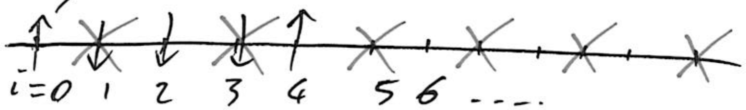
$\sigma_0 = +1$

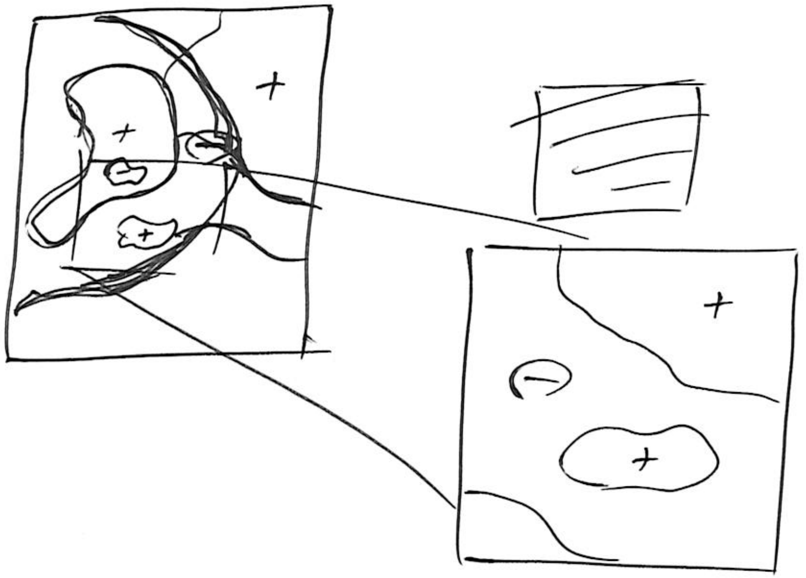


PBC: $\sigma_0 = \sigma_N$

$\sigma_1 = \sigma_{N+1}$

$\sigma_i = \sigma_{i+N}$

in 2d at the transition:



$$H = -J \sum_{i=1}^{N} \sigma_i \sigma_{i+1}$$

$$Z = \sum_{\{\sigma_i\}} e^{-\beta H}$$

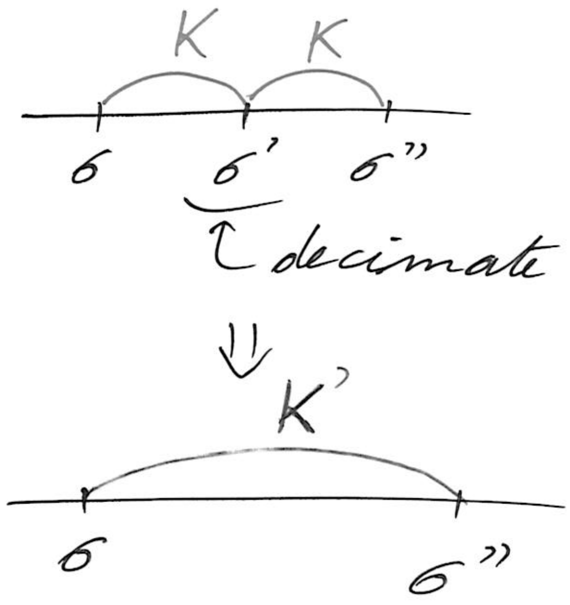
$$= \sum_{\{\sigma_i = \pm 1\}} e^{\overbrace{\beta J}^{=K} \sum_{i=1}^{N} \sigma_i \sigma_{i+1}}$$

$$Z = \sum_{\{\sigma_i \text{ with } i \text{ even} = \pm 1\}} e^{-\beta H'}$$

$$= \sum_{\{\sigma_{i\,\text{even}} = \pm 1\}} e^{K' \sum (\dots)}$$

1.1.1)

$$\sum_{\sigma'=\pm 1} e^{K\sigma\sigma' + K\sigma'\sigma''} = A\, e^{K'\sigma\sigma''}$$



