PCS - Stochastic processes - Exam 2022

A **1°** x obeys $p_0(x) = \frac{1}{\mu} e^{-x/\mu}$

$$\boxed{\langle x \rangle = \mu} \quad ; \quad \boxed{\langle x^2 \rangle = 2\mu^2}$$

$$V(x) = \langle x^2 \rangle - \langle x \rangle^2 = \mu^2$$

$$\boxed{\begin{aligned} \langle S_n \rangle &= n\mu \\ V(S_n) &= n\mu^2 \end{aligned}}$$

2° The CLT applies in a region of extention $n^{2/3}$ around $S_n = n\mu$ (see below, 5°)

$$\boxed{P(S_n) \simeq \frac{1}{\sqrt{2\pi n \mu^2}} \exp\left[-\frac{(S_n - n\mu)^2}{2n\mu^2}\right]}$$

3° Sanov theorem: $P(S_n) \doteq e^{-n\phi(z)}$; $z = \frac{S_n}{n}$

where $\phi(z) = D[q^* \| p_0]$, Kullback-Leibler

and $q^* = \text{argmin}\, D[q \| p_0]$ s.t. $\int q(x)dx = z$

We thus minimize $D[q\|p_0]$ with the constraints that $\int q(x)\,dx = 1$; $\int q(x)\,x\,dx = z$

$$\frac{\delta}{\delta q(x)}\left[\int dx \left(q(x) \log\frac{q(x)}{p_0(x)} + \lambda x\, q(x) + \mu\, q(x)\right)\right] = 0$$

$$\Rightarrow \log\frac{q(x)}{p_0(x)} + \lambda x + \mu = 0$$

$$\Rightarrow q(x) = \frac{1}{\mu} e^{-\mu^{-1}x - \lambda} e^{-\mu}$$

and since we need $\int_0^\infty q(x)\,x\,dx = z$, we get the only possibility

$$q(x) = \frac{1}{z} e^{-x/z}$$

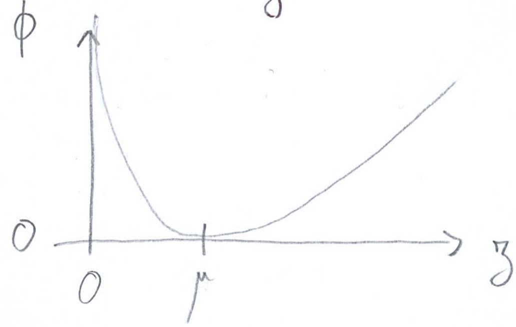
$$\Rightarrow \phi(z) = \int dx\, q(x) \log\left(\frac{q(x)}{p(x)}\right)$$

$$= \int dx\, \frac{1}{z} e^{-x/z}\left[\log\left(\frac{\mu}{z}\right) + x\left(\frac{1}{\mu} - \frac{1}{z}\right)\right]$$

$$\boxed{\phi(z) = -1 + \frac{z}{\mu} - \log\frac{z}{\mu}} \qquad (1)$$

$$\phi'(z) = \frac{1}{\mu} - \frac{1}{z} \qquad \text{and } \phi(\mu) = \phi'(\mu) = 0$$

$$\phi''(z) = \frac{1}{z^2}$$



4° The CLT predicts a quadratic $\phi(z)$:

$$P(S_n) \doteq \exp\left[-\frac{n}{2}\left(\frac{z-\mu}{\mu}\right)^2\right]$$

hence $\phi(z) = \frac{1}{2}\left(\frac{z-\mu}{\mu}\right)^2$

We can recover this from a Taylor expansion of Eq(1): for small δz,

$$\phi(\mu + \delta z) \sim \frac{1}{2\mu^2}(\delta z)^2$$

5° We have $\phi(z) \simeq \frac{1}{2}\phi''(\mu)(z-\mu)^2 + \frac{1}{6}\phi'''(\mu)(z-\mu)^3 + \dots$

