Discrete random walks

A - First results (obtained "by hand", no formalism)

1) Return probability: R.

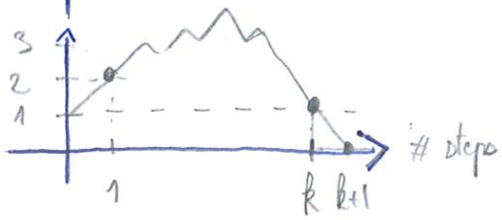
a) Bayes theorem: $R = p R_{\rightarrow} + q R_{\leftarrow}$

b) Consider a walker that starts from site 1; it can visit site 0 after n visits to site 1, with $n = 1, 2, \dots \infty \rightarrow$ the different values of n form a partition of all possible events:

$n = 1$

Proba q for this event

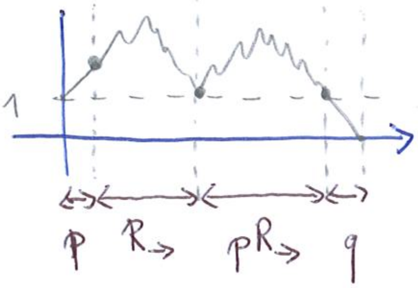
$n = 2$



walker starts at site 1, moves right, (proba p), then goes back to 1 (proba $R_{\rightarrow}$) and from site 1, moves left to 0 (proba q)

$\hookrightarrow$ proba $p R_{\rightarrow} q$

$n = 3$



Proba: $(p R_{\rightarrow})^2 q$

...

$$\Rightarrow \quad R_{\rightarrow} = q \sum_{n=1}^{\infty} (p R_{\rightarrow})^{n-1} = \frac{q}{1 - p R_{\rightarrow}}$$

$$\Leftrightarrow \quad p R_{\rightarrow}^2 - R_{\rightarrow} + q = 0$$

$$\Delta = 1 - 4pq = 1 - 4p(1-p) = 4p^2 - 4p + 1 = (2p-1)^2 = (p-q)^2$$

$$R_{\rightarrow} = \frac{1}{2p} \left(1 \pm |p-q| \right)$$

Root with $\oplus$ $\xrightarrow{p > 1/2}$ $\frac{1}{2p}(1 + p - q) = \frac{1}{2p}(p+p) = 1$: makes no sense (take $p \to 1$, walker should not come back)

$\xrightarrow{p < 1/2}$ $\frac{1}{2p}(1 - p + q) = q/p > 1$: no!

This root makes no sense.

Look at the other root:

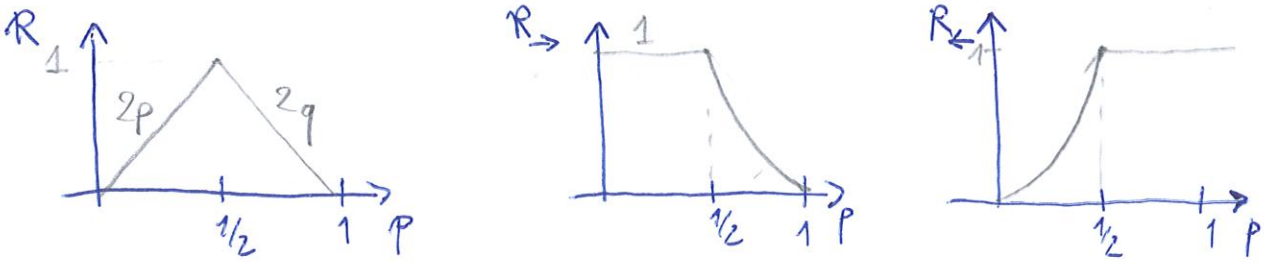
$$R_{\rightarrow} = \frac{1}{2p}\left(1 - p + q\right) = q/p \quad , \quad p \geq \frac{1}{2} \qquad \text{ok}$$

$$= \frac{1}{2p}\left(1 - q + p\right) = 1 \quad , \quad p \leq \frac{1}{2} \qquad \text{ok}$$

Thus $\boxed{R_{\rightarrow} = \frac{1}{2p}\left(1 - |p-q|\right)}$; $\boxed{R_{\leftarrow} = \frac{1}{2q}\left(1 - |p-q|\right)}$, permuting p & q

c) $R = p R_{\rightarrow} + q R_{\leftarrow}$; $\boxed{R = 1 - |p-q|}$; For $p = q = 1/2$, $R = 1$ return is certain.

