PCS - Stochastic Processes - Exam 2023

(A) 1°) $c_4 = \langle (X-\langle X\rangle)^4\rangle - 3\langle (X-\langle X\rangle)^2\rangle^2$

as will be shown below, when $\langle X\rangle = 0$

1°) a) $\langle e^{tX}\rangle = 1 + t\langle X\rangle + \frac{t^2}{2}\langle X^2\rangle + \frac{t^3}{3!}\langle X^3\rangle$

and here $\langle X\rangle = \langle X^3\rangle = 0$ ⇝ moment generating function

$$\log\langle e^{tX}\rangle = tc_1 + \frac{t^2}{2}c_2 + \frac{t^3}{3!}c_3 + \frac{t^4}{4!}c_4 + \dots$$

is the cumulant generating function

b) Here: $\langle e^{tX}\rangle = 1 + \frac{t^2}{2}\langle X^2\rangle + \frac{t^4}{24}\langle X^4\rangle + \dots$

$$\Rightarrow \log\langle e^{tX}\rangle = \frac{t^2}{2}\langle X^2\rangle + \frac{t^4}{24}\langle X^4\rangle + \dots - \frac{1}{2}\left(\frac{t^2}{2}\langle X^2\rangle + \dots\right)^2$$

$$= \frac{t^2}{2}\langle X^2\rangle + \frac{t^4}{24}\left[\langle X^4\rangle - 3\langle X^2\rangle^2\right] + O(t^6)$$

$$\Rightarrow c_2 = \langle X^2\rangle$$

$$\boxed{c_4 = \langle X^4\rangle - 3\langle X^2\rangle^2}$$

c) $\tilde{c}_4 = \dfrac{\langle X^4\rangle}{\langle X^2\rangle^2} - 3$

We also know that $\langle Y^2\rangle - \langle Y\rangle^2 \geq 0$ for all random variables Y. Thus

$\langle X^4\rangle \geq \langle X^2\rangle^2$ when $Y = X^2$

hence $\boxed{\tilde{c}_4 \geq -2}$ always.

Is it possible to realize the lower bound? Yes, since $\langle X^4\rangle = \langle X^2\rangle^2$ when X^2 takes a unique value, 1 (or a > 0 constant) this is the case for the distribution, of the Bernoulli type, where

$$X = \begin{cases} +1 & \text{with proba } 1/2 \\ -1 & \text{"} \quad \text{"} \quad 1/2 \end{cases}$$

2°) Seen in class, subdiffusion

$x_n \propto n^{1/\mu}$, $\mu < 2$

If $\langle \eta\rangle \neq 0$, one has to distinguish

$1 < \mu < 2$: $x_n \propto n\langle\eta\rangle$ grows linearly with n (like ballistic motion)

$0 < \mu < 1$: $x_n \propto n\langle\eta\rangle + (\dots)\, n^{1/\mu}$
$\propto n^{1/\mu}$ again

3°) Seen in class

(B) 1°) For Stratonovich:

$$\left\langle \frac{d\varphi(x(t))}{dt}\right\rangle = \langle \varphi'(x)\,\dot{x}\rangle$$
$$= \langle \varphi'(x)[\mu F(x) + \sqrt{2D}\,\eta(t)]\rangle$$
$$= \mu\langle\varphi'(x)F(x)\rangle + \sqrt{2D}\langle\varphi'(x)\eta(t)\rangle$$

$$\langle \varphi'(x)\eta(t)\rangle \underset{\text{Strato}}{=} \lim_{\Delta t\to 0}\frac{1}{\Delta t}\left\langle \frac{\varphi'(x(t)) + \varphi'(x(t+\Delta t))}{2} B_{\Delta t}\right\rangle$$

where $B_{\Delta t} = \int_t^{t+\Delta t}\eta(t')dt'$; $\langle B_{\Delta t}\rangle = 0$, $\langle B_{\Delta t}^2\rangle = \Delta t$