and the CLT holds provided

$$\left| n\,\phi'''(\mu)(z-\mu)^3 \right| \ll 1$$

$$\Leftrightarrow n\left|\frac{S_n}{n} - \mu\right|^3 \ll 1$$

$$\Leftrightarrow \boxed{|S_n - \mu n| \ll n^{2/3}}$$

6° $\langle e^{tS_n} \rangle = \int e^{tS_n} P(S_n)\, dS_n$

$$\doteq \int e^{tS_n} e^{-n\phi(z)} dS_n \qquad S_n = nz$$

$$\doteq e^{n \max_z [tz - \phi(z)]} \qquad \text{from saddle argument}$$

Besides, $\langle e^{tS_n}\rangle = \langle e^{tx}\rangle^n$

$$\Rightarrow \mathcal{K}(t) = \max_z \left[z t - \phi(z) \right]$$

which is inverted in

$$\boxed{\phi(z) = \max_t \left[z t - \mathcal{K}(t) \right]}$$

$$\mathcal{K}_n(t) = \frac{1}{n} \log \langle e^{tS_n}\rangle = \frac{1}{n}\log \langle e^{tx_1}\rangle^n$$

$$= \log \langle e^{tx_1}\rangle$$ does not depend on n since x_i are independent

$$\langle e^{tx_1}\rangle = \frac{1}{\mu}\int_0^\infty e^{tx} e^{-x/\mu}\, dx$$

$$= \mu\left(\frac{1}{\mu} - t\right)^{-1} = \frac{1}{1-\mu t}$$

$$\frac{1}{\mu} - t > 0 \Leftrightarrow \mu t < 1$$

$$\boxed{\mathcal{K}_n(t) = -\log(1-\mu t)}$$

To get $\phi(z)$: $\frac{d}{dt}\left(z t - \mathcal{K}(t)\right) = 0 = z + \frac{-\mu}{1-\mu t}$

$$\Leftrightarrow \quad \mu t^* = 1 - \frac{\mu}{z}$$

$$t^* = \frac{z-\mu}{\mu z}$$

$$\phi(z) = z t^* - \mathcal{K}(t^*)$$

$$= \frac{z-\mu}{\mu} + \log\left(1 - 1 + \mu/z\right)$$

$$\boxed{\phi(z) = \frac{z}{\mu} - 1 - \log\left(\frac{z}{\mu}\right)}$$ same result as with Sanov thm

7°) a) Taking again $S_n = n z$

$$P(S_n) \simeq \frac{1}{\mu} e^{-nz/\mu} e^{(n-1)\log[z n/\mu]} \left(\frac{e}{n-1}\right)^{n-1}$$

From Stirling, but not needed

$$\simeq e^{-n\phi(z)}$$

with $\phi(z) = \frac{z}{\mu} - \log\left(\frac{z}{\mu}\right) + \text{Cst}$

and we need that $\phi = 0$ at its minimum, which is for $z = \mu \Rightarrow \text{Cst} = -1$.

f) For large n, the most probable value of S_n is $n\mu$, and around it, the CLT applies:

$$\Pr\left[(n-\tfrac{1}{2})\mu \leq S_n \leq (n+\tfrac{1}{2})\mu\right]$$

$$= \int_{(n-\frac{1}{2})\mu}^{(n+\frac{1}{2})\mu} P(S_n)\, dS_n$$

$$= \int_{n-\frac{1}{2}}^{n+\frac{1}{2}} e^{-s} \frac{s^{n-1}}{(n-1)!}\, ds$$

$$\underbrace{= \int_{n-\frac{1}{2}}^{n+\frac{1}{2}} \frac{1}{\sqrt{2\pi n}} e^{-(s-n)^2/2n}\, ds}_{\sim \frac{1}{\sqrt{2\pi n}} \text{ for } n \text{ large}}$$ from CLT

One can show that

$$\int_{n-\frac{1}{2}}^{n+\frac{1}{2}} e^{-s}\frac{s^{n-1}}{(n-1)!}\, ds \sim e^{-n}\frac{n^{n-1}}{(n-1)!}$$

$$= e^{-n}\frac{n^n}{n!}$$

$$\Rightarrow e^{-n}\frac{n^n}{n!} \sim \frac{1}{\sqrt{2\pi n}}$$

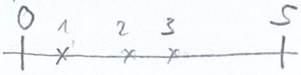
i.e. $\boxed{n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n}$ Stirling formula

Proof that $P_n(S_n) = \frac{1}{\mu}\frac{1}{(n-1)!} e^{-S_n/\mu} \left(\frac{S_n}{\mu}\right)^{n-1}$

$$S_n = \sum_{i=1}^n x_i$$

Each x_i is drawn according to an exponential law and thus, the largest n such that $S_n \leq S$ obey a Poisson distribution