$R = 2p, \; p < 1/2$
$= 2q, \; q > 1/2$

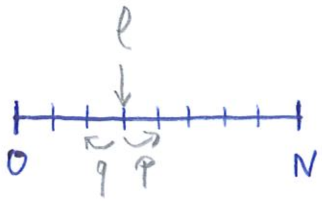


We have found, for instance with $p \geq \frac{1}{2}$: $R_{\rightarrow} = q/p$.
This also tells us that starting from site $\ell > 0$, the proba to visit the origin, some time in the future, is simply $R_{\ell \rightarrow} = \left(q/p\right)^{\ell}$
On the other hand, with $p \leq \frac{1}{2}$, $R_{\ell \rightarrow} = 1^{\ell} = 1$.

2) Exit time from an interval

a) $$P[T(\ell) = n+1] = p\, P[T(\ell+1) = n] + q\, P[T(\ell-1) = n]$$

$$P[T(0) = n] = \delta_{n,0} \quad ; \quad P[T(N) = n] = \delta_{n,0}$$



b) $$\langle T(\ell)\rangle = \sum_{n=0}^{\infty} n\, P[T(\ell) = n] = \sum_{n=1}^{\infty} n\, P[T(\ell) = n]$$

$$= \sum_{n=1}^{\infty} n \left\{ p\, P[T(\ell+1) = n-1] + q\, P[T(\ell-1) = n-1] \right\}$$

$$= \sum_{n=0}^{\infty} (n+1) \left\{ p\, P[T(\ell+1) = n] + q\, P[T(\ell-1) = n] \right\}$$

$$= \sum_{n=0}^{\infty} n \left\{ p\, P[T(\ell+1) = n] + q\, P[T(\ell-1) = n] \right\} + \sum_{n=0}^{\infty} 1 \cdot \left\{ pP + qP \right\}$$

$$\boxed{\langle T(\ell)\rangle = p \langle T(\ell+1)\rangle + q \langle T(\ell-1)\rangle + \underbrace{p+q}_{1}} \; ; \quad \langle T(0)\rangle = 0 , \quad \langle T(N)\rangle = 0$$

c) Resolution of the difference equation $\boxed{p\, x_{\ell+1} + q\, x_{\ell-1} - x_\ell = -1}$; $x_0 = x_N = 0$

For $p = q = 1/2$, we recognize a discrete Laplacian.

$\longrightarrow$ homogeneous equation : $p\, x_{\ell+1} + q\, x_{\ell-1} - x_\ell = 0$; take $x_\ell = r^\ell$

$p r^2 - r + q = 0$; already solved above.

we have found the two roots to be 1 and q/p ; we assume here $q \neq p$

The general solution is thus of form $\lambda + \mu \left(q/p\right)^\ell$

$\longrightarrow$ we need a particular solution, that we can take polynomial.

A constant does not work since $p + q - 1 = 0$, but a linear function works

$$x_\ell = b\ell \Rightarrow p\, b(\ell+1) + q\, b(\ell-1) - \cancel{b\ell} = -1 \Leftrightarrow pb - qb = -1$$

$$b = \frac{1}{q-p}$$

$\longrightarrow$ the complete solution reads :

$$\boxed{x_\ell = \lambda + \mu\left(q/p\right)^\ell} + \frac{\ell}{q-p}$$

Boundary conditions : $x_0 = x_N = 0 \Rightarrow \lambda + \mu = 0$; $\lambda + \mu\left(\frac{q}{p}\right)^N + \frac{N}{q-p} = 0$

$$\lambda\left[1 - \left(q/p\right)^N\right] = -\frac{N}{q-p}$$

Thus : $$\boxed{\langle T(\ell)\rangle = \frac{1}{q-p}\left[\ell - N\,\frac{1-\left(q/p\right)^\ell}{1-\left(q/p\right)^N}\right]}$$ for $q \neq p$ (ie $p \neq \frac{1}{2}$)

$\longrightarrow$ Particular case $p = q = 1/2$: the 2 roots above, 1 and q/p, become equal.