$x(t+\Delta t) = x(t) + \mu F\Delta t + \sqrt{2D}\,B_{\Delta t}$

$$\langle\varphi'(x)\eta(t)\rangle = \lim\frac{1}{\Delta t}\left\langle\left(\varphi'(x) + \frac{1}{2}\left[\varphi''(x) + \mu F\Delta t + \sqrt{2D}\,B_{\Delta t}\right]\right) \times B_{\Delta t}\right\rangle$$

since $\varphi'(x(t+\Delta t)) = \varphi'(x(t)) + (\mu F\Delta + \sqrt{2D}\,B_{\Delta t})\varphi''(x(t))$

$$\langle\varphi'(x)\eta(t)\rangle = \frac{1}{2}\sqrt{2D}\langle\varphi''(x)\rangle$$

$$\Rightarrow \left\langle\frac{d\varphi(x(t))}{dx}\right\rangle = \mu\langle\varphi'(x)F(x)\rangle + D\langle\varphi''(x)\rangle$$

We now compare to Itō-Döblin

$$\left\langle\frac{d\varphi(x(t))}{dt}\right\rangle = \langle\varphi'(x)\,\dot{x}\rangle + D\langle\varphi''(x)\rangle$$

which comes from

$$d\varphi = \varphi'(x)\,dx + \frac{1}{2}\varphi''(x)(dx)^2$$

Within Itō-Döblin formalism,

$$\langle f(x(t))\, \eta(t) \rangle = 0$$

$$\Rightarrow \boxed{\left\langle \frac{d\varphi(x(t))}{dt} \right\rangle = \mu \langle \varphi'(x) F(x) \rangle + D \langle \varphi''(x) \rangle}$$

same as for Stratonovich

2°) When $D(x)$, the noise is multiplicative, and the 2 routes differ. We repeat the above analysis, including fact that $D(x)$.

$\hookrightarrow \langle \varphi'(x) \sqrt{2D(x)}\, \eta(t) \rangle$; replace φ' by $\sqrt{2D(x)}\, \varphi'$

$$\hookrightarrow \langle \sqrt{D}\, \partial_x (\sqrt{D}\, \varphi') \rangle = \langle D(x) \varphi''(x) \rangle + \left\langle \sqrt{D} \frac{1}{2\sqrt{D}} D'(x) \varphi'(x) \right\rangle$$

and strato

$$\left\langle \frac{d\varphi}{dx} \right\rangle \overset{\text{S}}{=} \mu \langle \varphi'(x) F(x) \rangle + \langle D(x) \varphi''(x) \rangle + \frac{1}{2} \langle D'(x) \varphi'(x) \rangle$$

At Itō-Döblin level, we still have $\langle (dx)^2 \rangle = 2D\,dt$ and

$$\left\langle \frac{d\varphi}{dx} \right\rangle = \mu \langle \varphi'(x) F(x) \rangle + \langle D(x) \varphi''(x) \rangle$$

C

$$x_n = \eta_1 + \eta_2 + \cdots + \eta_n$$

1°) $\langle x_n \rangle = n \langle \eta \rangle = 0$

$$\langle x_n^2 \rangle = V(x_n) = V\left(\sum_{i=1}^n \eta_i\right) \overset{\text{independence}}{=} \sum_{i=1}^n V(\eta_i) = n\, V(\eta)$$

$$\boxed{\langle x_n^2 \rangle = \frac{2}{3} n}$$

2°) The mean and variance are finite, and the central limit theorem applies for typical fluctuations $\Rightarrow$ Gaussian

$$\boxed{P(x,n) \underset{n\to\infty}{\sim} \frac{1}{\sqrt{2\pi \frac{2n}{3}}} \exp\left\{ -\frac{3x^2}{4n} \right\}}$$

(b)

3°) Since the law of η is symmetric, the CLT region of typical fluctuations holds for $\boxed{|x| \text{ of order } n^{3/4}}$

4°) a) $\boxed{P(x,n) \doteq \exp\left[-n\, \phi\left(\frac{x}{n}\right) \right]}$

b) $\boxed{\phi(0) = 0}$; besides $z = 1 \Leftrightarrow x = n$, meaning that all steps η_i are $+1$:

$$P(x,n) = \left(\frac{1}{3}\right)^n = e^{-n \log 3}$$

$$\boxed{\phi(1) = \phi(-1) = \log 3}$$

c) We have $\langle e^{t x_n} \rangle = \langle e^{t \sum_{i=1}^n \eta_i} \rangle = \langle e^{t\eta} \rangle^n$