$\mathcal{P}(S/\mu)$. Take N, a random variable following $\mathcal{P}(S/\mu)$:



$$\Pr[N \geq n_0] = \Pr[S_{n_0} \leq S]$$

$$\Leftrightarrow \sum_{N=n_0}^{\infty} e^{-S/\mu} \frac{(S/\mu)^N}{N!} = \int_0^S P_{n_0}(s)\,ds$$

Then, take $\partial/\partial S$ on both sides:

$$P_{n_0}(S) = \frac{\partial}{\partial S} \sum_{N=n_0}^{\infty} e^{-S/\mu} \frac{(S/\mu)^N}{N!}$$

$$= \frac{1}{\mu} \sum_{N=n_0}^{\infty} \left(\frac{(S/\mu)^{N-1}}{(N-1)!} e^{-S/\mu} - \frac{(S/\mu)^N e^{-S/\mu}}{N!} \right)$$

$$= \frac{1}{\mu} \frac{1}{(n_0-1)!} e^{-S/\mu} \left(\frac{S}{\mu}\right)^{n_0-1}$$

B 1°) With Itō-Doblin

$$\frac{d}{dt}\langle \varphi(v) \rangle = \langle \varphi'(v)\, \dot{v} \rangle + \Gamma \langle \varphi''(v) \rangle$$

since Γ is here the diffusion coefficient

$\varphi(v) = v^2$ gives

$$\frac{d\langle v^2\rangle}{dt} = 2\langle v \dot{v} \rangle + 2\Gamma$$

$$= 2\left\langle v\left(-\gamma v + \sqrt{2\Gamma}\,\xi(t)\right)\right\rangle + 2\Gamma$$

$$= -2\gamma \langle v^2 \rangle + 2\Gamma$$

Since $\langle v(t)\xi(t)\rangle = 0$ at Itō-Doblin level

2°)

$$\frac{d}{dt}\left(\langle v^2\rangle - \frac{\Gamma}{\gamma}\right) = -2\gamma\left(\langle v^2\rangle - \frac{\Gamma}{\gamma}\right)$$

$$\Rightarrow \langle v^2\rangle - \frac{\Gamma}{\gamma} = A\, e^{-2\gamma t} \quad ; \langle v^2 \rangle = v_0^2 \text{ at } t=0$$

$$\Rightarrow A = v_0^2 - \frac{\Gamma}{\gamma}$$

$$\Rightarrow \langle v^2(t)\rangle = \frac{\Gamma}{\gamma}\left(1 - e^{-2\gamma t}\right) + v_0^2 e^{-2\gamma t}$$

$$\boxed{\langle v^2(t)\rangle = \left(v_0^2 - \frac{\Gamma}{\gamma}\right) e^{-2\gamma t} + \frac{\Gamma}{\gamma}}$$

3°) Within Stratonovich framework

$$\frac{d\langle v^2\rangle}{dt} = 2\langle v \dot{v}\rangle$$

$$= -2\gamma\langle v^2\rangle + 2\sqrt{2\Gamma}\underbrace{\langle v(t)\xi(t)\rangle}_{\neq 0}$$

$$\langle v(t)\xi(t)\rangle \rightarrow \left\langle \frac{v(t) + v(t+\Delta t)}{2\Delta t} \underbrace{\int_t^{t+\Delta t} \xi(t')\,dt'}_{B_{\Delta t}} \right\rangle$$

$$v(t+\Delta t) = v(t) - \gamma v(t)\Delta t + \sqrt{2\Gamma}\, B_{\Delta t} + \dots$$

$$\langle B_{\Delta t}\rangle = 0 \; ; \quad \langle B_{\Delta t}^2 \rangle = \Delta t$$

$$\rightarrow \left\langle \frac{1}{2\Delta t} v(t+\Delta t)\, B_{\Delta t} \right\rangle$$

$$= \frac{1}{2}\sqrt{2\Gamma}\,\langle B_{\Delta t}^2\rangle \frac{1}{\Delta t}$$

$$= \frac{1}{2}\sqrt{2\Gamma}$$

$$\Rightarrow \frac{d\langle v^2\rangle}{dt} = -2\gamma\langle v^2\rangle + 2\Gamma$$

as found above

4°) We know that the velocity pdf is gaussian at all times, hence the skewness is always 0. This can be recovered by calculation, but we need to adapt slightly Itō-Doblin calculus to a function $\varphi(v,t)$

$$\frac{d}{dt}\langle \varphi(v,t)\rangle = \left\langle \frac{\partial \varphi}{\partial t}\right\rangle + \left\langle \frac{\partial \varphi}{\partial v}\dot{v}\right\rangle + \Gamma\left\langle \frac{\partial^2\varphi}{\partial v^2}\right\rangle$$