The top/bottom expressions ~~is~~ are

- when $\sigma = \sigma''$
  - $\sum_{\sigma'} e^{K\sigma\sigma' + K\sigma'\sigma''} = \sum_{\sigma'} e^{2K\sigma\sigma'} = 2\,\mathrm{ch}(2K\cdot\sigma) = 2\,\mathrm{ch}\,2K$
  - $A\, e^{K'\sigma\sigma''} = A\, e^{K'}$

  $A\, e^{K'} = 2\,\mathrm{ch}\,2K$

- when $\sigma = -\sigma''$
  - $\sum_{\sigma'} e^{K\sigma'(\sigma+\sigma'')} = \sum_{\sigma'} e^{0} = 2$
  - $A\, e^{K'\sigma\sigma''} = A\, e^{-K'}$

  $A\, e^{-K'} = 2$

The two qties are equal iff

$$A^2 = 4\,ch(2K) \implies A = 2\sqrt{ch(2K)}$$

$$e^{2K'} = ch\,2K \implies e^{K'} = \sqrt{ch(2K)} \text{ or } K' = \frac{1}{2}\ln(ch\,2K)$$

1.1.2) $$Z = \sum_{\{\sigma_i\}} \prod_{i=1}^{N} e^{K\sigma_i\sigma_{i+1}}$$

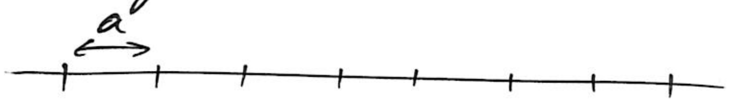
$$= \sum_{\{\sigma_i\}} \prod_{i=1}^{N/2} e^{K\sigma_{2i}\sigma_{2i+1} + K\sigma_{2i+1}\sigma_{2i+2}}$$

$$= \sum_{\{\sigma_i\}_{i\,even}} \prod_{i=1}^{N/2} \left(\sum_{\sigma_{2i+1}=\pm 1} e^{K\sigma_{2i}\sigma_{2i+1} + K\sigma_{2i+1}\sigma_{2i+2}} \right)$$

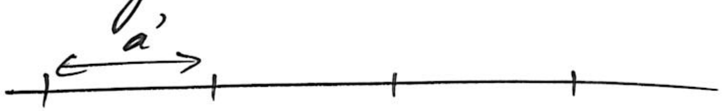
$$= \sum_{\{\sigma_i\}_{i\,even}} \prod_{i=1}^{N/2} A\, e^{K'\sigma_{2i}\sigma_{2i+2}}$$

