J. Phys. A: Math. Gen. **29** (1996) 2529–2551. Printed in the UK

The level splitting distribution in chaos-assisted tunnelling

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Received 17 July 1995, in final form 25 January 1996

Abstract. A compound tunnelling mechanism from one integrable region to another mediated by a delocalized state in an intermediate chaotic region of phase space was recently introduced to explain peculiar features of tunnelling in certain two-dimensional systems. This mechanism is known as chaos-assisted tunnelling. We study its consequences for the distribution of the level splittings and obtain a general analytical form for this distribution under the assumption that chaos assisted tunnelling is the only operative mechanism. We have checked that the analytical form we obtain agrees with splitting distributions calculated numerically for a model system in which chaos-assisted tunnelling is known to be the dominant mechanism. The distribution depends on two parameters. The first gives the scale of the splittings and is related to the magnitude of the classically forbidden processes, the second gives a measure of the efficiency of possible barriers to classical transport which may exist in the chaotic region. If these are weak, this latter parameter is irrelevant; otherwise it sets an energy scale at which the splitting distribution crosses over from one type of behaviour to another. The detailed form of the crossover is also obtained and found to be in good agreement with numerical results for models for chaos-assisted tunnelling.

1. Introduction

Under the denomination of 'quantum chaos', a large body of theoretical and experimental work has been devoted to the study of the specific features of a quantum system which can be traced back to the degree of non-integrability of the underlying classical dynamics [1–3]. An area receiving increasing attention is the subject of tunnelling. In fact, although often considered as a purely quantum phenomenon since it corresponds to classically forbidden processes, it appears amply clear now that tunnelling processes are strongly affected by the nature of the underlying classical dynamics [4–10].

Chaos can be present in systems with d degrees of freedom where d is greater than one (including non-autonomous one degree-of-freedom systems). A consequence of the rich set of dynamical possibilities in multidimensional systems is that quasi-degeneracies may exist analogous to those found in the standard symmetric double well problem even though no potential barrier is present. All that is truly necessary is a discrete symmetry. Indeed, unless the dynamics is entirely ergodic, some classical trajectories are trapped on d-dimensional invariant manifolds (invariant tori) inside the $2d$-dimensional phase space. If there is a discrete symmetry in a given system, say parity, then any torus which is not itself invariant under this symmetry operation will have at least one exact, distinct replica elsewhere in phase space. Semiclassical EBK quantization can be applied to these symmetrical tori with

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0305-4470/96/102529+23$19.50

the result that to this approximation these energy levels have degeneracies. It is important to note that this effect is distinct from ordinary symmetry induced eigenvalue degeneracy. The quasi-degeneracies are not exact because, generally, tunnelling effects break the degeneracy. This has been dubbed 'dynamical tunnelling' by Davis and Heller [11].

Rephrasing this in terms of wavefunctions dynamics makes it clearer why the term 'tunnelling' is appropriate even if there is no barrier. It is possible to construct what Arnold has termed *quasi-modes* [12] if there exist invariant tori fulfilling the EBK quantization conditions [13]. These are wave-functions semiclassically constructed on a single torus, and *they satisfy the Schrödinger equation to any order in* $\hbar$. Here, similarly to what happens when a potential barrier actually exists (e.g., a one-d conservative system), the actual eigenstates are not approximated by a single quasi-mode, but rather by a linear combination of quasi-modes constructed on symmetry related tori. Therefore, if one allows a quasi-mode constructed on one of the tori to evolve for a very long time it will eventually evolve into its symmetric partner, whereas classical trajectories remain indefinitely trapped on a single torus.

Our interest in higher-dimensional systems lies, however, mainly in the possibility of considering dynamics of a different nature. Even for integrable systems, no general theory of tunnelling in multidimensional systems is presently available. However, some theoretical and numerical studies [14, 15] clearly demonstrate that the tunnelling mechanism in this case is rather similar to what is observed for one-d systems. In particular the splitting between two quasi-degenerate doublets has a smooth exponential dependence in $\hbar$.

On the contrary, if chaotic and regular motion co-exist in the dynamics, as will generically be the case for low-dimensional Hamiltonian systems, the tunnelling between two symmetry related invariant tori separated by a significant chaotic region can lead to new behaviour which is quite different from the integrable dynamics case. From a now growing body of numerical work, either on one-dimensional time-dependent systems [4–6] or on two-dimensional conservative systems [7, 8, 10], it has become clear that the presence of chaos may be associated with certain qualitative features, namely: (i) great enhancement of the average splitting; (ii) extreme sensitivity to the variation of an external parameter; and (iii) strong dependence of the tunnelling properties on what is going on in the chaotic region separating the two tunnelling tori. A particularly striking piece of evidence of (iii) is given in [8] where it is observed that reducing drastically the classical transport in the chaotic sea from the neighbourhood of one torus to the one of its symmetric partners noticeably reduces the tunnelling rates.

In [10], it has been argued that a natural explanation of this unusual tunnelling behaviour is obtained through the assumption that in such systems tunnelling can take place in two fundamentally different ways, 'directly' or 'chaos-assisted'. In direct tunnelling, splittings are caused by the overlap of the semiclassical functions constructed on each symmetry related torus via the EBK scheme (the 'quasi-modes'). As discussed above, these quasi-modes are not eigenstates, but satisfy the Schrödinger equation to arbitrary order in $\hbar$. Nevertheless, they can be connected to other states through 'tunnelling' matrix elements which are exponentially small in $\hbar$. (See section II.A of [10] for an explicit example.) In contrast, in the 'chaos-assisted' regime, the picture is that tunnelling from one region to its symmetrical counterpart is mainly mediated by intermediate states associated with the chaotic region. This means that the tunnelling is dominated by the matrix elements between the quasi-modes and states semiclassically localized in the chaotic region, rather than by the matrix element directly connecting the two quasi-modes. This process involves coupling the regular states to the set of delocalized intermediate states in the chaotic sea and two tunnelling processes. The reason why it may nevertheless dominate is that the chaotic region may lie much 'closer' in phase space than the symmetrical regular region. The quasi-mode wavefunction overlap with a delocalized intermediate state can be much larger. Once in the chaotic region, there is nothing to prevent a particle from reaching a symmetrical regular region. In this process, not only do the two semiclassical states play a role, but the various delocalized chaotic eigenstates which might couple to them also become relevant. For this reason, the tunnelling amplitudes have a remarkable dependence on $\hbar$. On the one hand, they decay exponentially fast with $\hbar$, reflecting the smooth variation of the tunnelling amplitude from the integrable region to the chaotic sea. On the other hand, superposed on this smooth variation there is an extremely irregular fluctuation of the splittings due to the violent variation in the strength of the coupling to the close-lying chaotic states. This depends very sensitively on the smallness of the energy denominator, i.e. on whether or not a chaotic level lies close to the tunnelling doublet. In fact, these fluctuations can be so strong as to make any realistic assessment of the $\hbar$ dependence of the tunnelling amplitude impossible.