Here $\varphi(v,t) = (v - \langle v \rangle)^3$

$$\langle v \rangle = v_0 e^{-\gamma t}$$

$$\frac{\partial \varphi}{\partial t} = +3(v - \langle v\rangle)^2 \gamma \langle v \rangle$$

$$\partial_v \varphi = 3(v - \langle v \rangle)^2$$

$$\frac{d}{dt}\langle(v-\langle v\rangle)^3\rangle = 3\gamma\langle v\rangle\langle(v-\langle v\rangle)^2\rangle + \underbrace{\langle 3(v-\langle v\rangle)^2\dot{v}\rangle}_{-3\gamma\langle v(v-\langle v\rangle)^2\rangle} + \Gamma 6\underbrace{\langle v-\langle v\rangle\rangle}_{0}$$

$$= -3\gamma\langle(v-\langle v\rangle)^3\rangle$$

Hence a solution $\langle(v-\langle v\rangle)^3\rangle = B e^{-3\gamma t}$ and the initial condition is such that $B=0$

↳ vanishing skewness. We would get the same conclusion with the 4^{th} cumulant

C Geometric Brownian motion

1° At Stratonovich level, standard rules of calculus apply: $\frac{1}{S}\frac{dS}{dt} = \mu - \frac{\sigma^2}{2} + \sigma\xi(t)$

$$= \frac{d}{dt}\ln S$$

$$\Rightarrow \ln S(t) = \underbrace{\ln S(0)}_{0} + \left(\mu-\frac{\sigma^2}{2}\right)t + \sigma\underbrace{\int_0^t \xi(t')dt'}_{X(t)}$$

$X(t)$ is a Wiener process, $\langle X(t)\rangle = 0$

$$\langle X^2(t)\rangle = \sigma^2 t$$

(hence a diffusion coefficient $\sigma^2/2$).

$$\boxed{S(t) = \exp\left[\left(\mu-\frac{\sigma^2}{2}\right)t + X(t)\right]}$$

$X(t)$ is a gaussian process s.t $\langle X(t)\rangle = 0$ and $\langle X(t)X(t')\rangle = \sigma^2\min(t,t')$

2° For a gaussian X of mean m and variance v^2:

$$\langle e^{kX}\rangle = e^{km}\, e^{k^2v^2/2}$$

$$\Rightarrow \langle S^n(t)\rangle = e^{n(\mu-\sigma^2/2)t}\underbrace{\langle e^{nX}\rangle}_{e^{n^2\sigma^2 t/2}}$$

since $v^2 = \sigma^2 t$ here

$$\boxed{\begin{aligned}\langle S(t)\rangle &= e^{n\mu t} = e^{\mu t}\\ \langle S^2(t)\rangle &= e^{2\mu t + \sigma^2 t}\\ \langle S^n(t)\rangle &= e^{n\mu t + \frac{\sigma^2 t}{2}(n^2-n)}\end{aligned}}$$

$(\alpha = 1)$

3° The distribution of $X(t)$ is gaussian:

$$p_X(x) = \frac{1}{\sqrt{2\pi\sigma^2 t}}\exp\left[-\frac{x^2}{2\sigma^2 t}\right]$$

and $\log S = \left(\mu-\frac{\sigma^2}{2}\right)t + X$

$$p(S)\,dS = p_X(X)\,dX$$

$$\frac{d\log S}{dX} = 1 \Rightarrow \frac{dS}{dX} = S$$

$$\Rightarrow \boxed{p(S,t) = \frac{1}{S}\frac{1}{\sqrt{2\pi\sigma^2 t}}\exp\left[-\frac{\left[\log S - \left(\mu-\frac{\sigma^2}{2}\right)t\right]^2}{2\sigma^2 t}\right]}$$