We can no longer look for a solution in $\alpha r_1^\ell + \beta r_2^\ell$, but in $\alpha r_1^\ell + \beta \ell r_2^\ell$. Here, $r_1 = r_2 = 1 \rightarrow$ the general solution to the homogeneous equation is in $\alpha + \beta\ell$. the particular solution can be taken in ℓ^2

It has to be symmetric w r t $\ell = \frac{N}{2} \rightarrow \left[\left(\ell - \frac{N}{2}\right)^2 + Cst\right] \times \gamma$

The constant is set by the b.c :

$$\hookrightarrow \gamma\left\{\left(\ell - \frac{N}{2}\right)^2 - \left(\frac{N}{2}\right)^2\right\} = \gamma\, \ell(N-\ell)$$, to be plugged into $\frac{1}{2}x_{\ell+1} - x_\ell + \frac{1}{2}x_{\ell+1} = -1$

$$\Rightarrow \boxed{\langle T(\ell)\rangle = \ell(N-\ell)}$$

Alternatively, one may take the expression of $\langle T(\ell)\rangle$ for $p \neq q$, and take the limit $p \to \frac{1}{2}$ (ie $q \to \frac{1}{2}$ also); gives same result

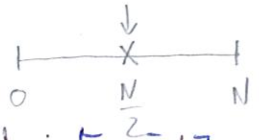
<u>Remark 1</u> it is possible to show that the walker leaves $[0,N]$ with proba one, when waiting enough. Indeed, denote $\pi(\ell)$ the proba to leave the interval (from its right or its left end), one day.

$$\pi(\ell) = p\,\pi(\ell+1) + q\,\pi(\ell-1) \quad \text{with} \quad \pi(0) = \pi(\ell) = 1$$

$$\Rightarrow \pi(\ell) = \alpha + \beta\left(\frac{q}{p}\right)^{\ell} \quad ; \quad \begin{cases} \alpha + \beta = 1 \\ \alpha + \beta\left(q/p\right)^{N} = 1 \end{cases} \Rightarrow \begin{cases} \alpha = 1 \\ \beta = 0 \end{cases}$$

$$\Rightarrow \pi(\ell) = 1$$

<u>Remark 2</u> : take $\ell = \frac{N}{2}$



; $\langle T(\frac{N}{2})\rangle = \frac{N^2}{4} \propto N^2$, diffusive scaling

Here, the diffusion coef is $\boxed{D = \frac{1}{2}}$

time $\propto \frac{\text{distance}^2}{D} \Rightarrow$ diff coefficient

↳ position $= \sum_{i=1}^{n} +1$ or -1

$$V(\text{position}) = n = 2 \times \underbrace{D}_{\frac{1}{2}} \times \underbrace{n}_{\text{time}}$$

Discrete random walks (3)

(B) Generating function formalism

1) a) $F_n(s|s_0) \leq P_n(s|s_0)$

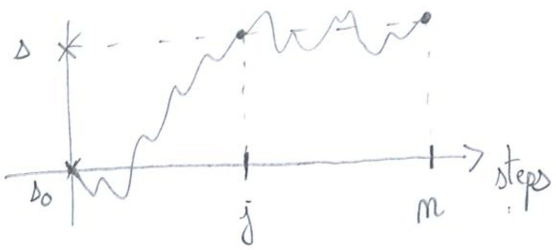
$\sum_s P_n(s|s_0) = 1$ the walker has to be somewhere

$\sum_n P_n(s|s_0)$ may diverge (mean # of visits, see below)

$\boxed{\sum_n F_n(s|s_0) = R(s|s_0) \leq 1}$: proba to reach site s, some time, starting at s_0

$\sum_s F_s(s|s_0) \leq 1$

b)



