$$H = \sum_I h_1(I) + \sum_{\langle I,J\rangle} h_2(I,J)$$

Questions 1.2.3 $\longrightarrow$ 1.2.6

this means that the summation over the σ_i's is at fixed S_I's.

Use $$\langle \dots \rangle = \frac{1}{Z_1} \sum_{\{\sigma_i\}}' \exp - \beta \sum_I h_1(I) \dots$$

$$Z = \sum_{\{S_I\}} Z_1 \langle e^{-\beta H_2} \rangle_1 \underset{N.-V.L}{\overset{\text{approx}}{\simeq}} \sum Z_1 e^{-\beta \langle H_2 \rangle_1}$$

$$\langle H_2 \rangle_1 = \langle \sum_{\langle I,J\rangle} h_2(I,J) \rangle = -K' \sum_{\langle I,J\rangle} S_I S_J$$

$$K' = 2K \left(\frac{e^{3K} + e^{-K}}{e^{3K} + 3e^{-K}} \right)^2$$

1.2.8. $$Z(K, N, a) = (e^{3K} + 3e^{-K})^{N'} Z(K', N', a')$$

$$K' = f(K) \qquad N' = \frac{N}{b^2}, \; a' = ba$$

$$b = \sqrt{3}$$

1.2.9. Search for the fixed points of $K' = f(K)$.

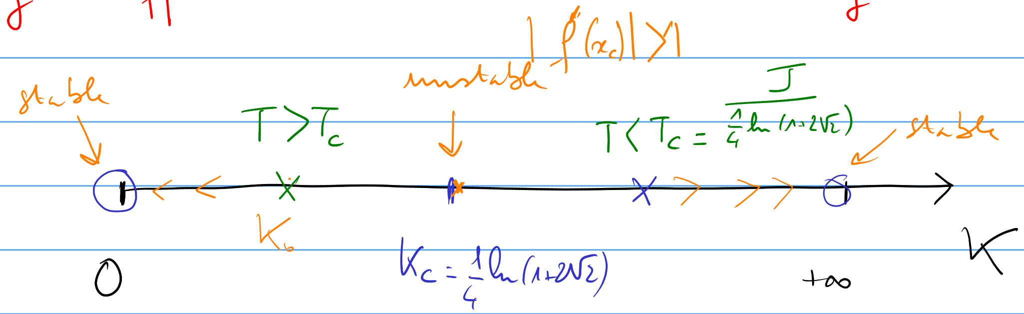
$$x \longmapsto f(x) = \boxed{2x \left(\frac{e^{4x} + 1}{e^{4x} + 3} \right)^2 = x}$$

Obvious fixed points $x^* = 0$

$x^* = +\infty$

$x^* = x_c = \frac{1}{4} \ln(1 + 2\sqrt{2})$

Big difference with the 1d Ising model



high t° fixed point

new feature in $d=2$.

$K_{n+1} = f(K_n)$

$K_0 = \underline{\underline{\beta}} J$

We denote by T_c the temp such that

$$\beta_c J = K_c = \frac{1}{4}\ln(1+2\sqrt{2}).$$

If $T > T_c$, we know the syst to be disordered. We expect, at large scales, the absence of order. At scales larger than ξ (correl. length) we see a bunch of effectively indep spins.

If $T < T_c$, the syst is ordered, especially if looked at from distances larger than ξ.

Beware, ξ is finite at $T \neq T_c$, and it is strictly 0 at $T = 0$ and at $T = \infty$.

Comparison with m f: $T_c^{mf} = 6J$

$$T_c^{NvL} = 2.98\,J < T_c^{mf}$$

$$T_c^{exact} = 3.6\,J$$

1.2.10 How about critical exponents?

$$\xi(T) = A_+ (T - T_c)^{-\nu} \qquad T \to T_c^+$$

$$= A_- (T_c - T)^{-\nu} \qquad T \to T_c^-$$

Let ξ_n be the corel. length at scale b^n, $b = \sqrt{3}$.

hypothesis $\quad \xi_n \sim |K_n - K_c|^{-\nu}$

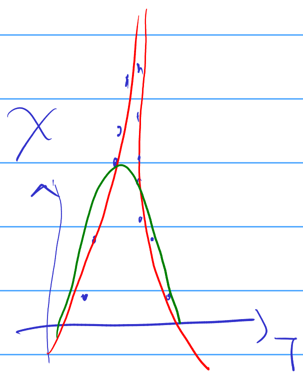
on condition, of course, that $K_0, K_1, \dots$ are close to K_c.