$$\Rightarrow \mathcal{K}_n(t) = \frac{1}{n} \log \langle e^{t x_n} \rangle = \log \langle e^{t\eta} \rangle$$

from the independence of the steps η_i

$$\langle e^{t\eta} \rangle = \frac{1}{3}\left[1 + e^t + e^{-t} \right] = \frac{1}{3}(1 + 2\cosh t)$$

$$\mathcal{K}(t) = \lim_{n\to\infty} \mathcal{K}_n(t) = \log\left(\frac{1 + 2\cosh t}{3} \right)$$

$$\phi(z) = \max_t \left[z t - \mathcal{K}(t) \right]$$

max $\Rightarrow z = \frac{d\mathcal{K}}{dt} = \frac{2 \sinh t}{1 + 2\cosh t} = \frac{e^t - e^{-t}}{1 + e^t + e^{-t}}$

$X = e^t$

$$z(1 + X + X^2) = X^2 - 1$$

$$X^2(1-z) - zX - 1 - z = 0$$

$-1 \leq z \leq 1$

$$X = \frac{z \pm \sqrt{z^2 + 4(1-z^2)}}{2(1-z)}$$

only (+) roots acceptable ($X > 0$)

$$X = \frac{z + \sqrt{4 - 3z^2}}{2(1-z)} = e^t$$

$$\Rightarrow t = \log\left[\frac{z + \sqrt{4 - 3z^2}}{2(1-z)} \right]$$

Check $z = 0$ $\Rightarrow t = 0$, $X = 1$

We note the symmetry :

$t \leftrightarrow -t$

$z \leftrightarrow -z$

$X \leftrightarrow 1/X$, X always positive

Note also that we expect $\boxed{\phi(z)=\phi(-z)}$, from the symmetry of the z-distribution. To get $\phi(z)$, we need to express $1+2\cosh t$ as a function of z

$$1+2\cosh t = 1+X+\frac{1}{X}$$

$$\frac{1}{X} = \frac{-z+\sqrt{4-3z^2}}{2(1+z)}$$ from the above symmetry. $(z \leftrightarrow -z)$

$$1+X+\frac{1}{X} = 1+\frac{z+\sqrt{}}{2(1-z)} + \frac{\sqrt{}-z}{2(1+z)}$$

$$= 1 + \frac{(1+z)(z+\sqrt{})+(1-z)(\sqrt{}-z)}{2(1-z^2)}$$

$$= 1 + \frac{\cancel{z}+\sqrt{}+z^2+\cancel{z\sqrt{}}+\sqrt{}-\cancel{z}-\cancel{z\sqrt{}}+z^2}{2(1-z^2)}$$

$$= 1 + \frac{\sqrt{}+z^2}{(1-z^2)}$$

$$= \frac{1+\sqrt{4-3z^2}}{1-z^2}$$ which is $z \leftrightarrow -z$ invariant ✓

Finally :

$$\boxed{\phi(z) = z\log\left[\frac{z+\sqrt{4-3z^2}}{2(1-z)}\right] - \log\frac{1+\sqrt{4-3z^2}}{1-z^2} + \log 3}$$

It can be checked that $\phi(z)=\phi(-z)$ (not obvious)

$\phi(0)=0$

$\phi(\pm 1) = \log 3$ ✓

d) The central limit behaviour is

$$P(x,n) \doteq \exp\left(-\frac{3x^2}{4n}\right)$$

$$\doteq \exp\left(-n\,\phi\left(\frac{x}{n}\right)\right)$$

Hence $\boxed{\phi(z) \sim \frac{3}{4}z^2}$ for $z\to 0$ (c)

Do we recover this from the Taylor expansion of the explicit $\phi(z)$? Perform expansion up to order 2:

$$\sqrt{4-3z^2} \overset{o}{\sim} 2\left(1-\frac{3}{8}z^2\right)$$

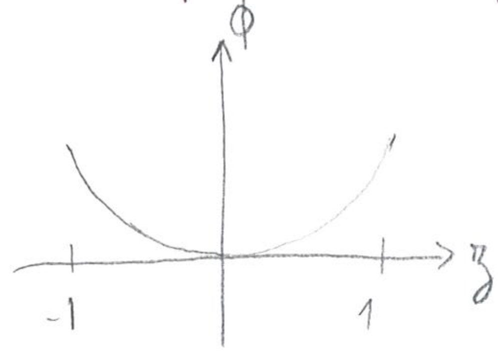
$$\phi(z) \sim z\log\left(\frac{z+2-3z^2/4}{2(1-z)}\right) - \log\frac{1+2-\frac{3}{4}z^2}{1-z^2} + \log 3$$