The complete picture of multidimensional tunnelling is much more difficult to treat. It was observed on the kicked Harper model by Roncaglia *et al* [16] that direct tunnelling can be dominant even in the presence of a significant chaotic region. Further, it is *a priori* possible to encounter more complicated dynamics. For instance, both direct and indirect tunnelling may be of the same importance or the dominant tunnelling mechanism may involve a larger number of classically forbidden processes. However, since cases have been shown to exist in which chaos-assisted tunnelling in the form described above dominates, we confine ourselves in the following to the simpler scenario described above.

That the chaos-assisted mechanism is actually the one taking place for tunnelling in the presence of chaos cannot at the present time be derived from the basic quantum mechanical law of evolution. The numerical evidence, as well as a far more careful and detailed discussion of the process, are given in [10]. It is worth stressing that one of the most important implications of the chaos-assisted mechanism described above is that it allows for a modelling of the splitting distribution in terms of ensembles of random matrices, as discussed in more detail in [10]. Therefore not only a qualitative interpretation of the numerically observed tunnelling behaviour is obtained, but also the statistical properties of the fluctuation effects can be *quantitatively* predicted theoretically. Comparison with groups of quasi-degeneracy splittings in quantum spectra is surprisingly accurate.

The purpose of this paper is the study of the resulting matrix ensembles. Therefore let us be more specific about the term 'splitting distribution' and how the theoretical predictions have been obtained in [10]. As mentioned above, a characteristic feature of tunnelling in the presence of chaos is the extreme sensitivity of quasi-degeneracy splittings to the variation of an external parameter. Within the 'chaos-assisted' interpretation, this is quite natural since the splitting of a given doublet may vary by orders of magnitude depending on whether a chaotic state is close to the tunnelling doublet or not. This implies that very small changes of external parameters, which leave the classical dynamics almost unaltered, may drastically change the splitting by shifting chaotic levels a distance of a few mean spacings. In any experimental setting the statistical behaviour of the splittings is likely to be of great relevance, even if one focuses on one single well defined doublet. In this case, the physically relevant quantity will be the distribution of splittings for an ensemble obtained by varying an external parameter over a range which is negligible on the classical scale, but still large on the quantum scale (i.e. many chaotic levels pass by the neighbourhood of the quasi-degenerate level). By analogy with the random matrix ensembles describing the spectral fluctuations of classically chaotic systems† (see e.g. [17, 18]), or to their generalization introduced to describe partly chaotic systems (see section 5 of [7]), one can build random matrix ensembles modelling the tunnelling distributions corresponding to a given phase space structure.

We shall not repeat here in detail the prescription given in [10] for constructing the random matrix ensemble relevant to a given classical configuration. A few examples of such ensembles are given below. Some points, though, should be stressed. First, we are not considering fully ergodic systems otherwise there would be no invariant tori. Hence the chaotic part of phase space cannot *a priori* be considered as structureless. A whole set of partial barriers should quite generally be present, preventing the motion in the chaotic region from being completely random. In order to quantify the efficiency of these partial barriers, additional time scales must be introduced apart from the mean Lyapunov exponent. As a consequence, the random matrix ensembles associated to the chaotic part of the phase space cannot be taken as structureless either, as demonstrated in [7]. One has to introduce 'transition ensembles' entirely specified by a set of 'transport parameters' $\Lambda_1, \Lambda_2, \ldots$. These transport parameters are fixed by the classical dynamics and are therefore not adjustable model parameters. In the chaos-assisted regime, the parameters which determine the tunnelling distribution are of two kinds:

(i) the variance v_t^2 of the tunnelling matrix elements; it describes the classically forbidden part of the tunnelling process and at the present time there exists no general theoretical way to evaluate it (see, however, [19]);

(ii) the set of 'transport parameters' $\Lambda_1, \Lambda_2, \ldots$, which are fixed by the classical dynamics inside the chaotic region.

A second point we would like to stress is that it is not necessary to solve analytically the so constructed random matrix ensembles to obtain a 'theoretical prediction' for the splitting distributions. The distributions are entirely specified by the random matrix ensemble and can be obtained concretely by performing a rather straightforward Monte Carlo calculation: i.e. taking at random a large number of matrices with the distribution specified by the ensemble, diagonalizing them numerically to obtain the splitting and constructing in this way a histogram of the splitting distribution. This was the procedure used in [10] to compare splitting distributions of doublets of a system of coupled quartic oscillators with those predicted by the proper random matrix ensemble.

Monte Carlo simulations, however, shed little light on the general features of the distribution. This is quite important in this context because, although all parameters but the tunnelling amplitude v_t can in principle be computed by studying the classical motion in the chaotic region, their practical evaluation requires a great deal of effort in the simplest situations and could turn out to be impossible for sufficiently complicated classical structures. Moreover, in experimental realizations, the Hamiltonian governing the dynamics may not be known in enough detail to allow fixing of the parameters of the ensemble with sufficient confidence.

It is therefore worthwhile to gain some further understanding of the splitting distribution determined by the ensembles of random matrices constructed in [10] and to obtain explicit expressions for these distributions. As we shall see, they can in fact be expressed in rather simple form. The resulting distributions have some very general features, which can be used as *the fingerprint of chaos assisted tunnelling* even when the precise structure of the chaotic motion is unknown. Because of the relative complexity of the derivation, we have

† These were originally introduced in nuclear physics without any intention of discussing the nature of the underlying classical dynamics.

chosen to organize this paper as follows. In section 2, we give the final result without any justifications and show how well our analytic findings compare with actual splitting distributions obtained in [10] for a system of coupled quartic oscillators. The remainder of the paper will be devoted to the derivation of this result. Section 3 will deal with the simpler case where the chaotic phase space can be taken as structureless. In section 4 we shall derive the splitting distributions in the case where effective partial barriers are present and discuss in more details the hypothesis and approximation used in the derivation. Section 5 will be devoted to some concluding remarks.

2. The splitting distribution

We are interested in a system possessing a discrete symmetry P and for which tunnelling takes place between two quasimodes Ψ_R^1 and Ψ_R^2 constructed on symmetrical invariant tori $\mathcal{T}_1$ and $\mathcal{T}_2 = P(\mathcal{T}_1)$. The eigenstates belong to a given symmetry class $+$ or $-$ depending on whether they are symmetric or antisymmetric under the action of P. We denote by Ψ_R^+ and Ψ_R^- the symmetric and antisymmetric combinations of the quasi-modes Ψ_R^1 and Ψ_R^2. If one neglects the direct coupling between the quasimodes, Ψ_R^+ and Ψ_R^- have the same mean energy E_R. The 'chaos-assisted' mechanism proposed in [7] assumes that the tunnelling from $\mathcal{T}_1$ to $\mathcal{T}_2$ originates from the (exponentially small) coupling between Ψ_R^+ (respectively Ψ_R^-) and chaotic states of same symmetry $|n,+\rangle$ $(n = 1, 2, \ldots)$ (respectively $|n,-\rangle$) semiclassically localized in the chaotic region surrounding the islands of stability containing $\mathcal{T}_1$ and $\mathcal{T}_2$. Therefore, in a basis where the chaotic part of the Hamiltonian is diagonal, the $+$ and $-$ sectors appear respectively as

$$\mathsf{H}^+ = \begin{pmatrix} E_r & v_1^+ & v_2^+ & \cdot & \cdot & \cdot \\ v_1^+ & E_1^+ & 0 & 0 & & \\ v_2^+ & 0 & E_2^+ & 0 & & \\ \cdot & 0 & 0 & \cdot & & \\ \cdot & & & & \cdot & \\ \cdot & & & & & \cdot \end{pmatrix} \qquad \mathsf{H}^- = \begin{pmatrix} E_r & v_1^- & v_2^- & \cdot & \cdot & \cdot \\ v_1^- & E_1^- & 0 & 0 & & \\ v_2^- & 0 & E_2^- & 0 & & \\ \cdot & 0 & 0 & \cdot & & \\ \cdot & & & & \cdot & \\ \cdot & & & & & \cdot \end{pmatrix}. \qquad (2.1)$$