$$Z(K, N, a) = A^{N/2} \cdot Z\left(K', N' = \frac{N}{2}, a' = 2a\right)$$

old system



new system



eff $\tilde{Z} = B \cdot Z$

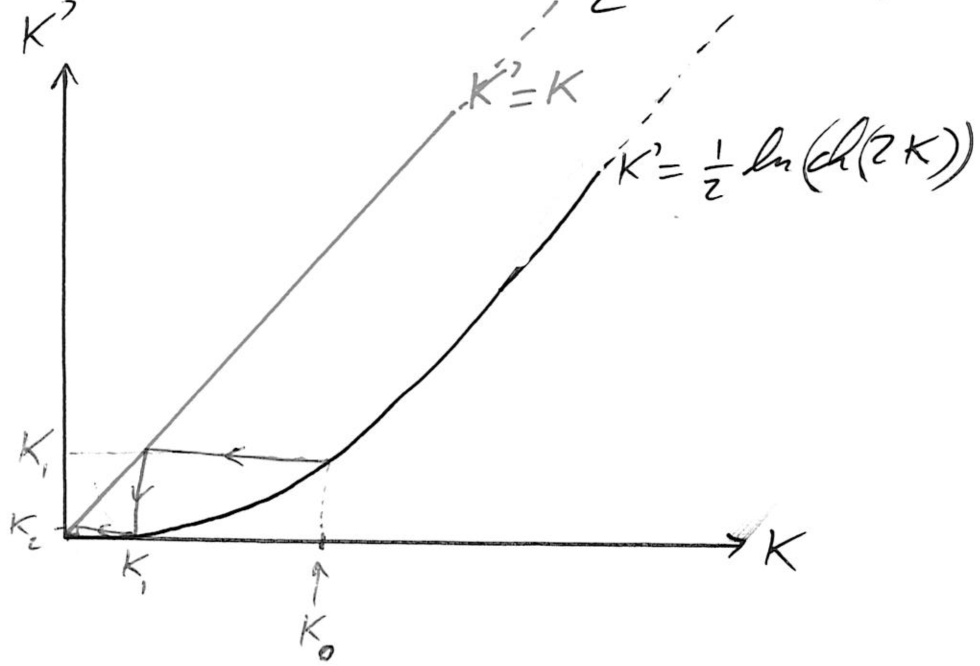
$$\tilde{F} = -k_B T \ln \tilde{Z}$$

$$= -k_B T \ln B - k_B T \ln Z$$

$$= \underbrace{-k_B T \ln B}_{\text{shift in the free energy.}} + F$$

So A brings a shift in the free energy $\rightarrow$ irrelevant.

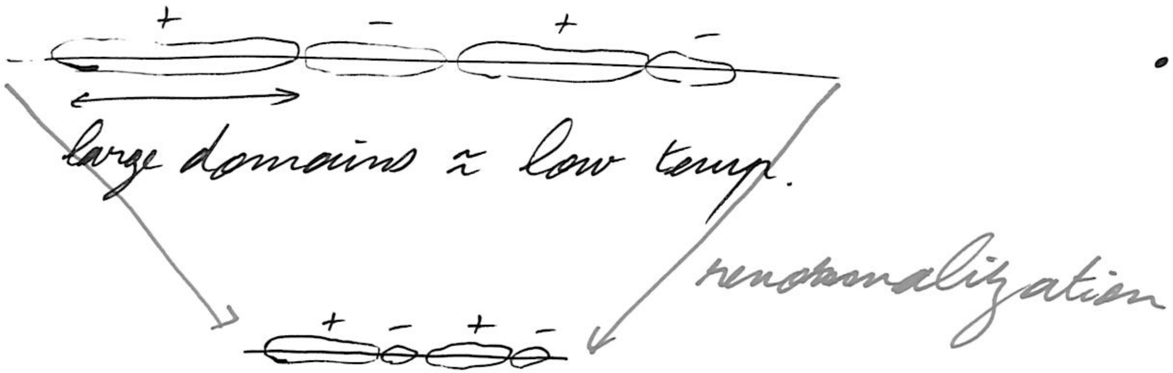
1.1.3) $K' = \frac{1}{2} \ln(\mathrm{ch}\, 2K)$



