{S}_i \cdot \vec{S}_j = \underbrace{\sin\theta_i}_{\sim\theta_i} \underbrace{\sin\theta_j}_{\sim\theta_j} \Big[\underbrace{\cos\varphi_i \cos\varphi_j + \sin\varphi_i \sin\varphi_j}_{\cos(\varphi_i - \varphi_j) \sim 1} \Big] + \underbrace{\cos\theta_i}_{1-\frac{\theta_i^2}{2}} \underbrace{\cos\theta_j}_{1-\frac{\theta_j^2}{2}}$$

$$\simeq \theta_i \theta_j + 1 - \frac{\theta_i^2}{2} - \frac{\theta_j^2}{2}$$

$$\simeq 1 - \frac{1}{2}(\theta_i - \theta_j)^2 \qquad (*)$$

Note that we get the same result with the XY model in the plane ($(d=2)$, $n=2$)



$$\vec{S}_i \cdot \vec{S}_j = \cos(\theta_i - \theta_j) \simeq 1 - \frac{(\theta_i - \theta_j)^2}{2}$$

and this result$^{(*)}$ is space dimension independent

Indeed: assume $\vec{S}_i$ and $\vec{S}_j = \vec{S}_i + \vec{\delta S}$ are close to each other

$$|\vec{S}_i| = |\vec{S}_j| = 1 \Rightarrow 1 = \vec{S}_j^{\,2} = \underbrace{\vec{S}_i^{\,2}}_{1} + 2\vec{S}_i \cdot \vec{\delta S} + (\vec{\delta S})^2 \Rightarrow 2\vec{S}_i \cdot \vec{\delta S} + |\delta S|^2 = 0$$

$$\cos(\vec{S}_i \cdot \vec{S}_j) = \vec{S}_i \cdot \vec{S}_j = \vec{S}_i \cdot (\vec{S}_i + \vec{\delta S}) = 1 + \vec{S}_i \cdot \vec{\delta S} = 1 - \frac{1}{2}|\delta S|^2$$
$$\simeq 1 - \frac{1}{2}(\theta_i - \theta_j)^2$$

This yields the simplified hamiltonian

$$\mathcal{H} \simeq + \frac{J}{2} \sum_{\langle ij \rangle} (\theta_i - \theta_j)^2 \xrightarrow[\text{limit}]{\text{continuous}} \frac{J}{2} \int \frac{d\vec{r}}{a^d} a^{+2} (\vec{\nabla}\theta)^2$$

Fluctuations are thus gaussian

$$\langle S_z \rangle = \langle \cos\theta \rangle \simeq 1 - \frac{\langle \theta^2 \rangle}{2} .$$ What is $\langle \theta^2 \rangle$ the variance of angle fluctuations?

Remember that for gaussian distribution with density

$$p(\vec{x}) \propto \exp\left(-\frac{1}{2} x_i \Gamma_{ij} x_j\right)$$ (Einstein convention of summation)

$$\langle x_i x_j \rangle = \left(\Gamma^{-1}\right)_{ij} \equiv G_{ij}$$ and $\langle e^{\vec{h}\cdot\vec{x}} \rangle = e^{\frac{1}{2} h_i \Gamma^{-1}_{ij} h_j}$, not useful here

In the present case: $\Gamma(\vec{r},\vec{r}\,') = -\beta J\, a^{2-d}\, \delta(\vec{r}-\vec{r}\,')\, \nabla^2_{\vec{r}}$

$$e^{-\beta H} = e^{-\frac{\beta J}{2} a^{2-d} \int d\vec{r} (\vec{\nabla}\theta)^2} = e^{+\frac{\beta J}{2} a^{2-d} \int d\vec{r}\, \theta\, \vec{\nabla}^2 \theta}$$ by parts

We have to find the inverse of the operator Γ, meaning $G(\vec{r},\vec{r}\,')$ s.t

$$\int d\vec{r}\,' \,\Gamma(\vec{r},\vec{r}\,')\, G(\vec{r}\,',\vec{r}\,'') = \delta(\vec{r}-\vec{r}\,'') ;$$ and $G(\vec{r}\,',\vec{r}\,'') = G(\vec{r}\,'-\vec{r}\,'')$ assuming invariance by translation