After diagonalization of H^+ and H^-, the regular levels will be shifted from an amount δ^+ and δ^-, respectively, giving the splitting

$$\delta = |\delta^+ - \delta^-|. \qquad (2.2)$$

Equation (2.1) merely summarizes the semiclassical picture one has of the chaos-assisted tunnelling, but no random matrix modelling has been introduced yet. This modelling follows by assuming that the statistical properties of the tunnelling are correctly reproduced if H^+ and H^- are replaced by an ensemble of matrices with some specified density.

As discussed in detail in [10], the natural choice for the tunnelling matrix elements $v_n^\pm$ describing the classically forbidden process is to take them as independent Gaussian variables with the same variance v_t^2 and, in the absence of any partial barrier, to use for the chaotic levels $E_1^+, E_2^+, \ldots, E_1^-, E_2^-, \ldots$ the classical ensembles of Wigner and Dyson which are known to model properly the spectral statistics of completely chaotic systems [20]. If time reversal invariance symmetry holds, as we shall assume in the following, this means that one should take $E_1^+, E_2^+, \ldots$ and $E_1^-, E_2^-, \ldots$ as two independent sequences, with a distribution given by the Gaussian orthogonal ensemble (GOE). Symbolically this ensemble is denoted by

$$\mathsf{H}^+ = \begin{pmatrix} E_R & \{v\} \\ \{v\} & (\mathrm{GOE})^+ \end{pmatrix} \qquad \mathsf{H}^- = \begin{pmatrix} E_R & \{v\} \\ \{v\} & (\mathrm{GOE})^- \end{pmatrix} \qquad (2.3)$$

where the subscripts + and − emphasize the independent nature of the distribution. In that case, we shall see in section 3 that the splitting distribution $p(\delta)$ is merely a truncated Cauchy law

$$p(\delta) = \begin{cases} \dfrac{4v_{\mathrm{t}}}{\delta^2 + 4\pi v_{\mathrm{t}}^2} & \delta < v_{\mathrm{t}} \\ 0 & \delta > v_{\mathrm{t}}. \end{cases} \tag{2.4}$$

As demonstrated in [7], such a simple statistical modelling of the chaotic states does not apply any more when structures, such as partial barriers, prevent classical trajectories to flow freely from one part of the chaotic phase space to another. In such classical configurations (which are presumably generic in systems where chaos and regularity coexist), transition ensembles have to be introduced to obtain a correct statistical description of chaotic states. Compared to the case where no barriers are present, the distribution of chaotic states will be modified in two ways. First, each of the parity sequences, taken separately, will usually not be distributed as a GOE anymore. However, as stressed in [10], and as will be made explicit in section 3, this has a negligible influence on the splitting distribution. More important is that such barriers may induce strong correlations between the two parity sequences of chaotic states. Consider a simple example for which a strong partial barrier separates the chaotic sea into two parts $\mathcal{R}_1$ and $\mathcal{R}_2$ which are symmetric images of each other under P. In this case, the relevant matrix ensemble can be symbolically written as [10]

$$\mathsf{H}^{\pm} = \begin{pmatrix} E_R & \{v\} \\ \{v\} & (\mathrm{GOE})_S \pm (\mathrm{GOE})_A(\Lambda) \end{pmatrix} \tag{2.5}$$

where the variance of the matrix elements of $(\mathrm{GOE})_S$ is chosen such that it has (in the neighbourhood of E_R) the same mean spacing D as the chaotic states, and the variance σ^2 of the matrix elements of $(\mathrm{GOE})_A(\Lambda)$ is fixed by the transport parameter Λ through

$$\frac{\sigma^2}{D^2} \equiv \Lambda. \tag{2.6}$$

The transport parameter is in turn semiclassically related to the classical flux Φ crossing the partial barriers by [10]

$$\Lambda = \frac{1}{4\pi^2} \frac{g\Phi}{(2\pi\hbar)^{d-1} f_1 f_2} \tag{2.7}$$

where $g = 1/2$ is the proportion of states in the corresponding symmetry class, $f_1 = f_2 = 1/2$ the relative phase space volume of region 1 and 2 and d the number of freedoms.

For very ineffective barriers, Λ is much larger than 1. $(\mathrm{GOE})_S + (\mathrm{GOE})_A(\Lambda)$ and $(\mathrm{GOE})_S - (\mathrm{GOE})_A(\Lambda)$ are two essentially independent ensembles and one recovers the truncated Cauchy law equation (2.4) for the splitting distribution. In the opposite extreme, a perfect barrier gives $\Lambda = 0$ (except for classically forbidden processes), and the + and − spectra are strictly identical. In particular, the displacements δ^+ and δ^- being the same give a null splitting. For small, but finite Λ, a typical level E_n^- will be usually found close to its symmetric analogue E_n^+, though slightly shifted. In this case, although the displacements δ^+ and δ^- are still Cauchy distributed, they are strongly correlated. This not only changes the scale of δ, but also its distribution.

It is useful to consider a qualitative description of the meaning of Λ. It can be thought of as the ratio $t_{\mathrm{H}}/t_{\mathrm{c}}$ of two time scales, one of which is purely classical, whereas the other is quantum mechanical: t_{c} is the time necessary to cross the barrier, that is, the typical time that a classical trajectory needs in order to go from one part of the chaotic sea to the other. The second time scale t_{H} is the Heisenberg time $\hbar/D$, where D is the average level spacing in the chaotic sea. As $\hbar \to 0$, of course, one expects Λ to go to infinity, as the Heisenberg time becomes classically infinite. Nevertheless, there may well be a very significant regime where the quantum dynamics is well described semiclassically and in which Λ is very small. In such cases, there exists a classical time scale comparable or larger than the Heisenberg time. This time scale is related to the time necessary to explore all of the available phase space. In this respect the situation is quite reminiscent of what happens in localization. The difference, of course, is that we only have a small number of weakly connected phase space regions and the long classical time has nothing to do with diffusion.