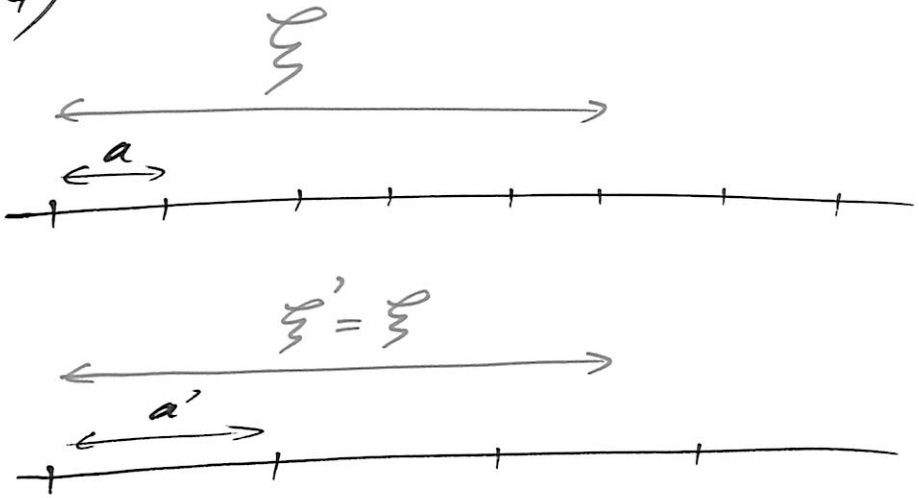
$K' \underset{K\to 0}{\sim} K^2$

2 fixed points:

- $K=0$ (high temperature)
  $\hookrightarrow = \beta J = \frac{J}{k_B T}$
- $K=+\infty$ (low temperature)



1.1.4)



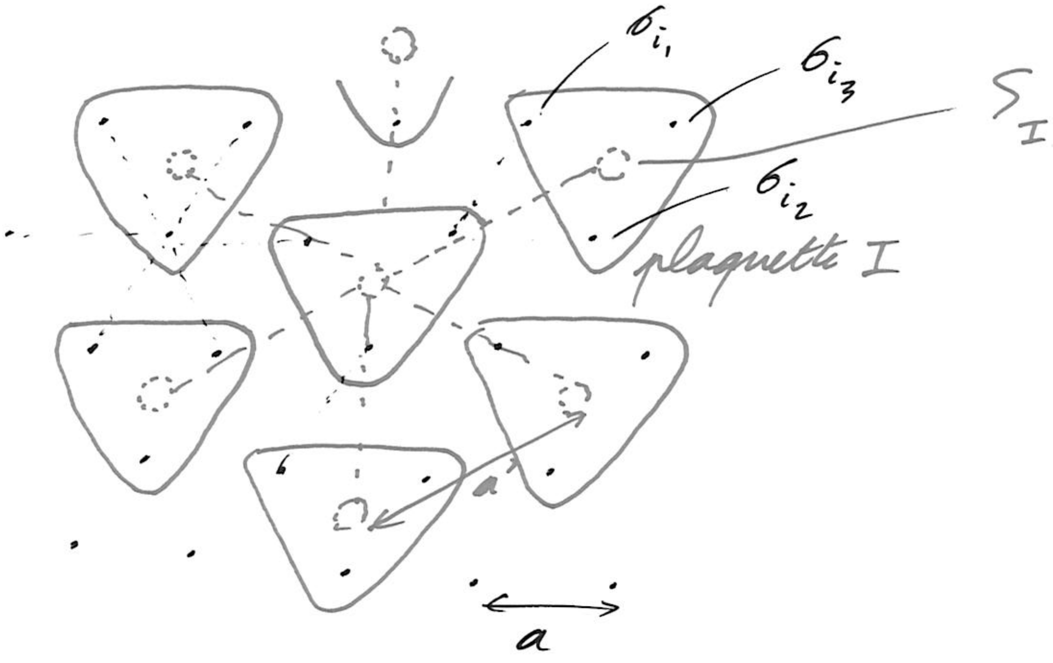
$\tilde{\xi} = \frac{\xi}{a}$ dimensionless correlation length

$$\tilde{\xi}' = \frac{\xi'}{a'} = \frac{\xi}{2a} = \frac{\tilde{\xi}}{2}$$

$$\begin{cases} \tilde{\xi}(K') = \dfrac{\tilde{\xi}(K)}{2} \\ K' = \dfrac{1}{2} \ln(\operatorname{ch} 2K) \end{cases} \rightsquigarrow \boxed{\operatorname{th} K' = (\operatorname{th} K)^2}$$