Here, this means $-\beta J a^{2-d} \nabla^2_{\vec{r}} G(\vec{r}) = \delta(\vec{r})$

$$\Rightarrow G(\vec{r}) = a^{d-2} \frac{kT}{J} \int \frac{d\vec{q}}{(2\pi)^d} \frac{e^{i\vec{q}\cdot\vec{r}}}{q^2}$$ where $\int$ "$\equiv$" $\int_{\frac{1}{L}}^{\frac{1}{a}}$ a fixed $L \to \infty$?

$$\Rightarrow \langle \theta^2 \rangle = G(\vec{0}) = a^{d-2} \frac{kT}{J} \underbrace{\int \frac{d\vec{q}}{(2\pi)^d} \frac{1}{q^2}}$$

$$\int_{\frac{1}{L}}^{\frac{1}{a}} dq\, q^{d-3} \propto \frac{a^{2-d} - L^{2-d}}{d-2} \propto \begin{cases} a^{2-d}, & d>2 \quad \text{converges in } L \\ L^{2-d}, & d<2 \quad \text{diverges with } L \end{cases}$$

Thus when $L \to \infty$, $\langle \theta^2 \rangle \to \infty$ for $d<2$, inconsistent

↳ <u>no long range order</u>

$\langle \theta^2 \rangle \propto \frac{kT}{J}$ for $d \geq 2$, small at small T, order is sustainable

The ordered phase is unstable wrt fluctuations of thermal origin, for $d<2$, hence no phase transition □

→ Mermin-Wagner

→ The absence of order for $d \leq 2$ is due to $\boxed{\text{Goldstone modes}}$, ie a global change in system's state, continuously parameterized (eg a rotation of all spins here) that does not cost energy. With discrete spins, there is no Goldstone mode.

The borderline dimensionality of two (the lower critical dimension) deserves a special treatment. For $d=2$, the above L^{2-d} becomes $\log L$, hence divergence and no order: there is no true long-range order, but a quasi-long-range order is possible. For $T<T_c$: correlations decay algebraically

$T>T_c$: " " exponentially, short-range.

This is a new type of phase transition, where topological defects play a key role

↳ Kosterlitz-Thouless transition (work in the 1970s, Nobel 2016).

Topological phase transition + Berezinskii

Order parameter and broken symmetry

When a phase transition occurs (eg in a magnetic system...), the low temperature phase does not feature the same symmetry as the Hamiltonian (invariance by arbitrary global rotation in XY model, or invariance by flipping all spins for Ising with $B=0$) → symmetry breaking

Assume we are Onsager, able to solve the 2d Ising model ($B=0$); how would we extract the order parameter? There are 3 solutions to the pb that with $M=\sum S_i$

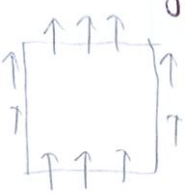
$$\langle M\rangle = \frac{1}{Z}\sum_{\mathcal{C}} e^{-\beta H} M = 0$$

⊛ through a vanishingly small field

$$\lim_{B\to 0^+}\lim_{N\to\infty}\frac{\langle M\rangle}{N} = m^*$$

but even Onsager cannot consider $B\neq 0$

⊛ through the boundary conditions (frozen ↑); take $N\to\infty$



$$m^* = \lim_{N\to\infty}\frac{\langle N_{up}\rangle - \langle N_{down}\rangle}{N}$$

$\langle M\rangle$

⊛ through correlations: non invasive solution!

$$\langle S(\vec{r}_1)\,S(\vec{r}_2)\rangle \xrightarrow[|\vec{r}_1-\vec{r}_2|\to\infty]{} (m^*)^2 \;\Bigg|\; \begin{array}{ll} 0, & T>T_c \\ m^{*2}\neq 0, & T<T_c \end{array}$$

Local order and correlation functions

We have seen that correlation functions hide the order parameter. They also contain key other information, useful to close the description of the phase transition.

$$\underbrace{G(\vec{r}_i,\vec{r}_j)}_{G_{ij}} = \langle S(\vec{r}_i)\, S(\vec{r}_j)\rangle - \underbrace{(m^*)^2}_{\lim\limits_{r_{ij}\to\infty} \langle S(\vec{r}_i) S(\vec{r}_j)\rangle}$$