Typically, the ensemble describing the classical structure of a system will be more complicated that the simple one given in equation (2.5). It is probable that transport from one regular island to its symmetric counterpart is affected through the joint effect of a whole set of moderately effective barriers rather than by the strong action of a single one. Thus one would expect to have much more structured ensembles, with a transport parameter Λ_n associated with each barrier. We shall see in section 4 that, when this is the case, the problem can still be handled much as above because all the information encoded in the transition ensemble (that is, essentially, the Λ_n's) can be summarized in a single parameter α, which is a weighted average of the variance of the $(E_n^+ - E_n^-)$.

The splitting distribution $p(\delta)$ therefore depends on three parameters: $(v_{\mathrm{t}})^2$, the variance of the tunnelling matrix elements, α^2 which measures the degree of correlation between the odd and even levels and D, the mean spacing of the chaotic levels to which the $\Psi_R^{\pm}$ are connected. If one consider effective barriers, α is smaller than D. Moreover, v_{t} being related to classically forbidden processes is usually extremely small and in particular much smaller than α. We shall therefore assume below $v_{\mathrm{t}} \ll \alpha < D$. Then, the main result of this paper is that, for this parameter range, the splitting distribution is given by

- for $v_{\mathrm{t}} < \delta$

$$p(\delta) = 0 \tag{2.8}$$

- for $v_{\mathrm{t}}^2/\alpha < \delta < v_{\mathrm{t}}$

$$p(\delta) = \frac{4v_{\mathrm{t}}}{\delta^2 + 4\pi v_{\mathrm{t}}^2} \qquad \text{(Cauchy)} \tag{2.9}$$

- for $\delta < v_{\mathrm{t}}^2/\alpha$

$$p(\delta) = 2\mu^{-1} G\left(\frac{\delta}{\mu}\right) \tag{2.10}$$

where the function G is the inverse Fourier transform of $\exp(-\sqrt{|q|})$, namely

$$G(x) \equiv \frac{1}{2\pi} \int \exp(\mathrm{i}qx) \exp(-\sqrt{q})\, \mathrm{d}q \tag{2.11}$$

and

$$\mu = \frac{\sqrt{32}\Gamma^2(3/4)}{\pi} \frac{\alpha v_{\mathrm{t}}^2}{D^2}. \tag{2.12}$$

As expected (see section IV.B. of [10]), only the smaller splittings are affected by the transport limitation, the distribution for larger splittings being unaffected. The asymptotic behaviour of the $\mu^{-1}G(\delta/\mu)$ is given by

$$p(0) = 2\mu^{-1}G(0) = \frac{1}{\sqrt{2}\Gamma^2(3/4)} \frac{D^2}{\alpha v_{\mathrm{t}}^2} \qquad \text{in } \delta = 0 \tag{2.13}$$

$$\mu^{-1}G(\delta/\mu) \simeq \frac{\Gamma(3/4)}{2^{1/4}\pi} \frac{\sqrt{\alpha}v_{\mathrm{t}}}{D} \frac{1}{\delta^{3/2}} \qquad \text{for } \delta \gg \alpha v_{\mathrm{t}}^2/D^2. \tag{2.14}$$

Therefore, for small enough α/D and v_t/D, the distribution $p(\delta)$ will, in a log–log plot, essentially consist of three straight lines: (i) a horizontal one (at $p(0)$, as given by equation (2.13)), for $0 < \delta < \alpha v_t^2/D^2$; (ii) a line of slope $(-3/2)$ in the range $\alpha v_t^2/D^2 < \delta < v_t^2/\alpha$; (iii) a slope (-2) characteristic of the Cauchy distribution for the range $v_t^2/\alpha < \delta < v_t$, after which the distribution abruptly falls to zero.

We digress briefly to give an intuitive picture of what is happening. First assume that there are no barriers. Then chaos-assisted tunnelling is essentially a compound process involving the two symmetrical states and those chaotic states which lie nearby. Fast tunnelling (large splittings) occur only if at least one of these states actually lies very near to the quasi-degenerate tunnelling state. This then yields a tunnelling process mediated by one single delocalized chaotic state. This process has a characteristic δ^{-2} distribution, as we shall show later. Here it is essential to realize that the chaotic state involved, since it has a well defined symmetry, will always directly couple from one torus to its symmetrical partner. On the other hand, if an efficient barrier exists, the chaotic states also come in quasi-degenerate doublets of opposite parity and of characteristic splitting α. Therefore, moderately fast processes will be mediated by such a doublet rather than by a single state. Again, we shall show that this leads to a universal behaviour of $\delta^{-3/2}$ as long as the doublet is identifiable as such. That is as long as the two energy denominators contribute by a roughly equal amount. However, for very fast processes, the tunnelling will again be mediated by a single state, namely the one nearest to the tunnelling doublet, and the δ^{-2} behaviour is again obtained. The details are given by the above formulae which also show a considerable amount of information for intermediate cases which cannot be derived in such a simple fashion.

Before going to the calculation of the above distribution, let us see how it compares to actual distributions of splittings obtained numerically for a system of two coupled quartic oscillators governed by an Hamiltonian of the form

$$H(\boldsymbol{p}, \boldsymbol{q}) = \frac{p_1^2 + p_2^2}{2m} + a(q_1^4/b + bq_2^4 + 2\lambda q_1^2 q_2^2). \tag{2.15}$$

Except for their presentation, we use here a log–log plot instead of a linear against log binning, the data used in figure 1 are exactly the same as those used in figure 13 of [10]; see this reference for more information on the system investigated. Here, we shall only say that each set of data has been obtained by numerically calculating the splittings between regular states constructed on a given identified pair of symmetrical invariant tori, for various values of the coupling λ†. The range of variation of λ is small on the classical scale (the classical dynamics remains essentially the same), but sufficiently large on the quantum scale that a good statistical significance is reached.

In figure 1 we display the comparison between the quartic oscillators data and the predicted form of the distribution equations (2.8)–(2.10) for two splitting distributions associated with two different pairs of symmetric invariant tori. The agreement is extremely good, especially if one considers that the distribution extends over more than six decades. Here a remark is in order. The parameters α and v_t used for the analytical curves in figure 1 are actually tunable parameters. This was already the case for v_t for the ensemble introduced in [10], since there is not yet any semiclassical theory allowing for the calculation of the matrix elements associated with such classically forbidden processes. Here, however, one has another tunable parameter α. In principle, this parameter is fixed once the the random matrix ensemble describing the statistical properties of the chaotic level is known, which is

† In practice, to increase the statistical significance of the distribution, data coming from close-lying tori have been combined.