When the walker is at s, it has first visited this site earlier than step n, at step j with $j = 1, 2, \dots n$ (for $n \geq 1$) and starting from s at step j) it has to be back at step n

$$\boxed{\begin{aligned} &\text{For } n \geq 1: \quad P_n(s|s_0) = \sum_{j=1}^{n} F_j(s|s_0)\, P_{n-j}(s|s) \\ &\text{For } n = 0: \quad P_0(s|s_0) = \delta_{s,s_0} \end{aligned}}$$

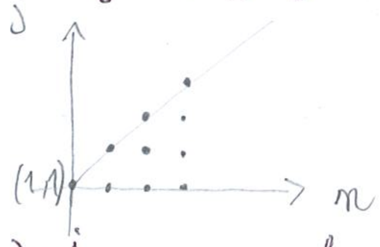
Law of total probability
$P_n(s|s_0) = \sum_j \text{Prob}(\text{be at } s \,|\, j)\, P(j)$

i.e $P_n(s|s_0) = \delta_{n,0}\delta_{s,s_0} + \sum_{j=1}^{n} F_j(s|s_0) P_{n-j}(s|s)$

interpreting the $\sum_1^0$ to be zero (for $n = 0$).

c) $$P(s|s_0;\xi) = \sum_{n=0}^{\infty} P_n(s|s_0)\,\xi^n = \delta_{s,s_0} + \sum_{n=1}^{\infty} P_n(s|s_0)\,\xi^n$$

$$= \delta_{s,s_0} + \sum_{n=1}^{\infty}\sum_{j=1}^{n} F_j(s|s_0) P_{n-j}(s|s)\,\xi^j\,\xi^{n-j}$$



$\begin{cases} 1 \leq n \leq \infty \\ 1 \leq j \leq n \end{cases} \iff \begin{cases} 1 \leq j \leq \infty \\ \underbrace{n-j}_{k} \geq 0 \end{cases}$

$$= \delta_{s,s_0} + \underbrace{\sum_{j=1}^{\infty}}_{=\sum_{j=0}^{\infty} \text{ since } F_0 = 0} \sum_{k=0}^{\infty} F_j(s|s_0)\,\xi^j\, P_k(s|s)\,\xi^k$$

$$\boxed{P(s|s_0;\xi) = \delta_{s,s_0} + F(s|s_0;\xi)\,P(s|s;\xi)}$$

2°) We have found that $P(s|s;\xi) = 1 + F(s|s;\xi) P(s|s;\xi)$

and we want that $R(s_0|s_0) = 1$. Note that

$$R(s|s) = \sum_{n=0}^{\infty} F_n(s|s_0) = \lim_{\xi \to 1} F(s|s;\xi)$$

$$F(s|s;\xi) = 1 - \frac{1}{P(s|s;\xi)} \quad ; \quad \boxed{R(s_0|s_0) = 1 \iff P(s_0|s_0;1) = \infty}$$

$$P(s|s;\xi) = \frac{1}{1 - F(s|s;\xi)} \qquad \boxed{R(s_0|s_0) = F(s|s;1)}$$ useful for part C.

3°) $I_n(s|s_0) = 1$ if site s occupied at time n; 0 otherwise ; $\langle I_n(s|s_0) \rangle = P_n(s|s_0)$

$\Rightarrow \sum_{n=0}^{\infty} I_n(s|s_0)$ is the total number of visits to site s.

$\Rightarrow \left\langle \sum_{n=0}^{\infty} I_n(s|s_0) \right\rangle = \sum_{n=0}^{\infty} P_n(s|s_0) = P(s|s_0;1)$ mean number of visits

the mean number of <u>returns</u> is $P(s_0|s_0;1) - 1 = \infty$ for recurrent walk

$= \dfrac{1}{1 - R(s_0|s_0)} - 1$ for transient " (non recurrent)

<u>Example</u> : take $R = 0,99$;

View each return like a new experiment, where walker comes back 99 times in 100 trials, and escapes to ∞ once.

$$\langle \#\text{ returns} \rangle = \frac{1}{0,01} - 1 = 99$$

More generally, for R given, $1-R$ is proba for not returning $= \dfrac{1}{\langle \#\text{return} \rangle + 1}$

1 trial not returning

$$\Rightarrow \langle \#\text{ return} \rangle = \frac{1}{1-R} - 1$$

4°) $F_n(s|s_0)$ is the proba to visit s for the first time at step n.