For $T \neq T_c$, $G(r) \doteq e^{-r/\xi}$

↳ equal to leading exponential order, ie $\log G \sim -r/\xi$, $r\to\infty$

$$\boxed{\xi \propto |T-T_c|^{-\nu}} \longrightarrow \infty \quad \text{for } T\to T_c$$

This divergence is related to that of the susceptibility through

$$kT\,\frac{\chi}{N} = \frac{\langle M^2\rangle - \langle M\rangle^2}{N} = \frac{1}{N}\sum_{ij} G_{ij} \simeq \int G(\vec{r})\, d\vec{r}, \text{ finite if } \xi \text{ is finite}$$

Thus χ infinite $\Rightarrow$ ξ infinite, happens at T_c

$\Leftarrow$ also holds

For $T=T_c$: $\boxed{G(r) \propto \dfrac{1}{r^{d-2+\eta}}}$, $r\to\infty$

Meaning of the correlation function

→ size of the correlated domains, ie something like a liquid droplet that appears in a gas, or a magnetic domain for magnets

→ yet, this formulation is incomplete, as revealed by **critical opalescence** experiments. We see milky white aspect (not transparent) in the test tube in some T window around T_c, not only at T_c. This window is for situations where $\xi > \lambda_{\text{visible}}$ and indicates that there are objects (correlated domains) of size λ_{visible}, to interact with light, and scatter it. This is a hint that ξ is the upper cutoff for the size of correlated domains: there are such domains for all sizes below ξ (down to molecular scale), but not above ξ.

→ If a macroscopic sample is cut in half: same behaviour, and free energy of the whole is the sum of the two. This can be repeated down to size $\sim \xi$.

The Mermin-Wagner sheds interesting light on the physics of the XY model, ie on the phenomenology at the borderline dimension of 2. ② bis

At small T, the behaviour is dominated by spin waves, and we can write

$$\mathcal{H} = -J \sum_{\langle \vec{x},\vec{y}\rangle} S_{\vec{x}} \cdot S_{\vec{y}} \simeq \text{cst} + \frac{J}{2} \sum_{\langle \vec{x},\vec{y}\rangle} \left(\theta_{\vec{x}} - \theta_{\vec{y}}\right)^2$$

Beyond $\langle \theta^2 \rangle$ that we computed, we can get the correlation function (we got it!)

$$\langle \theta_{\vec{x}} \theta_{\vec{y}} \rangle = G(\vec{x}-\vec{y}) \qquad \text{and here } d=2$$

$$K \equiv \beta J$$

$$= \frac{1}{K} \int \frac{d^2\vec{q}}{(2\pi)^2} \frac{e^{i\vec{q}\cdot\vec{r}}}{q^2}$$

$$= \underbrace{-\frac{1}{2\pi K} \log r}_{\text{Coulomb potential (apart from } K)} + \text{Cst} \qquad \left(// \text{ of } \frac{1}{4\pi r} \text{ in } 3d \right)$$

Then: $\langle S_{\vec{x}} \cdot S_{\vec{y}} \rangle = \langle \cos(\theta_{\vec{x}} - \theta_{\vec{y}}) \rangle = \mathrm{Re} \left\langle e^{i(\theta_{\vec{x}} - \theta_{\vec{y}})} \right\rangle = \left\langle e^{i(\theta_{\vec{x}} - \theta_{\vec{y}})} \right\rangle$

and we know that when X is a gaussian variable of mean 0 and variance σ^2

$$\langle e^{kX} \rangle = e^{k^2\sigma^2/2} \qquad \text{Here } X = \theta_{\vec{x}} - \theta_{\vec{y}}$$

$$\Rightarrow \langle S_{\vec{x}} \cdot S_{\vec{y}} \rangle = e^{-\langle (\theta_{\vec{x}} - \theta_{\vec{y}})^2 \rangle / 2}$$

$$= \exp\left[-\frac{1}{2}\langle \theta_{\vec{x}}^2 \rangle - \frac{1}{2}\langle \theta_{\vec{y}}^2 \rangle + \langle \theta_{\vec{x}} \theta_{\vec{y}} \rangle \right]$$

$$= \exp\left[-G(\vec{0}) + G(\vec{x}-\vec{y}) \right]$$

$$\propto \exp\left[+G(\vec{x}-\vec{y}) \right]$$

$$\propto \frac{1}{\|\vec{x}-\vec{y}\|^{\frac{1}{2\pi K}}}$$

→ Hugely different from Ising model (no exponential decay)