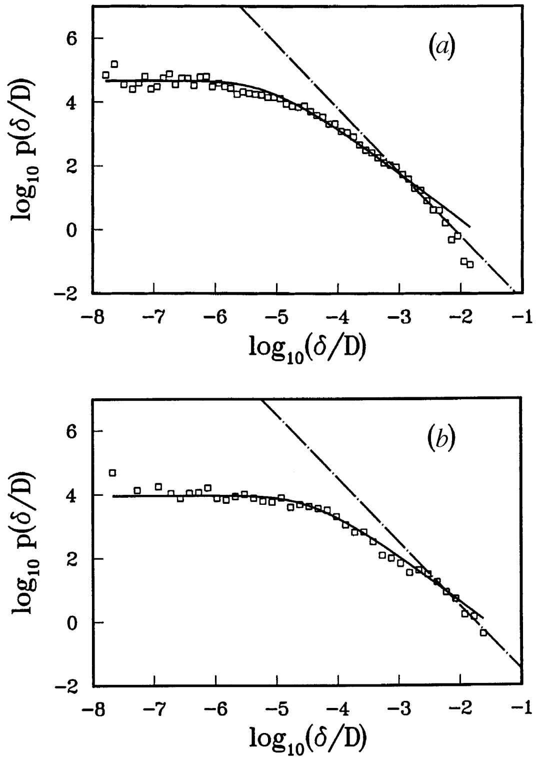

Figure 1. Comparison between the quartic oscillator's tunnelling splitting distribution (square symbols) and the predicted form equations (2.8)–(2.10) for two different groups of tunnelling tori. Except for their presentation, the quartic oscillator's data are the same as those in figure 13 of [10]. The transition from the G-like behaviour (full curve) to Cauchy-like behaviour (chained dot curve) is clearly seen, in spite of this latter being valid on a much shorter range: (*a*) group T_0 (using the notation of [10]), with $v_t = 1.1 \times 10^{-2}$; (*b*) group T_1 , with $v_t = 2.5 \times 10^{-2}$. The transport parameter has the same value $\alpha = 0.04$ in both cases, consistent with the fact that the partial barriers structure is the same in both cases. It has been taken into account that only a fraction $D_{\rm eff} = 0.36D$ of the states are actually participating to the tunnelling process.

the case for this particular system. In practice, however, there is usually no way to relate α analytically to, say, the set of transport parameters Λ_n's. Therefore α eventually plays the role of a tunable parameter. It should be borne in mind, however, that α and v_t only fix the scale of the distribution and, in particular, the place of the crossover from Cauchy to the G-like behaviour, but not its shape. Therefore, despite the presence of two parameters, the fact that the splitting distribution in figure 1 actually follows the prediction of equations (2.8)–(2.10) is a very stringent test of the relevance of the whole 'chaos-assisted' picture. After this attempt to put the results in perspective, we turn to their derivation.

3. The case without barriers

We now want to compute the distribution of the splittings in the case in which no barriers are present. Under these circumstances, it is sufficient to compute the distribution of δ_+, and hence δ_-, since these splittings are statistically independent.

To begin with, in the matrices $\mathbf{H}^+$ and $\mathbf{H}^-$ of equation (2.1), the matrix elements $v_i^\pm$

are associated with classically forbidden processes and are thus extremely small. Therefore, one can compute the displacement $\delta^{\pm}$ using a first-order perturbation result. One must take special care with the rare, but important case, where a chaotic level approaches E_R closely. This can be done using the exact two-by-two diagonalization result for each chaotic eigenstate and adding up the contributions. This gives

$$\delta^{\pm} = \frac{1}{2}\sum_{i=1}^{N}(E_R - E_i^{\pm})\left(1 - \sqrt{1 + \left(\frac{2v_i^{\pm}}{E_R - E_i^{\pm}}\right)^2}\right) \tag{3.1}$$

(to be understood in the $N \to \infty$ limit). In the absence of an $E_i^{\pm}$ near E_R, the above equation is equivalent to the usual perturbative result,

$$\delta^{\pm} \approx \sum_{i=1}^{N}\frac{|v_i|^2}{E_R - E_i^{\pm}} \tag{3.2}$$

but the full expression equation (3.1) has to be used to regularize it whenever any of the $(E^{\pm} - E_R)$ is of the order of v_{t}.

Although we shall give below a more detailed discussion ahead, the basic way we are going to use equation (3.1) is that the regularized form of $\delta^{\pm}(E_i^{\pm}, v_i^{\pm})$ prevents any splitting from being significantly larger than v_{t} and that, for $\delta^{\pm}$ smaller than v_{t}, equation (3.2) can be used safely. To clarify the discussion, we shall for the moment replace the $(v_i^{\pm})^2$ by their average value v_{t}^2, and justify below why this does not change the result. Without loss of generality we also set E_R equal to zero (E_R is not correlated to the chaotic spectrum, so it can be used as the origin of the energies). We shall in addition consider the normalized regular level shift x and energy level $e_1, e_2, \ldots$ (we drop the superscript $+$ or $-$ for the normalized quantities)

$$x = \frac{\delta^{\pm} D}{v_{\mathrm{t}}^2} \qquad e_i = \frac{E_i^{\pm}}{D}. \tag{3.3}$$

With these manipulations, $p(x)$ is in principle obtained for $x < D/v_{\mathrm{t}}$ (i.e. $\delta^{\pm} < v_{\mathrm{t}}$) as the integral

$$p(x) = \int \delta\left(x - \sum_{i=1}^{N}\frac{1}{e_i}\right)P(e_1, e_2, \ldots, e_N)\,\mathrm{d}e_1\,\mathrm{d}e_2 \ldots \mathrm{d}e_N \tag{3.4}$$

where $P(e_1, \ldots, e_N)$ is the joint probability of a GOE spectra with mean density equal to one in the centre of the semicircle. It happens, however, that the correlations of the chaotic states have no influence on $p(x)$, because the physics here is determined by the singular nature of the energy denominator which is not expected to be very sensitive to many-particle correlations.

To demonstrate this point, let us consider, for instance, the integral equation (3.4) except that we take for the chaotic states a Poisson distribution, i.e. that we neglect any correlations between them. In this case, introducing $\xi_i = Ne_i$, one can write

$$x = \frac{1}{N}\sum_{i=1}^{N}\xi_i^{-1} \tag{3.5}$$

that is the random variable x is the average of the ξ_i^{-1}, where the ξ_i's are independent variables with density of probability one at the origin. If the distribution of $1/\xi_i$ had a second moment (i.e. $\langle\xi^{-2}\rangle < \infty$), the usual central limit theorem would yield a Gaussian distribution for $p(x)$. Here the situation is quite different as this variance diverges. The distribution $p_0(y)$ of the $y_i = \xi_i^{-1}$ behaves as y^{-2} for $y \gg 1$ whatever the initial distribution of the ξ_i as long as that distribution is equal to one for ξ equal to zero†. From this follows through standard probabilistic arguments [21] that x has a non-singular limiting distribution, namely the Cauchy law

$$p(x) = \frac{1}{\pi^2 + x^2}. \tag{3.6}$$

Informally, this result can be obtained as follows. If the chaotic states are distributed independently, equation (3.4) reads

$$p(x) = \int_{-\infty}^{\infty} \prod_{i=1}^{N} \mathrm{d}y_i \; p_0(y_i) \delta\left(x - \frac{1}{N} \sum_{i=1}^{N} y_i\right). \tag{3.7}$$

