Tutorial 1 | Binder cumulants

(1) If $X \sim \mathcal{G}(0,\sigma)$, $p(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{x^2}{2\sigma^2}}$

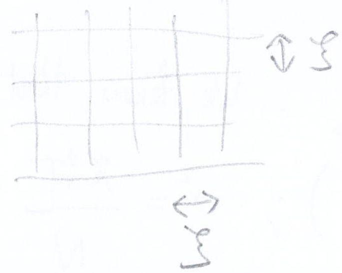
$$\frac{1}{\sqrt{2\pi\sigma^2}}\int x^2 e^{-\alpha x^2/2\sigma^2}\,dx = \frac{\sigma^2}{\alpha^{3/2}} \quad \text{and} \quad \frac{\partial}{\partial\alpha} \rightarrow \frac{1}{\sqrt{2\pi\sigma^2}}\int x^2\left(-\frac{x^2}{2\sigma^2}\right)e^{-\alpha x^2/2\sigma^2} = -\frac{3}{2}\frac{\sigma^2}{\alpha^{5/2}}$$

Then $\alpha = 1 \rightarrow \langle x^4\rangle = 3\langle x^2\rangle^2$

For $X \sim \mathcal{G}(m,\sigma)$, $m = \langle x\rangle \neq 0$, $\langle (x-m)^4\rangle = 3\langle (x-m)^2\rangle^2$

(2) $\langle X^2\rangle \simeq (\cancel{X}x^*)^2$ ; $\langle X^4\rangle \simeq {x^*}^4$ ; $\frac{\langle X^4\rangle}{\langle X^2\rangle^2} \simeq$

(3) Large $T$, spins correlated over a finite length $\xi$ ; for $L \gg \xi$, regroup spins into box size $\xi$



the different blocks are uncorrelated hence CLT holds for magnetization $s$.
thus $s$ is Gaussian, mean $0$ since $m_{sp} = 0$, and std dev $\chi kT/N$.

$$\Rightarrow U_L \underset{T\to\infty}{\rightarrow} 0$$

Small $T$, ie $T$ below $T_c$, $m_{sp} \neq 0$,
std deviation $= \frac{\chi kT}{N} \ll m_{sp}$ for $N$ large enough ($N = L^d$ here)
thus $U_L \rightarrow 1 - \frac{1}{3}\,1 = \frac{2}{3}$

(4) At $T_c$, $\xi \to \infty$, the pdf of $s$ cannot be gaussian : the CLT does not apply
(long range correlations $\langle s_i s_{i+j}\rangle \propto \frac{1}{|r_{ij}|^{d-2+\eta}} \propto \frac{1}{|r_{ij}|^{\eta}}$ in 2D where $\eta = 1/4$
while $\eta_{3D} \simeq 0{,}04$
+ the grouping in boxes of size $\xi$ fails
→ just one big infinite box!)
the statistics of $s$ becomes non trivial...
↳ curves cross at $T = T_c$ → interesting in practice.

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On general grounds, we expect $p_L(s) = L^a \tilde{p}\left(sL^b, \frac{L}{\xi}\right)$

Normalization $\Rightarrow a = b$.

At $T_c$, $\xi \to \infty$ and we have $p_L(s) = L^a \tilde{p}(sL^b, 0)$

hence the moments depend trivially on $L$, whatever $b$, and

$$\frac{\langle s^4\rangle_L}{\langle s^2\rangle_L^2} \text{ is } L \text{ independent.}$$

(5) $T<T_c$, take $T \lll T_c$ where $\langle s\rangle_L \simeq m_{sp}$. We know that the CLT holds for $s$, so that $p_L(s)$ is gaussian $g(m_{sp}, \sigma)$; $\sigma^2 = \frac{\chi k T}{N} \xrightarrow[N=L^d \to \infty]{} 0$

To know $U_L$, we have to compute

$\langle s^2\rangle_L$ and $\langle s^4\rangle_L$

We know that $\langle s^2\rangle = \langle s\rangle^2 + \sigma^2 = m_{sp}^2 + \sigma^2$

$$\langle s^3\rangle = \underbrace{c_3}_{\text{third cumulant } = 0 \text{ for a gaussian}} + 3\langle s^2\rangle\langle s\rangle - 2\langle s\rangle^3 = 3(m_{sp}^2 + \sigma^2)m_{sp} - 2m_{sp}^3$$

$$= m_{sp}^3 + 3m_{sp}\sigma^2$$

$$\langle s^4\rangle = \underbrace{c_4}_{0 \text{ for a gaussian}} + 4\langle s^3\rangle\langle s\rangle + 3\langle s^2\rangle^2 - 12\langle s^2\rangle\langle s\rangle^2 + 6\langle s\rangle^4$$

$$= 4[m_{sp}^3 + 3m_{sp}\sigma^2]m_{sp} + 3[m_{sp}^2+\sigma^2]^2 - 12[m_{sp}^2+\sigma^2]m_{sp}^2 + 6m_{sp}^4$$

$$\simeq 4m_{sp}^4 + 12m_{sp}^2\sigma^2 + 3(m_{sp}^4 + 2m_{sp}^2\sigma^2) - 12m_{sp}^4 - 12m_{sp}^2\sigma^2 + 6m_{sp}^4$$

$$\simeq m_{sp}^4 + 6m_{sp}^2\sigma^2$$

$$U_L = 1 - \frac{\langle s^4\rangle}{3\langle s^2\rangle^2} \simeq 1 - \frac{1}{3}\frac{m_{sp}^4 + 6m_{sp}^2\sigma^2}{(m_{sp}^2+\sigma^2)^2} \simeq 1 - \frac{1}{3}\left(1 + 6\frac{\sigma^2}{m_{sp}^2}\right)\left(1 - \frac{2\sigma^2}{m_{sp}^2}\right) \simeq \frac{2}{3} - \frac{4}{3}\frac{\sigma^2}{m_{sp}^2}$$