Let $y = \frac{1}{\tilde{\xi}}$ $\quad x = \ln(\operatorname{th} K)$; then $\forall x$ $y(2x) = 2y(x)$

Therefore $y \propto x \Longleftrightarrow \tilde{\xi}(K) \propto -\dfrac{1}{\ln(\operatorname{th} K)}$



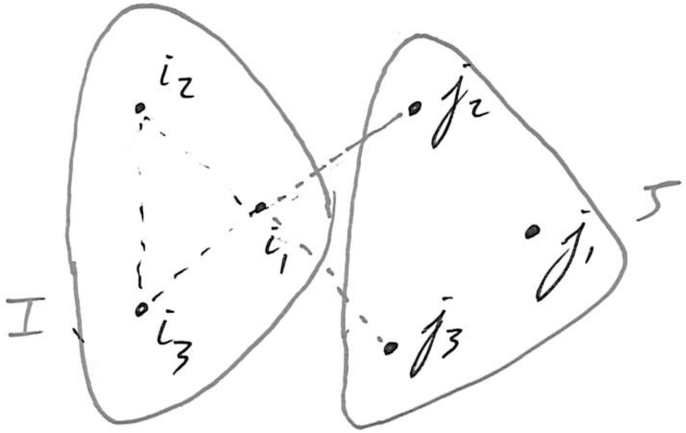
- $S_I = +1$ if a majority of $\{\sigma_{i_1} \ \sigma_{i_2} \ \sigma_{i_3}\}$ is >0
- — $= -1$ ———————— ———————— is <0

1.2.1) $N' = \dfrac{N}{3}$

area is conserved $N a^2 = N' (a')^2$

$$\Longrightarrow a' = \sqrt{3} \cdot a$$

1.2.2)



intra-plaquette:

$$h_1(I) = -J\left(\sigma_{i_1}\sigma_{i_2} + \sigma_{i_2}\sigma_{i_3} + \sigma_{i_3}\sigma_{i_1}\right)$$

inter-plaquette:

$$h_2(I,J) = -J\,\sigma_{i_1}\left(\sigma_{j_2} + \sigma_{j_3}\right)$$

1.2.3)

$$Z_1(\{S_I\}) = \sum_{\{\sigma_i\}}{}' e^{-\beta \sum_I h_1(I)}$$

sum over σ_i keeping the S_I fixed

$$= \prod_I \sum_{\{\sigma_{i_1}\sigma_{i_2}\sigma_{i_3}\}|S_I} e^{-\beta h_1(I)}$$

$$\sum_{\{\sigma_{i_1}\sigma_{i_2}\sigma_{i_3}\}|S_I} e^{-\beta h_1(I)} \qquad h_1(I) = -J\,(\sigma_{i_1}\sigma_{i_2} + \sigma_{i_2}\sigma_{i_3} + \sigma_{i_3}\sigma_{i_1})$$

Let $S_I = +1$

σ_{i_1}	σ_{i_2}	σ_{i_3}	$h_1(I)$	weight
+	+	+	$-3J$	e^{3K}
+	+	−	$+J$	e^{-K}
+	−	+	$+J$	e^{-K}
−	+	+	$+J$	e^{-K}

So $$\sum_{\{-\}|S_I=1} e^{-} = e^{3K} + 3e^{-K}$$

$$\sum_{\{-\}\,S_I=-1} e^{-} = e^{3K} + 3e^{-K}$$

$$Z_1(\{S_I\}) = \prod_{I=1}^{N/3} \left(e^{3K} + 3e^{-K}\right)$$

$$= \left(e^{3K} + 3e^{-K}\right)^{N/3} \qquad \text{indpt of } \{S_I\}$$

1.2.4) $$Z(K, N, a) = \sum_{\{\sigma_i\}} e^{-\beta(H_1 + H_2)}$$

$$= \sum_{\{S_I\}} \sum_{\{\sigma_i\}}{}' \, e^{-\beta H_1} \, e^{-\beta H_2}$$

$$= \sum_{\{S_I\}} Z_1 \times \underbrace{\frac{1}{Z_1} \sum_{\{\sigma_i\}}{}' \, e^{-\beta H_1} e^{-\beta H_2}}_{\left\langle e^{-\beta H_2} \right\rangle_1}$$