→ Scale invariant but with a temperature dependent exponent

→ No spontaneous order since $\xrightarrow[\|\vec{x}-\vec{y}\| \to \infty]{} 0$

On the other hand, at high temperature:

$$\langle S_{\vec{x}} \cdot S_{\vec{y}} \rangle \propto \exp\left(-\frac{\|\vec{x}-\vec{y}\|_1}{\xi}\right)$$

$$\xi = -\frac{1}{\ln K} \to 0$$

high T means K small

↳ same as Ising $-\frac{1}{\ln \tanh K}$!

Summary

$0 \qquad\qquad T_c \qquad\qquad \infty \qquad \to T$

$T < T_c$: $\langle S_{\vec{x}} \cdot S_{\vec{y}} \rangle \propto \frac{1}{|\vec{x}-\vec{y}|^{\varphi}}$

$\varphi = \frac{1}{2\pi K} \propto T \to 0$

$\varphi \underset{T \to T_c^-}{\to} 1/4$

quasi long-range order

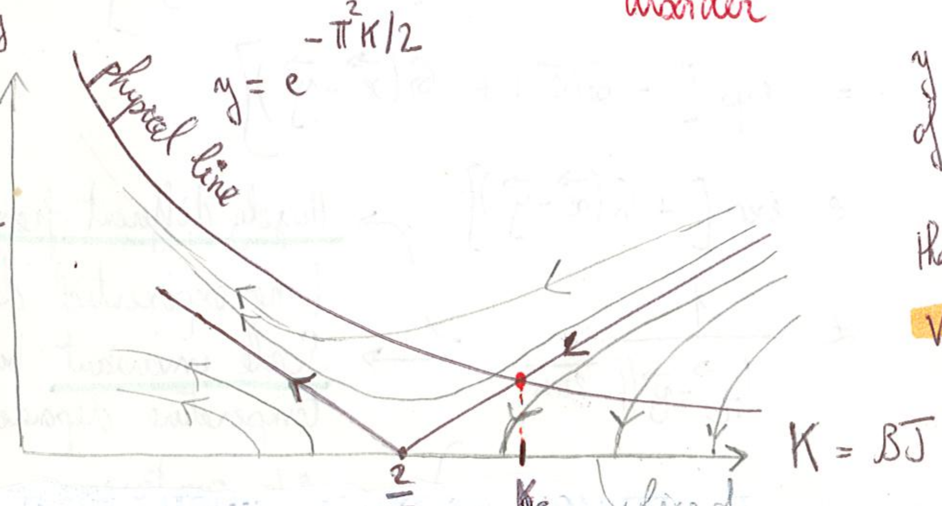
$T > T_c$: $\langle S_{\vec{x}} \cdot S_{\vec{y}} \rangle$ exponential, correlation length ξ

$\xi = -\frac{1}{\ln K} \to 0$

$\xi \propto \exp\left(\frac{c}{\sqrt{T-T_c}}\right)$, diverges $T \to T_c^+$

disorder

RG flow



control vortices/defects ≡ topological charges (cannot be removed by continuous deformation)

disorder

qlro

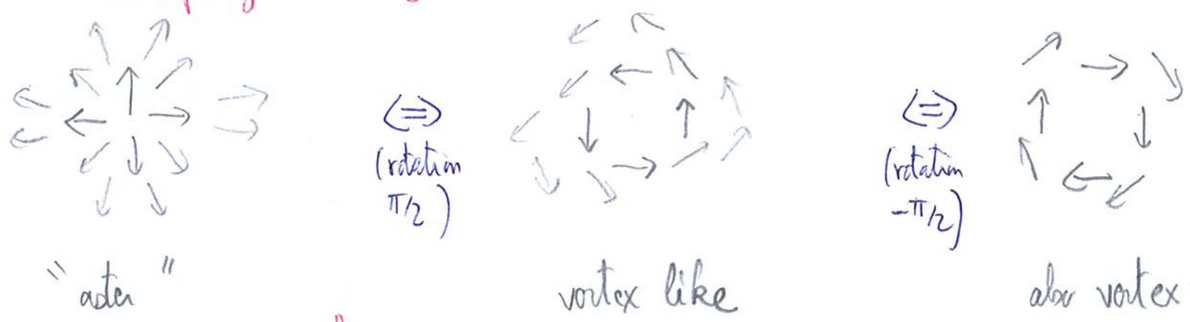
y controls the appearance of defects ≡ vortices (~ charges) that appear in pairs