$$\sim z\left[\log\left(1+\frac{z}{2}\right)-\log(1-z)\right] - \log\left(3(1-z^2/4)\right) + \log(1-z^2) + \log 3$$

$$\sim z\left(\frac{z}{2}+z\right) - \cancel{\log 3} + \frac{z^2}{4} - z^2 + \cancel{\log 3}$$

$$\sim \frac{3}{4}z^2$$

which is thus compatible with CLT finding ✓



e) Following Sanov, we are seeking the q distribution defined on $\{-1,0,1\}$ with arbitrary weights, such that $D(q\|p)$ is minimum, where p is the proba of the z jumps

$\frac{1}{3}$ for -1

$\frac{1}{3}$ " 0

$\frac{1}{3}$ " $+1$

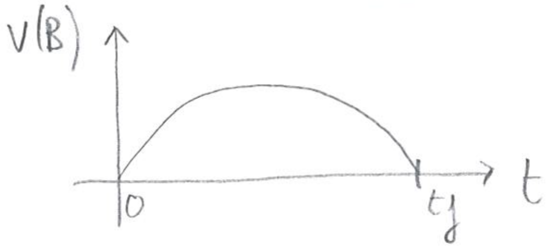
such that $\sum_{x=-1,0,1} x\,q(x) = z$.

Then $\phi(z) = D[q\|p]$.

Given z-constraint and normalization, only unknown is $q(1)$.

I Constrained processes & effective Lang. eq

1°) The variance vanishes at $t=0$ and $t=t_f$; it is maximal in between



2°) γ is an inverse time-scale;

$$\boxed{D = \frac{kT}{m\gamma}}$$, fluctuation-dissipation relation

3°) $$\partial_t P = -\frac{1}{m\gamma}\partial_x (FP) + D\partial_x^2 P$$
$$= D\,\partial_x\left[-\beta F P + \partial_x P\right] \quad ; \quad \beta = \frac{1}{kT}$$

4°) We are here considering $x_f = 0$

$$\tilde{P}(x,t) = \frac{P(x_1=0, t_1=0;\ x_2=x, t_2=t,\ x_3=0, t_3=t_f)}{\int dx \quad ''}$$

Since $x(t)$ is Markovian:

$$\tilde{P}(x,t) = \frac{P(0,t_f|x,t)\,P(x,t|0,0)}{\int dx\, P(0,t_f|x,t)\,P(x,t|0,0)}$$

$$\boxed{\tilde{P}(x,t) = \frac{P(0,t_f|x,t)\,P(x,t|0,0)}{P(0,t_f|0,0)}}$$

5°) $Q(x,t)$ obeys the backwards FP equation

$$\boxed{\partial_t Q = -D\left[\beta F(x)\partial_x Q + \partial_x^2 Q\right]}$$

6°) $$\tilde{P}(x,t) = \frac{Q(x,t)\,P(x,t|0,0)}{P(0,t_f|0,0)}$$

$$\partial_t \tilde{P} = \frac{D}{P(0,t_f|0,0)}\Big\{ Q\left[\partial_x(-\beta F P) + \partial_x^2 P\right] + P\left[-\beta F \partial_x Q - \partial_x^2 Q\right]\Big\}$$

We then reconstruct $\tilde{P}$ from P and Q, making use of

d

$$\partial_x^2(PQ) = P\partial_x^2 Q + 2\partial_x P\partial_x Q + Q\partial_x^2 P$$

$$\frac{\partial_t \tilde{P}}{D} = \frac{1}{P(0,t_f|0,0)}\Big\{-\partial_x(\beta F P Q) + \boxed{Q\partial_x^2 P} - P\partial_x^2 Q\Big\}$$

$$\boxed{\partial_x^2(PQ) - P\partial_x^2 Q - 2\partial_x P\partial_x Q}$$

$$= \frac{1}{P(0,t_f|0,0)}\Big\{-\partial_x(\beta F P Q) + \partial_x^2(PQ) - 2P\partial_x^2 Q - 2\partial_x P\partial_x Q\Big\}$$

