In the vicinity of a fixed point of RG:

$$\tilde{f}(g_1, g_2, \dots) = b^{-d}\, \tilde{f}(g_1 b^{y_1}, g_2 b^{y_2}, \dots)$$

For a magnetic system, where the only relevant scaling fields are t and the magnetic field h:

$$\tilde{f}(t, h, g_3, \dots) = b^{-d}\, \tilde{f}(t\, b^{y_1}, h\, b^{y_2}, g_3\, b^{y_3}, \dots) \qquad (*)$$

holds $\forall\, b > 1$

to be compared to $\tilde{f} \propto |t|^{2-\alpha}$ for $h = 0$

We can take here $|t| b^{y_1} = 1$, ie $b = |t|^{-1/y_1}$

$$\Rightarrow \tilde{f}(t, 0, \dots) \propto |t|^{d/y_1} \qquad \Rightarrow \qquad \frac{d}{y_1} = 2 - \alpha$$

Then, consider magnetization $m = -\dfrac{\partial f}{\partial h}$

$$(*) \Rightarrow m = b^{y_2} b^{-d}\, \psi(t\, b^{y_1}, h\, b^{y_2}, \dots) \quad \text{where } \psi(x, y, \dots) = \frac{\partial \tilde{f}}{\partial y}(x, y, \dots)$$

Taking again $b = |t|^{-1/y_1}$, we see that for $h = 0$.

$$m \propto |t|^{(y_2 - d)/(-y_1)} \qquad \Rightarrow \qquad \beta = \frac{d - y_2}{y_1}$$

Similarly, for $t = 0$ and $h \neq 0$:

$$m = b^{y_2 - d}\, \psi(0, h\, b^{y_2})$$

Taking $h\, b^{y_2} = 1$, we have

$$m \propto h^{\frac{d - y_2}{y_2}} \qquad \Rightarrow \qquad \delta = \frac{y_2}{d - y_2}$$

Turning to the susceptibility:

$$\chi = \left.\frac{\partial m}{\partial h}\right|_t = b^{2y_2}\, b^{-d}\, \varphi(t\, b^{y_1}, h\, b^{y_2}); \quad \varphi(x, y) = \frac{\partial \psi}{\partial y}(x, y, \dots)$$

At $h = 0$, $\chi \propto \left(|t|^{-\frac{1}{y_1}}\right)^{2y_2 - d}$

$$\Rightarrow \qquad \gamma = \frac{2y_2 - d}{y_1}$$

We are thus recovering all the hyper-scaling relations found with more heuristic arguments, such as :

$$
\begin{cases}
\nu d = 2 - \alpha \\
(1+\delta)\beta = \nu d \\
-\gamma = \nu d - 2\beta\delta
\end{cases}
\qquad \text{since} \quad \boxed{\nu = \frac{1}{y_1}}
$$

<u>Behaviour of the correlation function under RG</u>

- Near a fixed point :

$$G(\vec{r}, t, h, g_3, \dots) = \gamma(b)\, G\left(\frac{\vec{r}}{b}, t\, b^{y_1}, h\, b^{y_2}, g_3\, b^{y_3}, \dots\right)$$

The "contrast factor" $\gamma(b)$ can be found by considering $h = 0$, $t = 0$

where $G(\vec{r}) \propto \dfrac{1}{r^{d-2+\eta}} = \gamma(b)\, G\left(\dfrac{\vec{r}}{b}\right) \Rightarrow \gamma(b) = \dfrac{1}{b^{d-2+\eta}}$

For $t = 0$ and $h \neq 0$, this yields an interesting information :

$$G(r, t, 0 \dots) = G\left(\frac{r}{b}, 0, h\, b^{y_2}\right)$$

Taking $h\, b^{y_2} = 1$:

$$G(r, t, 0) = G\left(\frac{r}{h^{-1/y_2}}, 0, 1\right)$$

which is telling us that : $\xi \propto h^{-1/y_2}$

We can check that we recover the same exponent as with pure scaling arguments :

$$\frac{1}{y_2} = \frac{\nu}{\beta\delta} \qquad \text{indeed : } \frac{\nu}{\beta\delta} = \frac{1}{y_1} \frac{1}{\frac{d-y_2}{y_1} \frac{y_2}{d-y_2}} = \frac{1}{y_2}$$

$$\boxed{\xi \propto h^{-\nu/\beta\delta}} \quad \text{for } t = 0$$