↳ LOG-NORMAL LAW

When t is large, and for $|\log S| \ll \sigma^2 t$, we have

$$p(S,t) \propto \frac{1}{S}\exp\left[\frac{2(\log S)(\mu-\sigma^2/2)}{2\sigma^2}\right]$$

$$\propto \frac{1}{S}\, S^{-\frac{1}{2}+\mu/\sigma^2}$$

$$\propto S^{-\frac{3}{2}+\mu/\sigma^2}$$

4° Fokker-Planck:

$$\boxed{\partial_t p(S,t) = -\partial_S\left[\left(\mu-\frac{\sigma^2}{2}\right)S\,P(S,t)\right] + \frac{\sigma^2}{2}\partial_S\left[S\,\partial_S(SP)\right]}$$

5° Constraint $S \geqslant S_{min}$ added (a "wall")

Steady state for:

$$\partial_S\left[\left(\mu-\frac{\sigma^2}{2}\right)S\,P(S)\right] = \frac{\sigma^2}{2}\partial_S\left[S\,\partial_S(SP)\right]$$

and equilibrium corresponds to a vanishing current, i.e.

$$\left(\mu-\frac{\sigma^2}{2}\right) s P(s) = \frac{\sigma^2}{2} s\, \partial_s (sP)$$

$$\Rightarrow \frac{1}{sP}\frac{dsP}{ds} = \frac{2}{\sigma^2}\left(\mu - \frac{\sigma^2}{2}\right)/s = \frac{1}{s}\left(\frac{2\mu}{\sigma^2}-1\right)$$

$$\log sP = \log\left(s^{-1+2\mu/\sigma^2}\right) + \text{cst}$$

$$\boxed{P(s) = \frac{C}{s^{2-2\mu/\sigma^2}} \quad , \quad s \geq S_{min}}$$

which is normalizable since

$$\underbrace{1-2\mu/\sigma^2}_{\text{Lévy index}} \geq 0 \Leftrightarrow \mu - \frac{\sigma^2}{2} \leq 0, \text{ true}$$

6°) Working with $z \equiv \log S$, we have

$$\dot{z} = +\left(\mu - \frac{\sigma^2}{2}\right) + \sigma \xi(t)$$

and we can draw a parallel with a colloidal object in a gravitational field $-g$ $(g \geq 0)$, at temp T, described by the overdamped Langevin equation:

$$\dot{z} = \tilde{\mu}(-gm) + \sqrt{2D}\, \xi(t)$$

where $\tilde{\mu} = D/kT$ is the mobility, and m is the colloid mass. We know that the equilibrium distribution/density is barometric:

$$\tilde{p}(z) \propto e^{-\frac{mgz}{kT}} \propto e^{-\frac{\tilde{\mu} m g z}{D}}$$

Analogy: $\boxed{\begin{aligned} -mg\tilde{\mu} &= \mu - \sigma^2/2 \\ \sqrt{2D} &= \sigma \end{aligned}}$

$$\Rightarrow \tilde{p}(z) \propto e^{+\left(\mu-\sigma^2/2\right)\frac{2}{\sigma^2} z} = e^{z\left(\frac{2\mu}{\sigma^2}-1\right)}$$

Finally: $p(S) = \frac{1}{S}\tilde{p}(\log S) \propto \frac{1}{S} S^{2\mu/\sigma^2 - 1}$, as above

7°) $\dot{S}(t) = \mu' S + \sigma S(t)\xi(t)$

$$\text{Itō} \Rightarrow \boxed{\partial_t p(S,t) = -\partial_S\left[\mu' S p\right] + \frac{\sigma^2}{2}\partial_S^2\left[S^2 p\right]}$$

Hence the current is

$$j = \mu' S p - \frac{\sigma^2}{2}\partial_S\left(S^2 p\right)$$

On the other hand, we got the Stratonovich

$$j = \left(\mu - \frac{\sigma^2}{2}\right) S p - \frac{\sigma^2}{2} S\, \partial_S (S p)$$

We impose the two currents to coincide (ie that the 2 Fokker-Planck equations are the same):

$$\mu' S p - \frac{\sigma^2}{2}\left[2 S p + S^2 \partial_S p\right] = \left(\mu - \frac{\sigma^2}{2}\right) S p - \frac{\sigma^2}{2} S p - \frac{\sigma^2}{2} S^2 \partial_S p$$

$$\Rightarrow \mu' = \sigma^2 + \mu - \frac{\sigma^2}{2} - \frac{\sigma^2}{2}$$

$\boxed{\mu' = \mu}$ Establishes correspondence Itō $\Leftrightarrow$ Stratonovich.