$$K_{n+1} = f(K_n)$$
$$K_c = f(K_c)$$

$$K_{n+1} - K_c = f(K_n) - f(K_c)$$
$$\simeq (K_n - K_c) f'(K_c)$$

$$\xi_{n+1}^{-\frac{1}{\nu}} \simeq \xi_n^{-\frac{1}{\nu}} f'(K_c)$$

$$\Longrightarrow \quad (\xi_{n+1} / \xi_n) = f'(K_c)^{-\nu} = 1/b \quad \Longrightarrow$$



ν such that $b = f'(K_c)^{\nu}$

$\boxed{\nu = 1.13}$ after explicit evaluation of $f'(K_c)$.

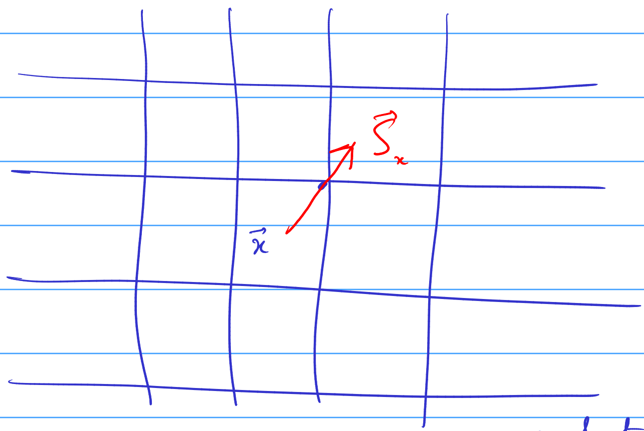
$\nu_{mf} = 1/2 \qquad \nu_{exact} = 1$

The Kosterlitz Thouless transition

2d, a = lattice spacing, $L \times L$, N spins

$$\vec{S}_{\vec{x}} = \text{a 2d vector} = (\cos\theta_{\vec{x}}, \sin\theta_{\vec{x}})$$

$\vec{x} \in$ 2d lattice



a dot product

$$H = -J \sum_{\langle \vec{x}, \vec{y} \rangle} \vec{S}_{\vec{x}} \cdot \vec{S}_{\vec{y}}$$

2.2. Correlations at high temperatures

2.2.1. Rotating each spin by θ_0 leaves H invariant.
The system is also translationally invariant

2.2.2. What is the gs?
All spins aligned in one direction, the latter being arbitrary.
It's an infinitely degenerate gs.

2.2.3. $H = \frac{J}{2} \sum_{\langle \vec{x}\vec{y} \rangle} (\theta_{\vec{x}} - \theta_{\vec{y}})^2$

In the gs, spins are aligned, locally, then
$\theta_{\vec{x}} - \theta_{\vec{y}}$ is small, and

$$\vec{S}_{\vec{x}} \cdot \vec{S}_{\vec{y}} = \cos(\theta_{\vec{x}} - \theta_{\vec{y}}) \simeq 1 - \frac{1}{2}(\theta_{\vec{x}} - \theta_{\vec{y}})^2 + \dots$$

2.2.4. $$\langle \theta_{\vec{x}} \theta_{\vec{y}} \rangle = \frac{1}{K} G(\vec{x} - \vec{y})$$

G is such that $- \sum_{\mu = x,y} \left[G(\vec{n} + a\vec{e}_\mu) + G(\vec{n} - a\vec{e}_\mu) - 2G(\vec{n}) \right] = \delta_{\vec{n},\vec{0}}$

$$Z = \int \prod_{\vec{x}} d\theta_{\vec{x}} \exp - \frac{\beta J}{2} \sum_{\langle \vec{x}, \vec{y} \rangle} (\theta_{\vec{x}} - \theta_{\vec{y}})^2$$

Recognize a Gaussian distribution as in the (beloved) Prerequisites.