Since $R(s|s_0) = \sum_{n=0}^{\infty} F_n(s|s_0)$, $\dfrac{F_n(s|s_0)}{R(s|s_0)}$ is the conditional proba to visit s for the first time at step n, given that site s is indeed visited.

$$\Rightarrow \boxed{\tau(s|s_0) = \sum_{n=0}^{\infty} n \frac{F_n(s|s_0)}{R(s|s_0)} = \frac{F'(s|s_0;1)}{F(s|s_0;1)}}$$

Discrete random walks ④

Ⓑ 5°) Application to biased r.w

$$P(0_0|0_0;\xi) = \sum_{n=0}^{\infty} P_n(0_0|0_0)\,\xi^n \qquad ; \quad P_n(0_0|0_0) = 0 \text{ if } n \text{ is odd}$$
$$= \binom{n}{n/2} p^{n/2} q^{n/2} \text{ if } n \text{ even}$$

$$= \sum_{i=0}^{\infty} P_{2i}(0|0_0)\,\xi^{2i}$$

$$= \sum_{i=0}^{\infty} \binom{2i}{i} p^i q^i \xi^{2i}$$

$$P(0_0|0_0;\xi) = \left(1 - 4pq\xi^2\right)^{-1/2}$$

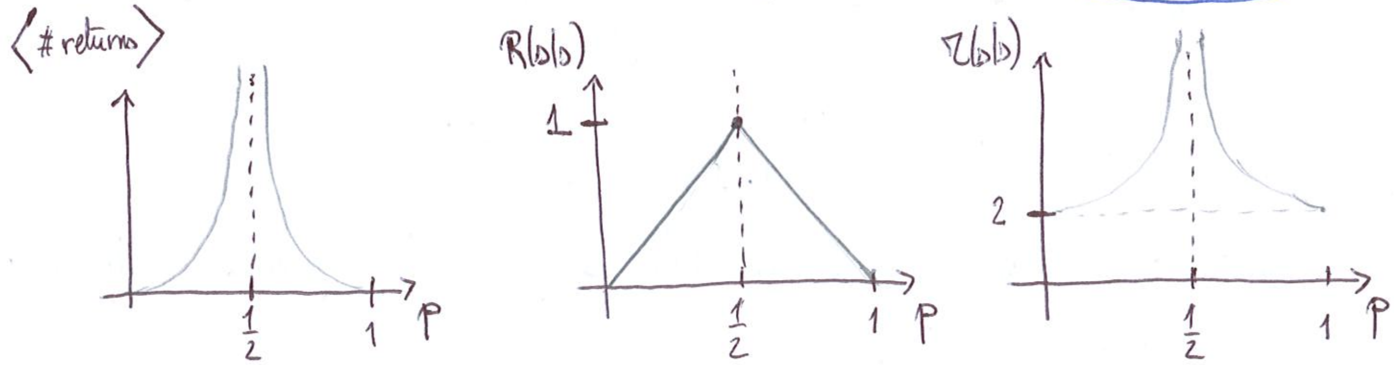
$$F(0_0|0_0;\xi) = 1 - \frac{1}{P(0_0|0_0;\xi)} = 1 - \left(1 - 4pq\xi^2\right)^{1/2}$$

$$\Rightarrow R(0_0|0_0) = F(0_0|0_0;1) = 1 - (1-4pq)^{1/2} = 1 - |p-q|$$ We recover the result of Ⓐ

Mean # of returns $= P(0_0|0_0;1) - 1 = \dfrac{1}{|p-q|} - 1$

Conditional mean first passage time:

$$\tau(0_0|0_0) = \frac{F'(0_0|0_0;1)}{F(0_0|0_0;1)} = \frac{1}{1-|p-q|}\,\frac{1}{2}\left(\frac{4pq\,2\xi}{\sqrt{1-4pq\xi^2}}\right)_{\xi=1} = \frac{4pq}{(1-|p-q|)\,|p-q|}$$



Note that $\tau(0|0) \searrow$ when $p \nearrow$ and $\to 1$: the return probability decreases, which does not affect $\tau(0|0)$, conditioned onto those walks that return.