D) Martingales etc.

1°) 1) $\dot{X}(t) = \sqrt{2D}\,\xi(t)$

$$\Rightarrow X(t) = X(s) + \underbrace{\sqrt{2D}\int_s^t \xi(t')\,dt'}_{B_{t-s}}$$

$$\langle B_{t-s}\rangle = 0$$

$$\langle B_{t-s}^2\rangle = 2D(t-s)$$

$$\langle X^2(t) - 2Dt \,|\, X(s)\rangle$$

$$= \langle X^2(s) + B_{t-s}^2 + 2X(s)B_{t-s} - 2Dt \,|\, X(s)\rangle$$

$$= X^2(s) + 2D(t-s) + 0 - 2Dt$$

$$= X^2(s) - 2Ds$$

Hence $X^2(t) - 2Dt$ is a martingale

D-1) 2) From Doob stopping time theorem

$$\langle X^2(\tau) - 2D\tau \rangle = X^2(0) - 0 = 0$$

$$\Rightarrow \boxed{\langle \tau \rangle = \frac{1}{2D} \langle X^2 \rangle = \frac{a^2}{2D}}$$

3) Use again $X(t) = X(s) + B_{t-s}$

$$\langle X^4(t) - \varphi(t) | X(s) \rangle$$

$$= \langle X^4(s) + 4X^3(s) B_{t-s} + 6X^2(s) B^2_{t-s} + 4X(s) B^3_{t-s} + B^4_{t-s} - \varphi(t) | X(s) \rangle$$

$$= X^4(s) + 6X^2(s) \langle B^2_{t-s} | X(s) \rangle + \langle B^4_{t-s} | X(s) \rangle - \varphi(t)$$

and we see that the right-hand side depends on $X^2(s)$, while the left side does not depend explicitly on $X^2(t)$ $\Rightarrow$ we cannot find martingales of the proposed form

4) Proceed as above: since B_{t-s} is gaussian of 0 mean: $\langle B^4_{t-s} \rangle = 3 \cdot \langle B^2_{t-s} \rangle^2$

$$= 3 \left(2D(t-s) \right)^2$$

$$\langle X^4(t) - 12Dt X^2(t) + \psi(t) | X(s) \rangle$$

$$= X^4(s) + 12D(t-s) X^2(s) + 12D^2(t-s)^2 - 12Dt \left[X^2(s) + 2D(t-s) \right] + \psi(t)$$

$$= X^4(s) - 12Ds X^2(s) + 12D^2(t-s)^2 - 12D^2 t(t-s) \times 2 + \psi(t)$$

We thus require that:

$$\psi(t) + 12D^2(t-s)^2 - 24D^2 t(t-s) = \psi(s)$$

$$\Leftrightarrow \psi(t) + D^2(t-s) 12 \left[t - s - 2t \right] = \psi(s)$$

$$\Leftrightarrow \psi(t) - \psi(s) = D^2 \cdot 12 \left(t^2 - s^2 \right)$$

Hence $\boxed{\psi(t) = 12 D^2 t^2}$

5) Doob again:

$$\langle X^4(\tau) - 12D\tau X^2(\tau) + 12D^2\tau^2 \rangle = 0$$

$$\Rightarrow a^4 - 12Da^2 \langle \tau \rangle + 12D^2 \langle \tau^2 \rangle = 0$$

$$12D^2 \langle \tau^2 \rangle = -a^4 + 12a^2 \frac{a^2}{2} = 5a^4$$

$$\boxed{\langle \tau^2 \rangle = \frac{5a^4}{12D^2}}$$

6) $$\langle e^{\theta X(t)} | X(s) \rangle = e^{\theta X(s)} e^{\theta^2 2D(t-s)/2}$$

since $X(t)$ is $g\left(X(s), \sqrt{2D(t-s)}\right)$

Hence $\boxed{e^{\theta X(t)} e^{-Dt\theta^2} \text{ is a martingale}}$

$$\phi(t) = \theta^2 D t$$

7) $$\langle e^{\theta X} e^{-\theta^2 D\tau} \rangle = 1$$

$$\Rightarrow \langle e^{-\theta^2 D\tau} \rangle \left(\frac{1}{2} e^{\theta a} + \frac{1}{2} e^{-\theta a} \right) = 1$$

$$\langle e^{-\theta^2 D\tau} \rangle = \frac{1}{\cosh(\theta a)}$$

$$\Rightarrow \boxed{\langle e^{-nD\tau/a^2} \rangle = \frac{1}{\cosh(\sqrt{n})}}$$

