ISING MODEL IN 1D (3)

Correlation function from graph analysis

The trick is to write $e^{K s s'} = ch(K) + s s' sh(K)$

Then $Z = \sum_{\{S_i\}_{1\le i\le N}} e^{\sum_{i=1}^{N-1} \beta J S_i S_{i+1}}$ free boundary conditions $S_1 = S_N$

$$= \sum_{\{S_i\}_{1\le i\le N}} \prod_{i=1}^{N-1} \left[ch(\beta J) + S_i S_j\, sh(\beta J) \right] = (ch\,\beta J)^{N-1} \sum_{\{S_i\}} \prod_{i=1}^{N-1} \left[1 + S_i S_{i+1}\, th\,\beta J \right]$$

As soon as a bond is picked, yields 0 contribution since it necessarily has dangling ends.
Hence, no bonds should be picked $\Rightarrow$ $\boxed{Z_{free} = (ch\,\beta J)^{N-1} \sum_{\{S_i\}_{1\le i\le N}} 1 = 2^N (ch\,\beta J)^{N-1}}$

as already found. It is also straightforward to find Z with periodic b.c.
Now, 2 graphs contribute: the empty one as before, and the fully covered

$$Z_{p.b.c} = (ch\,\beta J)^{N-1} \left(1 + (th\,\beta J)^{N-1} \right) 2^{N-1}$$

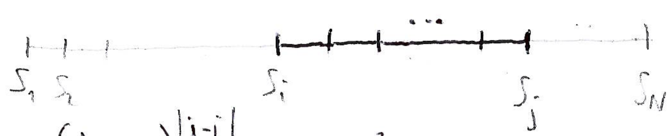
N-1 degrees of freedom now ($S_1 = S_N$)

$$\boxed{Z_{p.b.c.} = \left[(ch\,\beta J)^{N-1} + (sh\,\beta J)^{N-1} \right] 2^{N-1}}$$

Next: correlation function

$$\langle S_i S_j \rangle = \frac{(ch\,\beta J)^{N-1}}{Z} \sum_{\{S_i\}} S_i S_j \prod_{k=1}^{N-1} \left(1 + S_k S_{k+1}\, th\,\beta J \right)$$

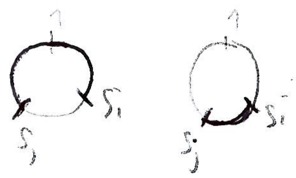
The only graph that contributes is

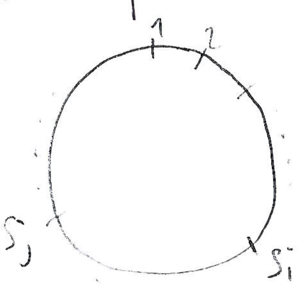


$$\langle S_i S_j \rangle = \frac{(ch\,\beta J)^{N-1}}{Z} \sum_{\{S_i\}} (th\,\beta J)^{|j-i|} \underbrace{S_i^2 S_{i+1}^2 \cdots S_j^2}_{1}$$

$$= \left[th(\beta J) \right]^{|j-i|}$$

What about periodic boundary conditions? Only two graphs contribute





$$\Rightarrow \langle S_i S_{i+k} \rangle = \frac{(th)^k + (th)^{N-k-1}}{1 + (th)^{N-1}}$$