If we now take the Fourier transform of equation (3.7), the result factorizes and one obtains

$$\hat{p}(q) \equiv \int_{-\infty}^{\infty} p(x) \mathrm{e}^{\mathrm{i}qx} \, \mathrm{d}x = \hat{p}_0(q/N)^N \tag{3.8}$$

where $\hat{p}_0(q)$ is the Fourier transform of $p_0(y)$. The large-y behaviour of the latter leads to a singularity of $\hat{p}_0(q)$, which, by reason of the symmetry of $p_0(y)$, must be located at the origin. Further, this singularity is of the type of a discontinuous derivative. For any symmetric function $f(y)$ with the same large y behaviour as $p_0(y)$, $f(y) - p_0(y)$ decreases more rapidly than y^{-2}. This implies that $\hat{f}(q) - \hat{p}_0(q)$ has a continuous derivative. The jump in $q = 0$ of the derivative of $\hat{p}_0(q)$ must therefore be the same as for $f(y) = (1 + x^2)^{-1}$, the Fourier transform of which is $\exp(-\pi|q|)$. Noting that because of the normalization, $\hat{p}_0(0) = 1$, one has

$$\hat{p}_0(q) = 1 - \pi|q| + o(q) \qquad (q \ll 1) \tag{3.9}$$

and therefore

$$\hat{p}(q) = \lim_{N \to \infty} \left(1 - \frac{\pi|q|}{N}\right)^N = \exp(-\pi|q|) \tag{3.10}$$

from which the result follows immediately by inverse Fourier transformation.

At the opposite extreme, one can consider the most rigid spectrum and see what happens if the chaotic states are distributed uniformly. In that case, $p(x)$ can be written as

$$p(x) = \int_{-1/2}^{+1/2} \mathrm{d}e \, \delta\left(x - \sum_{n=-\infty}^{+\infty} \frac{-1}{n+e}\right) \tag{3.11}$$

which, using the equality [22]

$$\mathrm{cotg}(\pi x) = \frac{1}{x} + \frac{2x}{\pi} \sum_{k=1}^{\infty} \frac{1}{x^2 - k^2}$$

readily gives

$$p(x) = \int_{-1/2}^{+1/2} \mathrm{d}E \, \delta(x - \pi \, \mathrm{cotg}(\pi E)) = \frac{1}{x^2 + \pi^2} \tag{3.12}$$

that is the very same Cauchy distribution as for the Poissonian case. There is no doubt that if the two extremes of totally uncorrelated and maximally correlated spectra give the same result, the correlation between chaotic states plays no role. Therefore, as demonstrated in figure 2, the splittings are also Cauchy distributed when the chaotic states are GOE

† Therefore, the result will not be affected by a secular change of the mean density of states away from the origin.

distributed. Indeed, this result can actually be shown using supersymmetric techniques [23] and also turns out to follow from results on S-matrix ensembles for the one-channel case [24] under quite general conditions for both the v_i and the energies E_i. In fact, it turns out that the sums involved in computing the K-matrix in a one-channel system are exactly of the type we are interested in and their distribution can be found exactly under the assumption that the ensemble is ergodic and analytic in the energy. For details see [24, 25].

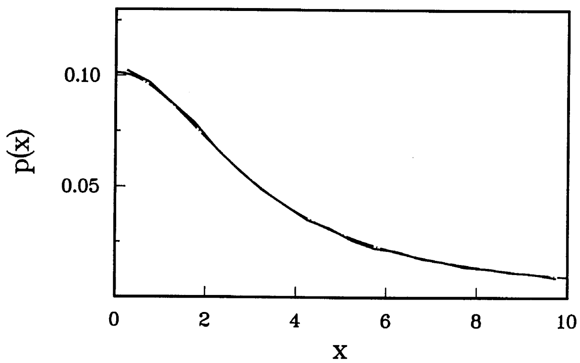

Figure 2. Comparison between a Monte Carlo calculated distribution of the reduced variable $x = \delta^{\pm}(D/v_t^2)$ (full curve) and the Cauchy law equation (3.6) (chained dot curve). The Monte Carlo result is obtained from the numerical diagonalization of 10^5 matrices of size 80×80, which matrix elements are taken at random with the distribution specified by the ensemble equation (2.3) (using $v_t/D = 10^{-4}$). It thus takes fully into account the GOE correlations of the chaotic states. Nevertheless, and although a linear scale has been used to emphasize the centre of the distribution where the effects of correlations should be the strongest, the two curves are essentially indistinguishable.

Before turning to the more difficult case of problems where transport limitations play a role in the tunnelling mechanism, let us come back to a couple of points not treated in the above discussion. Since we have seen that correlations between chaotic states are of little importance, we shall discuss these points under the assumption that there are no such correlations. The first concerns the fact that the tunnelling matrix elements are randomly distributed following a Gaussian law, instead of being constant as assumed in the above discussion. However, it can easily be checked that, in the Poissonian case, this simply amounts to performing first the integral over the tunnelling matrix elements distribution. The second point concerns the need to use the regularized form equation (3.1) instead of its non-degenerate approximation equation (3.2). Let us now see the effect of using this more correct formula which takes quasi-degeneracies fully into account by treating the corresponding 2×2 matrix exactly. In this case the relevant function of ξ_i is equal to

$$y_i = \xi_i \left((C_N)^2 - \sqrt{(C_N)^4 + \left(\frac{2v_i}{\xi_i}\right)^2} \right) \qquad C_N = \frac{D^2 N^2}{v_t^2} \tag{3.13}$$

which is equal to ξ_i^{-1} for $\xi_i \gg C_N^{-1}$ but saturates to a value of $1/v_t$ for smaller values of ξ_i. This implies that $\hat{p}_0(q)$ has a singularity of the type described above only for q less than $v_t/(ND)$. This in turn involves a departure of $\hat{p}(q)$ from pure exponential behaviour when q becomes of the order of v_t/D and hence for normalized splittings of the order of D/v_t, which in unnormalized units correspond to splittings of order v_t. We recover in this way the intuitive picture discussed above, namely that the basic role of the regularization is to prevent splittings of size larger than the root-mean-square deviation of the v_i, whereas the distribution for smaller values remains unaffected. For the distribution $p(x)$ displayed in figure 2, the effect of this regularization cannot be observed because it only affects the range $x \geqslant v_t^{-1} = 10^4$ which is not covered in this linear scale. It is however clearly seen in figure 1, as well as in figures 3 and 4 of the following section.

4. The case of efficient barriers

Let us now consider the more complicated case of systems for which transport limitation induces strong correlation between the symmetry classes. As we have emphasized in the previous section, the correlation between chaotic states inside a symmetry class has little or no influence on the distribution of the shifts, δ^+ and δ^-, of the regular levels, due to their coupling with the chaotic states. Therefore, even in the case where there exist efficient barriers to transport in the chaotic region, the shifts $\delta_\pm$ should be still distributed according to the Cauchy distribution derived in the preceding section. In fact, the main effect of such barriers is to induce strong correlations between the E_i^+ and the E_i^-, which only affect the distribution of the splittings themselves $\delta = |\delta^+ - \delta^-|$. Another consequence of the presence of partial barriers which may also influence the distribution of splittings is that it may yield some inhomogeneity of the variance of the tunnelling matrix elements, as well as of the correlation between chaotic states. We shall come back to this point at the end of the section when comparing our findings with exact results calculated numerically using Monte Carlo techniques.