When $p \to 1$, the only event leading to a return is a sequence of 2 steps, left/right. The only "possible" returns are in 2 steps.

Note that $\tau(0|0) \to \infty$ for $p \to \frac{1}{2}$: although the walker always comes back to its starting point, it takes ∞ time on average.

Ⓒ Translationaly invariant random walks

1°) a) $P_{n+1}(\vec{\ell}) = \sum_{\vec{\ell}'} P_n(\vec{\ell}-\vec{\ell}')\, p(\vec{\ell}')$

b) $\tilde{P}_{n+1}(\vec{k}) = \tilde{P}_n(\vec{k})\,\lambda(\vec{k})$; $\tilde{P}_0(\vec{\ell}) = 1 \Rightarrow \boxed{\tilde{P}_n(\vec{k}) = \lambda^n(\vec{k})}$

$$\boxed{P_n(\vec{\ell}) = \int_{[-\pi,\pi]^d} \frac{d^d\vec{k}}{(2\pi)^d}\, e^{-i\vec{k}\cdot\vec{\ell}}\, \lambda^n(\vec{k})}$$

↳ this stems from the geometry of the lattice on which the r.w. takes place. In all generality, the Brillouin Zone (BZ) appears here.

c) $$P(\vec{\ell},\xi) \equiv \sum_n \xi^n P_n(\vec{\ell}) = \int_{BZ} \frac{d\vec{k}}{(2\pi)^d}\, e^{-i\vec{k}\cdot\vec{\ell}} \sum_{n=0}^{\infty} \left(\lambda(\vec{k})\,\xi\right)^n$$

Note that $|\lambda| \leqslant \sum_{\vec{\ell}} p(\vec{\ell}) = 1$; $\lambda(\vec{0}) = 1$

$$\boxed{P(\vec{\ell},\xi) = \int_{BZ} \frac{d\vec{k}}{(2\pi)^d}\, e^{-i\vec{k}\cdot\vec{\ell}}\, \frac{1}{1-\xi\lambda(\vec{k})}}$$

for $|\xi\lambda(\vec{k})| < 1$

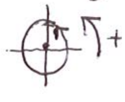
ie $|\xi| < 1$

2°) With the 1D random walk:

$$\lambda(k) = p\, e^{ik} + q\, e^{-ik} \qquad \text{(lattice spacing is unity)}$$

$$P(\ell,\xi) = \frac{1}{2\pi}\int_{-\pi}^{\pi} dk\, \frac{e^{-ik\ell}}{1-\xi\left(p e^{ik} + q e^{-ik}\right)}$$

Case $\ell > 0$; change $k \to -k$ and then use $z = e^{ik}$; z runs over the unit circle $\mathscr{C}$



$$P(\ell,\xi) = \frac{1}{2\pi}\oint_{\mathscr{C}} \underbrace{\frac{dz}{iz}}_{dk}\, \frac{z^\ell}{1-\xi\left(p z^{-1} + q z\right)}$$

$$= -\frac{1}{2i\pi}\oint_{\mathscr{C}} \frac{z^\ell}{z^2 - \frac{1}{\xi q} z + \frac{p}{q}}\, \frac{1}{\xi q}$$

and we will use the residue theorem

Roots of $A(z) = \xi q z^2 - z + \xi p$; $\Delta = 1 - 4\xi^2 pq > 0$

$$z_\pm = \frac{1 \pm \sqrt{1-4pq\xi^2}}{2\xi q}$$

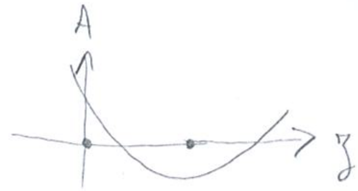
Discrete random variables ⑤.