Gaussian weight $\exp - \frac{K}{2} \frac{1}{2} \sum_{\vec{x}} \underbrace{\sum_{\vec{y} \text{ nn of } \vec{x}} (\theta_{\vec{x}} - \theta_{\vec{y}})^2}$

$$(\theta_{\vec{x}+a\vec{e}_x} - \theta_{\vec{x}})^2 + (\theta_{\vec{x}+a\vec{e}_y} - \theta_{\vec{x}})^2$$
$$(\theta_{\vec{x}-a\vec{e}_x} - \theta_{\vec{x}})^2 + (\theta_{\vec{x}-a\vec{e}_y} - \theta_{\vec{x}})^2$$

$$= \exp - \frac{K \cdot 2}{4} \sum_{\vec{x}} (\vec{\nabla}\theta)^2, \quad \vec{\nabla}\theta = \left[\theta_{\vec{x}+a\vec{e}_x} - \theta_{\vec{x}}, \theta_{\vec{x}+a\vec{e}_y} - \theta_{\vec{x}} \right]$$

a discrete gradient

$$= \exp - \frac{K}{2} \sum_{\vec{x}} (\vec{\nabla}\theta)^2 = \exp - \frac{K}{2} \sum_{\vec{x}} \theta_{\vec{x}} (-\Delta_{\vec{x}} \theta_{\vec{x}})$$

by parts (Abel transformation)

where $\Delta_{\vec{x}} \theta_{\vec{x}}$ = discrete Laplacian

$$= \theta(\vec{x} + a\vec{e}_x) + \theta(\vec{x} - a\vec{e}_x) + \theta(\vec{x} + a\vec{e}_y) + \theta(\vec{x} - a\vec{e}_y) - 4\theta(\vec{x})$$

We know that the angles are distributed with

$$P(\{\theta_{\vec{x}}\}) \propto \exp - \frac{K}{2} \sum_{\vec{x}\vec{y}} \theta_{\vec{x}} \overbrace{[-\Delta_x \delta_{\vec{x},\vec{y}}]}^{\Gamma_{\vec{x},\vec{y}}} \theta_{\vec{y}}$$

Kronecker Delta

$$\langle \theta_{\vec{x}} \theta_{\vec{y}} \rangle = \frac{1}{K} \underbrace{(-\Delta)^{-1}_{\vec{x}\vec{y}}}_{\text{by definition, this is } G(\vec{x}-\vec{y})}$$

because $\sum_{\vec{z}} (-\Delta^{-1})_{\vec{x}\vec{y}} \cdot G(\vec{y}-\vec{z}) = \delta_{\vec{x},\vec{z}}$

What do I mean by $(-\Delta^{-1})$?

In $d=1$,

$$(\Delta)_{ij} = \delta_{i+1,j} + \delta_{i-1,j} - 2\delta_{ij}$$

$$\sum_j (\Delta)_{ij} f_j = f_{i+1} + f_{i-1} - 2f_i = \Delta f$$

Be Wise, discretize

2.2.4 $\boxed{\langle \theta_{\vec{x}} \theta_{\vec{y}} \rangle = \frac{1}{K} G(\vec{x}-\vec{y})}$

2.2.5. $C(\vec{x},\vec{y}) = \langle \vec{S}_{\vec{x}} \cdot \vec{S}_{\vec{y}} \rangle = \langle \cos(\theta_{\vec{x}} - \theta_{\vec{y}}) \rangle$

$$C(\vec{x},\vec{y}) = \mathrm{Re} \langle e^{i(\theta_x - \theta_{\vec{y}})} \rangle = \mathrm{Re}\, e^{i \underbrace{\langle \theta_{\vec{x}} - \theta_{\vec{y}} \rangle}_{=0} - \frac{1}{2} \langle (\theta_{\vec{x}} - \theta_{\vec{y}})^2 \rangle_c}$$

because we know that the θ's are Gaussian, so that the cumulant expansion stops at order 2.

$$\langle (\theta_{\vec{x}} - \theta_{\vec{y}})^2 \rangle_c = \langle (\theta_x - \theta_{\vec{y}})^2 \rangle - \langle (\theta_{\vec{x}} - \theta_{\vec{y}}) \rangle^2$$

$$\langle e^{i\lambda x}\rangle = e^{i\lambda\langle x\rangle + \frac{1}{2}\underbrace{(i\lambda)^2}_{-\lambda^2}\langle x^2\rangle_c}$$

$$C(\vec{x},\vec{y}) = \cancel{\mathrm{Re}}\, e^{-\frac{1}{2}\langle(\theta_{\vec{x}}-\theta_{\vec{y}})^2\rangle}$$