$$= \quad '' \quad \Big\{-\partial_x(\beta F P Q) + \partial_x^2 PQ - 2\partial_x\left(P\partial_x Q\right)\Big\}$$

$$= -\partial_x(\beta F\tilde{P}) + \partial_x^2\tilde{P} - 2\partial_x\left[\frac{\tilde{P}}{Q}\partial_x Q\right]$$

$$\boxed{\frac{\partial \tilde{P}}{\partial t} = -D\partial_x\left[\left(\beta F + 2\partial_x \log Q\right)\tilde{P}\right] + D\partial_x^2\tilde{P}}$$

7°) The corresponding Langevin equation is

$$\boxed{\dot{x} = \frac{1}{m\gamma}\left[F(x) + 2kT\partial_x \log Q\right] + \sqrt{2D}\,\xi(t)}$$

hence an additional effective potential

$$-2kT\log Q(x,t) = V_{eff}(x,t)$$

In the low probability regions where Q is small, V_{eff} becomes large $\Rightarrow$ penalty. When $t \to t_f$, $Q(x,t)$ goes to a $\delta(x)$, meaning that V_{eff} will be large except at 0 $\Rightarrow$ this will force the walker to go back to the origin, as demanded.

8°) $F=0 \Rightarrow \partial_t Q = -D\partial_x^2 Q$ with

$$Q(x,t_f) = \delta(x)$$

Using the shifted time $\tau = t_f - t$, we have

$$\partial_\tau Q = +D\partial_x^2 Q$$, usual diffusion

$$Q(x,\tau) = \frac{1}{\sqrt{2\pi\, 2D\tau}}\exp\left[-\frac{x^2}{4D\tau}\right]$$

Hence $Q(x,t) = \frac{1}{\sqrt{4\pi D(t_f-t)}} \exp\left[-\frac{x^2}{4D(t_f-t)}\right]$

from which we can compute $V_{eff} = -2kT \log Q$

and $-\partial_x V_{eff} = 2kT \frac{(-x)\times 2}{4D(t_f-t)}$

this yields the conditioned Langevin equation

$$\boxed{\dot{x} = \frac{-x}{t_f-t} + \sqrt{2D}\,\xi(t)}$$

the walker evolves in a harmonic potential

$$V_{eff} = \frac{2kT}{4D}\frac{x^2}{t_f-t} = \frac{kT}{2D}\frac{x^2}{t_f-t}$$

which has a time dependent stiffness in $\frac{1}{t_f-t}$, divergent for $t \to t_f$. This forces the Brownian object to go back to 0 exactly at t_f, no matter what the detailed trajectory is.

9°) We have $F=0$, but we look at excursions for which the particle cannot visit $x \leq 0$ region. It is like having an external potential that is ∞ for $x \leq 0$. Otherwise, for $x>0$, one has to solve again

$$\boxed{\partial_t Q = -D\partial_x^2 Q}$$

with $Q(x,t_f) = \delta(x-x_f)$

and $Q(x=0,t) = 0$

this diffusion equation can be solved by a method of "images", analogous to that used in electrostatics: we solve $\partial_t Q = -D\partial_x^2 Q$ in the whole space, but with

$$Q(x,t_f) = \delta(x-x_f) - \delta(x+x_f)$$

Symmetry garanties that $Q(x=0,t)=0 \quad \forall t$

so that restricting to $x>0$, we get the desired solution.

$$Q(x,t) = \frac{1}{\sqrt{4\pi D(t_f-t)}}\left\{\exp\left(-\frac{(x_f-x)^2}{4D(t_f-t)}\right) - \exp\left(-\frac{(x_f+x)^2}{4D(t_f-t)}\right)\right\}$$

a) We need $\partial_x \log Q$ for $x_f \to 0$. We compute first the derivative, and then take the limit

$$+2D\partial_x \log Q = \frac{\frac{x_f-x}{t_f-t} e^{-\frac{(x_f-x)^2}{4D(t-t_f)}} + \frac{x_f+x}{t_f-t} e^{-(\cdot\cdot t)}}{e^{-\frac{(x_f-x)^2}{4D(t_f-t)}} - e^{-(x_f+x)^2/4D(t_f-t)}}$$

$$\underset{x_f\to 0}{\sim} \frac{2D}{x} - \frac{x}{t_f-t}$$

(expand the exp in vicinity of $x_f=0$)