4.1. Derivation of equations (2.8)–(2.10)

To lighten the notation somewhat, we use in this subsection scaled energies $e_i^\pm = E_i^\pm/D$ (for which the mean density of states around the regular level is therefore one), and note $\bar{v}_t = v_t/D$ and $\bar{\alpha} = \alpha/D$. We use the following modelling of our problem. First, we shall ignore any correlation between chaotic states inside each symmetry class and merely require that the mean density of chaotic states be equal to D around $E_R = 0$. To normalize the number of states to N, we choose the variables e_i^+ and e_i^- both distributed according to a Gaussian law, which we take to be $N^{-1}\mathrm{e}^{-\pi(e_i^\pm/N)^2}$. In this way, the density at zero is $1/N$ for each level (and thus the total density is one). Since we shall consider the $N \to \infty$ limit, this can be thought of as a flat distribution on the scale of a mean level spacing, the Gaussian form being just introduced to normalize in a proper way the distribution. Again, we take the chaotic states to be identically distributed random variables, the e_i^+ are independent of each other for different values of i. The correlations between the two parity sequences of chaotic states must be implemented which we shall do by assuming that the $(e_i^+ - e_i^-)$ have a Gaussian distribution characterized by its width $\bar{\alpha}$. As discussed in section 2, $\bar{\alpha}$ is related to a characteristic time necessary for a classical trajectory to travel from one regular island to its symmetric counterpart. To justify this construction, let us consider for instance the ensemble of equation (2.5) introduced in section 2. In this case, it was seen that the Hamiltonians of the two symmetry sectors are related to one another by adding a GOE matrix, the off-diagonal elements of which have variance $(2\sigma)^2$, related through equations (2.6) and (2.7) to the classical flux Φ crossing the partial barrier. As is well known, the resulting spectrum is formally the same as the result of letting the levels move according to an interacting Brownian motion during a time $\Lambda = \sigma^2/D^2$, as described by Dyson [26]. However, for short times (i.e. small Λ), it is generally accepted that an interacting diffusion process can be replaced by a free one [27], which here means that the

$(E_i^+ - E_i^-)$ follow a Gaussian distribution, of width $\alpha = \sqrt{2}(2\sigma)$. This also follows more specifically from the results shown in [28]. There it was shown that a randomly distributed sequence diffusing over a short time Λ only acquires correlations on a scale of $\sqrt{\Lambda}$. (Note that for $\alpha^2 \gg D^2$, this modelling becomes essentially meaningless. For instance it is not possible anymore to specify unambiguously which $E_{i'}^-$ is to be associated with a given E_i^+ when considering the variance of their difference. This is not of great importance, though, since the final result shows that in this case the truncated Cauchy behaviour described in section 3 is recovered.) Thus the joint probability distribution function of the e_i^+ and the e_i^- is

$$P(e_i^+, e_i^-) = \left(\frac{1}{\sqrt{2\pi}\bar{\alpha}N}\right)^N \prod_{i=1}^N \exp\left(-\pi\left(\frac{e_i^+}{N}\right)^2 + \frac{1}{2\bar{\alpha}^2}(e_i^+ - e_i^-)^2\right). \tag{4.1}$$

Note the asymmetric treatment of e_i^+ and e_i^-. This simply means that, because of the very strong correlation between the two sets of eigenvalues, the large-scale distribution of one set entirely determines that of the other.

For more structured ensembles than the one of equation (2.5), the actual diffusion process, while 'turning on' the transport parameters Λ_n's from zero to their actual value, may be significantly more complicated. Nevertheless, because the final splitting results from the average effect of the coupling with a large number of chaotic states, it is natural to assume that a kind of central limit theorem is involved and that the form equation (4.1) can also be used in practice (we shall discuss this question in greater detail in the next subsection). As mentioned in section 2, however, there will not necessarily be a simple relationship between α and the transport parameters of these ensembles.

Now the problem is to compute the splitting distribution. We shall disregard in the following the complications created by the inclusion of the complete expression equation (3.1) for the shifts, since this only causes a cut-off at values of δ equal to v_t, as was already discussed in the previous section. As a further simplification, we shall take for a moment the tunnelling matrix elements as being constant (equal to v_t) and shall come back later to the (slight) modifications to the result due to averaging over their Gaussian distribution. Introducing the scaled variable $X = (\delta^+ - \delta^-)/D$, $(\delta/D = |X|)$ the distribution of X is obtained as

$$p(X) = \int \prod_{i=1}^N \mathrm{d}e_i^+ \, \mathrm{d}e_i^- \, P(e_i^+, e_i^-)\, \delta\left(X - \bar{v}_t^2 \sum_{i=1}^N \left(\frac{1}{e_i^+} - \frac{1}{e_i^-}\right)\right). \tag{4.2}$$

Again, this integration can be factorized by introducing the Fourier transform $F(q)$ of $p(X)$ and everything can be reduced to quadratures. The details are a trifle tedious and are therefore relegated to appendix A. The final result is

$$p(X) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(q)\mathrm{e}^{-\mathrm{i}qx}\,\mathrm{d}x \tag{4.3}$$

$$F(q) = \exp\left(-\frac{\bar{\alpha}}{\sqrt{2\pi}}\Psi(\tilde{q})\right) \qquad \tilde{q} = \frac{\bar{v}_t^2}{\bar{\alpha}}q. \tag{4.4}$$

Here $\Psi(\tilde{q})$ is given by the expression

$$\Psi(\tilde{q}) = \int_{-\infty}^{\infty} \mathrm{d}y\, \Phi\left(\frac{\sqrt{8}\tilde{q}}{|1-y^2|}\right) \tag{4.5}$$

$$\Phi(z) = 2\int_0^{\infty} \frac{\mathrm{d}t}{t^3}(1-\cos zt)\mathrm{e}^{-1/t^2}. \tag{4.6}$$

For ease of reference we give the following integrals, which are derived for completeness in appendix B:

$$\int_{-\infty}^{\infty} \Phi(1/y)\,\mathrm{d}y = \pi^{3/2} \tag{4.7}$$

$$\int_{-\infty}^{\infty} \Phi(1/y^2)\,\mathrm{d}y = \sqrt{2\pi}\,\Gamma(3/4). \tag{4.8}$$

An asymptotic study of the function $\Phi(z)$ for $z \gg 1$ and $z \ll 1$ yields

$$\Phi(z) = \begin{cases} -z^2 \ln z + \mathrm{O}(z^2) & z \ll 1 \\ +1 + \mathrm{O}(z^{-1}) & z \gg 1 \end{cases} \tag{4.9}$$