$$z_+ z_- = \frac{p}{q} > 0$$

The 2 roots have the same sign

$$A(0) = \xi p > 0$$

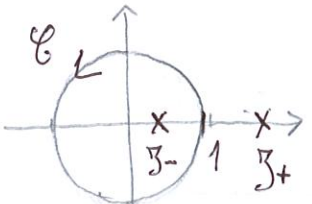
$$A(1) = \xi(p+q) - 1 \leqslant 0$$



$$\Rightarrow 0 < z_- < 1 < z_+$$

and we apply the residue theorem:

$$\oint_{\mathcal{C}} f(z)\,dz = 2i\pi \sum_{z_k,\ \text{residues}} \mathrm{Res}(f, z_k)\, \overbrace{\mathrm{Ind}_{\mathcal{C}}(z_k)}^{\#\text{ of turns made by } \mathcal{C} \text{ around } z_k}$$



$$P(\ell, \xi) = -\frac{z_-^{\ell}}{\xi q (z_- - z_+)}$$

Only the residue at z_- does matter. Note that we took $\ell \geqslant 0$ hence z^{ℓ} is not singular for $z = 0$

$$P(\ell, \xi) = + \left(\frac{1 - \sqrt{1 - 4pq\xi^2}}{2\xi q}\right)^{\ell} \frac{\cancel{2}\,\xi q}{\cancel{\xi q}\; 2\sqrt{1 - 4pq\xi^2}} \quad ; \ \ell \geqslant 0$$

For getting the result for $\ell < 0$, we need to exchange p and q. Thus

$$P(\ell, \xi) = \left(\frac{1}{2\xi q}\right)^{\ell} \frac{\left(1 - \sqrt{1 - 4pq\xi^2}\right)^{\ell}}{\sqrt{1 - 4pq\xi^2}} \qquad \ell \geqslant 0$$

$$= \left(\frac{1}{2\xi p}\right)^{|\ell|} \frac{\left(1 - \sqrt{1 - 4pq\xi^2}\right)^{|\ell|}}{\sqrt{1 - 4pq\xi^2}} \qquad \ell \leqslant 0$$

For $\ell = 0$: $$= \frac{1}{\sqrt{1 - 4pq\xi^2}}$$, OK, see Ⓑ-5° $\Rightarrow R(0) = 1 - \frac{1}{P(0,1)} = 1 - \sqrt{\ } = 1 - |p - q|$ OK

Then, for $\ell \neq 0$: $P(\ell; \xi) = F(\ell; \xi) P(0, \xi)$, see Ⓑ 3°

$$F(\ell; \xi) = \frac{P(\ell; \xi)}{P(0; \xi)}$$

$$F(\ell; \xi) = \begin{cases} \left(\frac{1}{2\xi q}\right)^{\ell} \left(1 - \sqrt{1 - 4pq\xi^2}\right)^{\ell} \\ \left(\frac{1}{2\xi p}\right)^{|\ell|} \left(1 - \sqrt{1 - 4pq\xi^2}\right)^{|\ell|} \end{cases} \qquad \ell \neq 0$$

and we remember that $R(\ell) = R(s_0+\ell | s_0) \equiv \sum_n F_n(s_0+\ell|s_0) = F(\ell; \xi = 1)$

$$R(\ell) \overset{\ell>0}{=} \left(\frac{1}{2q}\right)^{\ell} \left(1-|p-q|\right)^{\ell}$$

$$R(\ell) \overset{\ell<0}{=} \left(\frac{1}{2p}\right)^{|\ell|} \left(1-|p-q|\right)^{|\ell|}$$

$\boxed{\text{Summary}}$: take $p > q$

$$\ell > 0 \to R(\ell) = \left(\frac{1}{2q}\right)^{\ell} \left(1-p+q\right)^{\ell} = 1$$

$$\ell = 0 \to R(0) = 1 - |p-q|$$

$$\ell < 0 \to R(\ell) = \left(\frac{1}{2p}\right)^{|\ell|} \left(1+q-p\right)^{\ell} = \left(\frac{q}{p}\right)^{|\ell|}$$

Note that $R(1) = R(\rightarrow)$; $R(-1) = R(\leftarrow)$ found above.