$$\langle(\theta_{\vec{x}}-\theta_{\vec{y}})^2\rangle = \langle\theta_{\vec{x}}\theta_{\vec{x}}\rangle + \langle\theta_{\vec{y}}\theta_{\vec{y}}\rangle - 2\langle\theta_{\vec{x}}\theta_{\vec{y}}\rangle$$
$$= \frac{2}{K}\left[G(\vec{0}) - G(\vec{x}-\vec{y})\right]$$

$$\Longrightarrow \quad C(\vec{x},\vec{y}) = \exp\left(\underbrace{\left[G(\vec{x}-\vec{y}) - G(\vec{0})\right]}\,/K\right)$$

$$\bar{G}(\vec{x}-\vec{y}) \simeq -\frac{1}{2\pi}\ln\|\vec{x}-\vec{y}\|$$

$$C(\vec{x},\vec{y}) = \|\vec{x}-\vec{y}\|^{-\frac{1}{2\pi K}}$$

Hugely different from the Ising model (no exponential decay!)

Scale invariant, but with a nonuniversal decay exponent ($\frac{1}{2\pi K}$).

There is no spontaneous magnetization because

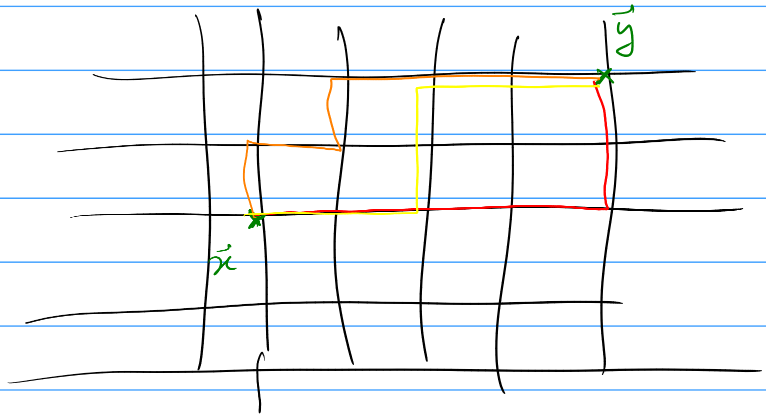
$$\langle\vec{S}_{\vec{x}}\cdot\vec{S}_{\vec{y}}\rangle \xrightarrow[\|\vec{x}-\vec{y}\|\to\infty]{} 0$$

2.2.6 A useful identity:

$$\int\frac{d\theta}{2\pi}\cos(\theta_1-\theta)\cos(\theta-\theta_2) = \frac{1}{2}\cos(\theta_1-\theta_2)$$

(to do on your own!)

2.2.7. $\mathcal{N}(\vec{x},\vec{y})$ = # of shortest paths on the lattice connecting $\vec{x}$ to $\vec{y}$



$\|\vec{x}-\vec{y}\|_1$

$$\mathcal{N}(\vec{0},\vec{r}) = \binom{|r_x|+|r_y|}{|r_x|} < 2^{|r_x|+|r_y|}$$

$$(1+1)^{|r_x|+|r_y|} = 2^{|r_x|+|r_y|} = \underbrace{\sum_{n=0}^{|r_x|+|r_y|} \binom{|r_x|+|r_y|}{n}}_{\text{a sum of } >0 \text{ terms}}.$$

2.2.8. $$Z = \int \prod_{\vec{x}} \frac{d\theta_{\vec{x}}}{2\pi} \underbrace{e^{K\sum_{\langle \vec{x},\vec{y}\rangle} \cos(\theta_{\vec{x}}-\theta_{\vec{y}})}}_{\prod_{\langle \vec{x},\vec{y}\rangle} \underbrace{e^{K\cos(\theta_{\vec{x}}-\theta_{\vec{y}})}}_{1+K\cos(\theta_{\vec{x}}-\theta_{\vec{y}})+\frac{K^2}{2}\cdots}}$$

if $K \to 0$
$\{ T \to +\infty$

Use the trick of 2.2.6 to arrive at 2.2.7 and then to Eq (5) of the text.

$$P(\{\sigma_i\}) = \prod_i \delta_{\sigma_i,1}$$

$$G_{ij} = \langle \sigma_i \sigma_j \rangle - \langle \sigma_i \rangle \langle \sigma_j \rangle \sim \frac{e^{-|i-j|/\xi}}{|i-j|^{d-2+\eta}}$$

~~Dualité~~ Kramers - Wannier duality