$$\frac{1}{Z_1(\{S_I\})} \sum_{\{\sigma_i\}}{}' \, e^{-\beta H_1} A = \langle A \rangle_1$$

avg of A w.r.t. H_1

1.2.5)
$$\left\langle e^{-\beta H_2} \right\rangle_1 = e^{\langle -\beta H_2 \rangle_{1,c} + \frac{1}{2}\langle (-\beta H_2)^2 \rangle_{1,c} + \cdots}$$

by definition of $\langle \cdots \rangle_c$

$$= e^{-\beta \langle H_2 \rangle_{1,c}} \times e^{\frac{\beta^2}{2}\underbrace{\langle (H_2)^2 \rangle_{1,c}}_{\text{approximate to } 0}} \times \cdots$$

1.2.6) The cumulants of order 2 or higher describe the correlations btw the internal spin variables (at fixed $\{S_I\}$) belonging to ≠ plaquettes.

$$\langle x y \rangle_c = \langle x y \rangle - \langle x \rangle \langle y \rangle = 0$$

true if x, y statistically indpt.

(12)

1.2.7) $$\langle \sigma_{i_1} \rangle_1 = \frac{(+1)\cdot e^{3K} + (+1) e^{-K} + (+1) e^{-K} + (-1)\cdot e^{-K}}{e^{3K} + 3e^{-K}} \quad \text{if } S_I = +1$$

for $S_I = +1$ $$\langle \sigma_{i_1} \rangle_1 = \frac{e^{3K} + e^{-K}}{e^{3K} + 3e^{-K}}$$

for $S_I = -1$ $$\langle \sigma_{i_1} \rangle_1 = -\frac{e^{3K} + e^{-K}}{e^{3K} + 3e^{-K}}$$

So $$\langle \sigma_{i_1} \rangle_1 = \frac{e^{3K} + e^{-K}}{e^{3K} + 3e^{-K}} \, S_I$$

1.2.8) We want to compute $\langle H_2 \rangle_1 = \sum_{\langle I,J \rangle} \langle h_2(I,J) \rangle$

$$\langle h_2(I,J) \rangle_1 = -J \langle \sigma_{i_1} (\sigma_{j_2} + \sigma_{j_3}) \rangle_1$$
$$= -J \left[\langle \sigma_{i_1} \sigma_{j_2} \rangle_1 + \langle \sigma_{i_1} \sigma_{j_3} \rangle_1 \right]$$

Because H_1 doesn't couple spins from $\neq$ plaquettes

$$\langle \sigma_{i_1} \sigma_{j_2} \rangle_1 = \langle \sigma_{i_1} \rangle_1 \cdot \langle \sigma_{j_2} \rangle_1$$

$$\langle h_2(I,J) \rangle_1 = -J \langle \sigma_{i_1} \rangle_1 \left(\langle \sigma_{j_2} \rangle_1 + \langle \sigma_{j_3} \rangle_1 \right)$$

$$= -J \, S_I S_J \cdot 2 \left(\frac{e^{3K} + e^{-K}}{e^{3K} + 3e^{-K}} \right)^2$$

So if $\underline{K' = 2K \left(\frac{e^{3K} + e^{-K}}{e^{3K} + 3e^{-K}} \right)^2}$, then

$$Z(K, N, a) = \sum_{\{S_I\}} \cdot Z_1 \cdot e^{\sum_{\langle IJ \rangle} K' S_I S_J}$$

$$= \left(e^{3K} + 3e^{-K} \right)^{N'} \cdot Z(K', N', a')$$