Vortex unbinding at K_c

Defects in the XY model, representing spins

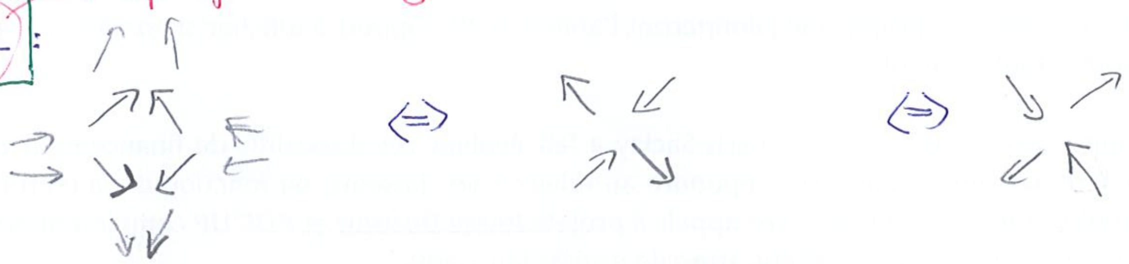
$\boxed{\text{Defect } +1}$: all configs are equivalent (up to a rotation of each spin, which does not change energy)

→ topological charge

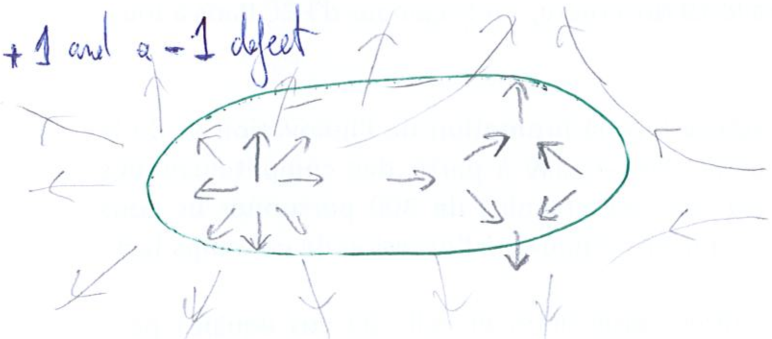


→ topological charge

$\boxed{\text{Defect } -1}$:



A +1 and a −1 defect



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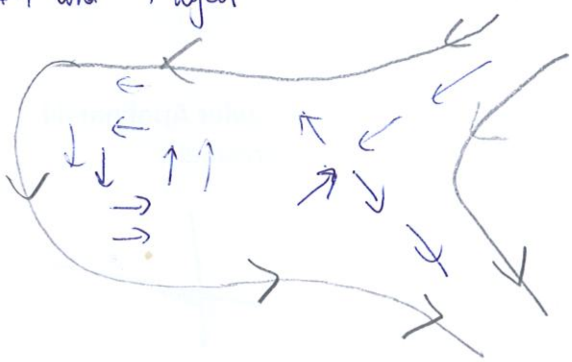
Another +1 and −1 defect



Conclusion on introduction

- Phase transition $\Leftrightarrow$ discontinuity or singularity in free energy derivative
- No phase transition in a finite system
  what about computer simulations $\rightarrow$ see tutorial on Binder cumulants
  $\rightarrow$ finite-size scaling in ch. IV
- Phase transition means long-range order $(T<T_c)$
  Although the correlation function is short-range outside T_c (exponential)
- Correlation length ξ diverges at T_c:
  microscopic details no longer matter $\rightarrow$ universality
  $\rightarrow$ Renormalization Group treatment
- Notion of lower critical dimension, below which fluctuations destroy order
  1 for discrete spins, 2 in continuous case (Mermin-Wagner)
  ↳ Goldstone modes destabilize order
- Beyond universality, Ising also directly useful for
  - binary alloys
  - lattice gas models, for liq/gas transition or adsorption of H onto iron
  - fads / hypes
  - neuronal activity
  - protein folding
  - bird flocking