3) Polya theorem

Each lattice point has $\frac{1}{2d}$ neighbours onto which the walker can jump, with equiprobability: the structure factor of the walk then reads

$$\lambda(\vec{k}) = \frac{1}{2d} \sum_{j=1}^{d} \left(e^{ik_j} + e^{-ik_j}\right)$$

where k_j is the j^{th} Cartesian coordinate of $\vec{k}$

$$= \frac{1}{d} \sum_{j=1}^{d} \cos(k_j)$$

a) Here again, we have

$$P(\vec{\ell}, \xi) = \int_{BZ} \frac{d\vec{k}}{(2\pi)^d} \frac{e^{-i\vec{k}\cdot\vec{\ell}}}{1 - \frac{\xi}{d}\sum_{j=1}^{d} \cos(k_j)}$$

and the Brillouin Zone (BZ) is cubic: $[-\pi, \pi]^d$

The trick here is to exponentiate:

$$\frac{1}{1 - \frac{\xi}{d}\sum_j \cos(k_j)} = \int_0^{\infty} dt \, e^{-t\left[1 - \frac{\xi}{d}\sum_{j=1}^{d} \cos(k_j)\right]}$$

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$$P(\vec{\ell},\xi) = \frac{1}{(2\pi)^d}\int_0^\infty dt \int_{BZ} d\vec{k}\; e^{-t}\, e^{\frac{\xi t}{d}\sum_{j=1}^{d}\cos(k_j) - i\sum_{j=1}^{d} k_j \ell_j}$$

$$= \frac{1}{(2\pi)^d}\int_0^\infty e^{-t} dt \underbrace{\int dk_1 \cdots dk_d \prod_{j=1}^{d} e^{\frac{\xi t}{d}\cos k_j - i k_j \ell_j}}_{\prod_{j=1}^{d}\int_{-\pi}^{\pi} dk\; e^{+\frac{\xi t}{d}\cos k - i k \ell_j}}$$

$$= \int_0^\infty dt\, e^{-t}\prod_{j=1}^{d}\int_0^{2\pi}\frac{dk}{2\pi}\exp\left(\frac{\xi t}{d}\cos k\right)\cos(k\ell_j)$$

↳ the function integrated is 2π-periodic

$\Rightarrow \int_0^{2\pi} = \int_{-\pi}^{\pi}$

$$= \int_0^\infty dt\, e^{-t}\prod_{j=1}^{d}\underbrace{\int_0^{\pi}\frac{dk}{\pi}\, e^{(\xi t/d)\cos k}\cos(k\ell_j)}_{I_{\ell_j}(\xi t/d) \,=\, I_{|\ell_j|}(\xi t/d)} \quad \text{thus } z = \frac{\xi t}{d}$$

$$\boxed{P(\vec{\ell},\xi) = \int_0^\infty dt\, e^{-t}\prod_{j=1}^{d} I_{|\ell_j|}\left(\frac{\xi t}{d}\right)}$$

k) We have shown that the walk is recurrent (ie the return proba to the starting site is 1) iff $P(\vec{0},\xi=1) = +\infty$. We thus have to see when the above integral diverges.

$$P(\vec{0},\xi=1) = \int_0^\infty dt\, e^{-t}\left[I_0\left(\frac{\xi t}{d}\right)\right]^d$$

$$e^{-t}\left[I_0\left(\frac{\xi t}{d}\right)\right]^d \underset{t\to\infty}{\sim} e^{-t}\left[\frac{e^{\xi t/d}}{\sqrt{2\pi\,\xi t/d}}\right]^d \underset{t\to\infty}{\overset{\xi=1}{\sim}} \frac{1}{(2\pi t/d)^{d/2}} \propto t^{-d/2}$$

Hence $P(\vec{0},1) = \infty \iff \frac{d}{2} \leq 1$

$\iff d \leq 2$: Pólya's theorem

The r.w is recurrent in dimensions 1 and 2, and not recurrent (transient) for dimension 3, 4, ...