We recover the "bridge" term $-\frac{x}{t_f-t}$ that forces back to 0, plus a force that diverges as $\frac{1}{x}$ when the particle approaches the "wall" ($x\to 0$, ie the forbidden region $x<0$). This term makes sure that no trajectory goes to $x<0$ and that we indeed have an excursion.

b) See above comment: quite remarkably, the two contributions ($\frac{2D}{x}$ and $-\frac{x}{t_f-t}$) are decoupled (additive)

10°) The end point $x(t_f)$ is left free, and no longer constrained to be 0. Yet, the constraint $x>0$ remains at all times. We denote the corresponding Q by $Q_M \Rightarrow$ meander. It is obtained by integrating (9) over all x_f values, from 0 to ∞.

$Q_M(x,t) = \int_0^\infty Q(x,t)\,dx_f$;

We introduce $\tilde{x} = \frac{x}{\sqrt{4D(t_f-t)}}$

$$y = \frac{x_f-x}{\sqrt{4D(t_f-t)}} \quad ; \quad z = \frac{x_f+x}{\sqrt{4D(t_f-t)}}$$

$$Q_\Pi(x,t) = \frac{1}{\sqrt{\pi}} \int_0^\infty e^{-\frac{(x_f - x)^2}{4Dt}} \frac{dx_f}{\sqrt{4Dt}} - \frac{1}{\sqrt{\pi}} \int_0^\infty e^{-\frac{(x_f + x)^2}{4Dt}} \frac{dx_f}{\sqrt{4Dt}}$$

$$= \frac{1}{\sqrt{\pi}} \int_{-\tilde{x}}^\infty e^{-y^2} dy - \frac{1}{\sqrt{\pi}} \int_{\tilde{x}}^\infty e^{-z^2} dz$$

$$= \frac{1}{\sqrt{\pi}} \left[\cancel{\int_0^\infty e^{-y^2} dy} + \underbrace{\int_{-\tilde{x}}^0 e^{-y^2} dy}_{\int_0^{\tilde{x}} e^{-y^2} dy} - \left(\cancel{\int_0^\infty e^{-z^2} dz} - \int_0^{\tilde{x}} e^{-z^2} dz \right) \right]$$

$$= \frac{2}{\sqrt{\pi}} \int_0^{\tilde{x}} e^{-y^2} dy$$

$$Q_\Pi(x,t) = \mathrm{erf}\left(\frac{x}{\sqrt{4D(t_f - t)}} \right)$$

The effective force follows: $\partial_x \log Q_\Pi(x,t) = \frac{2}{\sqrt{\pi}} \frac{1}{\sqrt{4D(t_f-t)}} \frac{e^{-\frac{x^2}{4D(t_f-t)}}}{\mathrm{erf}\left(\frac{x}{\sqrt{4D(t_f-t)}}\right)}$,

which yields a Langevin equation:

$$\underbrace{\boxed{\dot{x} = \frac{2}{\sqrt{\pi}} \sqrt{\frac{D}{t_f - t}} \frac{e^{-\frac{x^2}{4D(t_f-t)}}}{\mathrm{erf}\left(\frac{x}{\sqrt{4D(t_f-t)}}\right)} + \sqrt{2D}\, \xi(t)}}$$

since $\mathrm{erf}(t) \sim \sqrt{\frac{2}{\pi}}\, t$ for small t,
this force behaves, for small x
as $\cancel{\frac{2}{\sqrt{\pi}}} \sqrt{\frac{D}{\cancel{t_f - t}}} \frac{1}{\cancel{\frac{2}{\sqrt{\pi}}} \frac{x}{\sqrt{4D(\cancel{t_f-t})}}} \sim \frac{2D}{x}$, as before for preventing the particle to enter $x < 0$ zone.

We may wonder why a force in $\frac{2D}{x}$ would not be enough for generating meanders: it would not lead to the proper statistics, hence the more complex form in $\frac{\exp(\ )}{\mathrm{erf}(\ )}$