8) The above is the moment generating function

Take $\tilde{\tau} = D\tau/a^2$:

$$\langle e^{-n\tilde{\tau}} \rangle = \frac{1}{\cosh \sqrt{n}}$$

$$\Big\downarrow n \to 0$$

$$1 - n\langle \tilde{\tau} \rangle + \frac{n^2}{2} \langle \tilde{\tau}^2 \rangle$$

$$\cosh \sqrt{n} = 1 + \frac{1}{2} n + \frac{1}{24} n^2 + O(n^3)$$

$$\frac{1}{\cosh \sqrt{n}} \underset{0}{\sim} 1 - \frac{1}{2} n - \frac{1}{24} n^2 + \frac{n^2}{4}$$

$$\sim 1 - \frac{n}{2} + \frac{n^2}{6} \frac{5}{4}$$

and we recover $\langle \tilde{\tau} \rangle = \frac{1}{2}$; $\langle \tilde{\tau}^2 \rangle = \frac{5}{12}$

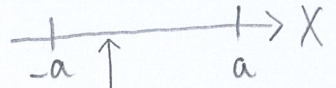
$$D\langle \tau \rangle = \frac{a^2}{2} \; ; \; D^2 \langle \tau^2 \rangle = \frac{5a^4}{12}$$

D-2c) Feynman-Kac

$$Q(x_0) = \left\langle e^{-\int_0^\tau V(x(t'))dt'} \right\rangle$$

$$\boxed{D \frac{d^2 Q(x_0)}{dx_0^2} - V(x_0) Q(x_0) = 0}$$

with boundary condition $Q(a) = Q(-a) = 1$
and we also require $Q(x_0) = Q(-x_0)$



x_0, start point of walker
Absorbed at $X = \pm a$

Since we are interested in $\left\langle e^{-nD\tau/a^2} \right\rangle$,
we choose $\boxed{V(x) = nD/a^2}$

$$\Rightarrow \frac{d^2 Q}{dx_0^2} = \frac{n}{a^2} Q(x_0)$$

$$Q(x_0) = \frac{\cosh(\sqrt{n}\, x_0/a)}{\cosh(\sqrt{n})}$$

Finally, we have to take $x_0 = 0$, starting point of the random walk, and we recover

$$\boxed{Q(0) = \left\langle e^{-nD\tau/a^2} \right\rangle = \frac{1}{\cosh\sqrt{n}}}$$

D-3c) 1) $\partial_t p(x,t|X(s),s) = D\partial_x^2 p(x,t|X(s),s)$

2) $\langle f(X(t),t) | X(s) \rangle$
$= \int f(x,t)\, p(x,t|X(s),s)\, dx$

3) We demand that
$\langle f(X(t),t) | X(s) \rangle$ does not depend on t.

$$\Rightarrow \partial_t \langle \cdots \rangle = 0$$
$$= \int \partial_t f(x,t)\, p(x,t|X(s),s)\, dx$$
$$+ \underbrace{\int f(x,t)\, \partial_t p(\ldots)\, dx}$$
$$= +D \int f(x,t)\, \partial_x^2 p \, dx$$
$$= +D \int p(x,t|X(s),s)\, \partial_x^2 f \rightarrow \text{by parts}$$

$$\Rightarrow \partial_t \langle \cdots \rangle = \int \left(\partial_t f + D\partial_x^2 f \right) p(x,t|X(s),s)\, dx$$

It is then sufficient to take

$$\partial_t f + D \partial_x^2 f = 0 \qquad (2)$$

Above, we considered

$f(x,t) = x^2 - 2Dt$ (a)

$x^4 - 12Dt x^2 + 12D^2 t^2$ (b)

$e^{\theta x - \theta^2 Dt}$ (c)

Check that (2) is obeyed:

(a) : $-2D \quad 2D = 0$, OK

(b) : $-12Dx^2 + 24D^2 t$
$D[4 \cdot 3x^2 - 12Dt \cdot 2] = 0$, OK

(c) : $-D\theta^2 e^{\theta x - \theta^2 Dt}$
$+ D\theta^2 e^{\theta x - \theta^2 Dt} = 0$, OK

And indeed, (2) guarantees that $f(X(t),t)$ is a martingale