First return, last return and tread distributions

1°)

Only 2 sites remain
↳ we have 2 survival paths among the 8 possible

In general, we have $N_+ - N_-$ survival paths (assuming $N_+ > N_-$) among the N.
Note that this does not depend on random permutations of the steps

2°) Proba $\dfrac{N_+ - N_-}{N}$ to choose a survival path, with $N_+ = N - N_-$.

3°) N_- is fixed (thus $N_+ = N - N_-$ also): there are $\binom{N}{N_-}$ paths associated to this value of N_-, among which a fraction $\dfrac{N-2N_-}{N}$ survives.
The total number of paths is thus

$$\frac{N-2N_-}{N}\binom{N}{N_-}$$

4°) We take N even; we need $N_+ > N_-$ for a possible survival path; hence $N_- = 0, 1, \dots, \frac{N}{2} - 1$. The total number of survival paths, ie those that always stay to the right of the origin is

$$\sum_{N_-=0}^{N/2-1} \frac{N-2N_-}{N}\binom{N}{N_-} = \sum_{N_-=0}^{N/2-1} \binom{N-1}{N_- -1} + \binom{N-1}{N_-} - \underbrace{2\frac{N_-}{N}\binom{N}{N_-}}$$

$$\frac{N_-}{N}\frac{N!}{N_-!(N-N_-)!} = \frac{(N-1)!}{(N_- -1)!(N-N_-)!} = \binom{N-1}{N_- -1}$$

convention $\binom{N-1}{-1} = 0$

$$= \sum_{N_-=0}^{N/2-1} \binom{N-1}{N_-} - \binom{N-1}{N_- -1}$$

$$= \binom{N-1}{N/2 - 1}$$

Without the assumption of N even, we would get a total $\binom{N-1}{\left\lfloor \frac{N-1}{2} \right\rfloor}$

5°) The total number of paths remaining to the left of the origin is the same as to the right.
Each path has proba $\frac{1}{2^N}$, hence

$$\boxed{S(N) = \frac{1}{2^{N-1}}\binom{N-1}{\frac{N}{2}-1}} \overset{N=2n}{=} \frac{1}{2^{2n}}\binom{2n}{n}$$

6°) For large N, we use Stirling approximation

$$S(N) = \frac{1}{2^{N-1}} \frac{\frac{N}{2} N!}{N \left(\frac{N}{2}\right)!} \frac{1}{\left(\frac{N}{2}\right)!} \sim \boxed{\sqrt{\frac{2}{\pi N}}} \qquad \text{since } \binom{N}{\frac{N}{2}} \sim 2^N \sqrt{\frac{2}{\pi N}}$$

$$\boxed{S(N) \underset{N\to\infty}{\longrightarrow} 0}$$

Hence the probability to return to the origin is 1

The walk is reccurent

7°) $t = N\tau_0$; $s(t) \sim \sqrt{\frac{2\tau_0}{\pi t}}$, $\boxed{F(t) = -\frac{ds}{dt} \underset{t\to\infty}{\sim} \sqrt{\frac{\tau_0}{2\pi t^3}} \propto t^{-3/2}}$

Lévy index $1+\mu = \frac{3}{2}$

$\mu = \frac{1}{2}$

Thus the mean return time diverges

8°) After $2n_e$ steps, the proba to be back to origin is $\binom{2n_e}{n_e} \frac{1}{2^{2n_e}} \underset{n_e\to\infty}{\sim} \frac{1}{\sqrt{\pi n_e}}$

9°) The proba to return for the last time at step $2n_e$
is the product of
- the proba to sit at the origin at step $2n_e$
- and the proba of "surviving" $2n - 2n_e$ steps hereafter

Hence the proba of last return (for large n_e, large n)

$$\frac{1}{\sqrt{\pi n_e}} \times \sqrt{\frac{2}{\pi(2n - 2n_e)}} = \frac{1}{\pi\sqrt{n_e(n-n_e)}}$$

With $\tau = \frac{n_e}{n}$; $p_{last}(\tau)\, d\tau = \frac{1}{\pi\sqrt{n_e(1-n_e)}}\, dn_e$ ie $\boxed{p_{last}(\tau) = \frac{1}{\pi\sqrt{\tau(1-\tau)}}}$

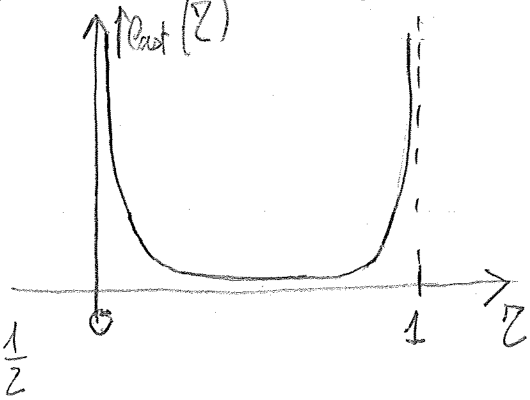
$\frac{d\tau}{dn_e} = \frac{1}{n}$

$$\int_0^a p_{last}(\tau)\, d\tau = \int_0^a \frac{d\tau}{\pi\sqrt{\tau(1-\tau)}} \overset{y=\sqrt{\tau}}{=} \int_0^{\sqrt{a}} \frac{2y\, dy}{\pi\sqrt{y^2(1-y^2)}} = \frac{2}{\pi}\int_0^{\sqrt{a}} \frac{dy}{\sqrt{1-y^2}} = \frac{2}{\pi} \arcsin\left(\sqrt{a}\right)$$

All this counterintuitive!

Symmetry $\tau \Leftrightarrow 1-\tau$
not obvious.

$\Pr\left(\tau > \frac{1}{2}\right) = \Pr\left(\tau < \frac{1}{2}\right) = \frac{1}{2}$



Singular maxima
at $\tau = 0$ and 1 !

Mean value is least
probable.