(the prefactors given here are actually correct, but we shall not need them, and the order of magnitude is easy to obtain). Therefore, $\Psi(\tilde{q})$ basically gives a measure of the domain of y such that $\tilde{q}/(1-y^2)$ is larger than one. For $\tilde{q} \gg 1$, this is obviously of order $\sqrt{\tilde{q}}$. It can be evaluated more precisely by noting that in this range of $\tilde{q}$, $\Phi(\sqrt{8}\tilde{q}/|1-y^2|) \simeq \Phi(\sqrt{8}\tilde{q}/y^2)$. This is the case, because in the range of y where $|1-y^2|^{-1} \not\simeq y^{-2}$, i.e. for y not large, both $\tilde{q}/|1-y^2|$ and $\tilde{q}/y^2$ are much larger than one if $\tilde{q}$ is and therefore Φ saturates to its asymptotic value one in any case. One therefore finds for $\tilde{q} \gg 1$

$$\Psi(\tilde{q}) \approx \int_{-\infty}^{\infty} \mathrm{d}y\,\Phi\left(\frac{\sqrt{8}\tilde{q}}{y^2}\right) = 2^{3/4}\Gamma(3/4)\sqrt{2\pi|\tilde{q}|}. \tag{4.10}$$

and

$$F(q) \approx F_\infty(q) = \exp\left(-2^{3/4}\Gamma(3/4)\sqrt{\bar{\alpha}\bar{v}_{\mathrm{t}}^2|q|}\right) \qquad \text{for } q \gg \frac{\bar{\alpha}}{\bar{v}_{\mathrm{t}}^2}. \tag{4.11}$$

For $\tilde{q} \ll 1$ on the other hand, $\tilde{q}/(1-y^2)$ is large only in the neighbourhood of $y = 1$. From this it follows that one can restrict oneself to the range of integration $y \sim \pm 1$. One obtains

$$\begin{aligned}\Psi(\tilde{q}) &\approx 2\int_{1-\tilde{q}}^{1+\tilde{q}} \mathrm{d}y\,\Phi\left(\frac{\sqrt{8}\tilde{q}}{(1-y)(1+y)}\right) \\ &\approx 2\int_{-\infty}^{\infty} \mathrm{d}t\,\Phi\left(\frac{\sqrt{2}\tilde{q}}{t}\right) = (2\pi)^{3/2}\tilde{q}\end{aligned}$$

and therefore

$$F(q) \approx F_0(q) = \exp(-2\pi\bar{v}_{\mathrm{t}}^2|q|) \qquad \text{for } q \ll \frac{\bar{\alpha}}{\bar{v}_{\mathrm{t}}^2}. \tag{4.12}$$

It remains to perform the inverse Fourier transform equation (4.4) and to deduce the asymptotic behaviour of $p(X)$ from that of $F(q)$. For large X, $p(X)$ is dominated by the singularities of its Fourier transform, which here means the derivative discontinuity at the origin. Therefore for $\bar{v}_{\mathrm{t}} > X \gg \bar{v}_{\mathrm{t}}^2/\bar{\alpha}$ one can use in equation (4.4) the asymptotic $q \ll \bar{\alpha}/\bar{v}_{\mathrm{t}}^2$ approximation $F_0(q)$ of $F(q)$. Applying the inverse Fourier transformation yields an almost perfect Cauchy distribution of the form

$$p(X) = \frac{2\bar{v}_{\mathrm{t}}}{X^2 + 4\pi^2\bar{v}_{\mathrm{t}}^2}. \tag{4.13}$$

The above result amounts to adding independently the variable δ^+ and δ^-, distributed as given by equation (3.6), and to neglect the correlations between the two symmetry classes. As mentioned in [10], this is indeed quite natural since splittings larger than $v_{\mathrm{t}}^2/\bar{\alpha}$ are due to chaotic levels lying closer than a distance $\bar{\alpha}$ from the regular level. Since $\bar{\alpha}$ can be viewed as the scale on which chaotic levels are correlated, chaotic states contributing to $p(X)$ for $X \gg \bar{v}_t^2/\bar{\alpha}$ can therefore be considered as essentially decorrelated from their symmetric counterparts. In the language of section 2, the splittings we are looking at here are so large that the tunnelling is always mediated by a single state.

Consider now the range $X \ll \bar{v}_t^2/\bar{\alpha}$, for which the splitting distribution is affected by correlation between symmetry classes. In that case, the term $\exp(-\mathrm{i}qX)$ in the inverse Fourier transformation equation (4.3) is essentially constant in all the range $q \leqslant \alpha/v_t^2$ where $F(q)$ differs from its asymptotic behaviour $F_\infty(q)$. Noting that

$$\frac{1}{2\pi}\left(\int_{-\infty}^{\infty} F_\infty(q)\exp(-\mathrm{i}qx)\,\mathrm{d}q\right) = \lambda^{-1}G(X/\lambda) \qquad \lambda = \sqrt{8}\Gamma^2(3/4)\bar{\alpha}\bar{v}_t^2 \tag{4.14}$$

where $G(x)$ is the inverse Fourier transform of $\exp(-\sqrt{q})$ as defined in equation (2.11), one therefore has

$$p(X) = \lambda^{-1}G(X/\lambda) + K \tag{4.15}$$

where K is the constant

$$K \equiv \frac{1}{2\pi}\int_{-\infty}^{\infty}\mathrm{d}q\,(F(q) - F_\infty(q)). \tag{4.16}$$

For small $\bar{\alpha}$, however, K is of order $\bar{\alpha}^2/\bar{v}_t^2$ whereas $\lambda^{-1}G(X/\lambda)$ ranges from order $1/(\bar{\alpha}\bar{v}_t^2)$ at $X = 0$ to $\bar{\alpha}^2/\bar{v}_t^2$ at its lowest value, i.e. at the crossover $X \sim \bar{v}_t^2/\bar{\alpha}$ between the Cauchy-like and G-like behaviour. Therefore K can usually be neglected, although in some special circumstances it shows up as a small plateau between these two regimes; we shall disregard it from now on. Then, the large-X behaviour of $\lambda^{-1}G(X/\lambda)$ is dominated by the $\sqrt{q}$ singularity at the origin of $F_\infty(q)$, so that it goes as $X^{-3/2}$ as $X \to \infty$. This is in fact the hallmark of the $X \ll \bar{v}_t^2/\bar{\alpha}$ regime we are discussing.

Finally, one has to take into account the fact that the tunnelling matrix elements are not constant, but randomly distributed. As can be seen in the derivation of equation (4.4) (see the remark below equation (A4)) this merely amounts to replacing the expression for $\Psi(q)$ given in equation (4.5) by the function $\bar{\Psi}(q)$ defined as follows,

$$\bar{\Psi}(q) = \langle\Psi(v^2 q/\alpha)\rangle_v \tag{4.17}$$

where the brackets denote averaging over the v's, which we take to have a Gaussian distribution with variance v_t^2. This new function is of course much more complicated than the original one, but its asymptotic behaviour for small or large values of q is readily obtained. Indeed, for $q \ll 1$, $\Psi(q)$ is proportional to $|q|$, so that the average is obtained by replacing v by v_t. On the other hand, for $q \gg 1$, $\Psi(q)$ is proportional to $\sqrt{|q|}$, so that its average over a Gaussian distribution is obtained by replacing v by $\sqrt{2/\pi}\,v_t$. Using these facts together with the above estimates for the behaviour of $p(X)$ one finally obtains the result stated in equations (2.8)–(2.10) in section 2. Note a few trivial differences. There we consider splittings as being always positive whereas in the above computation we treated positive and negative splittings separately. This introduces a factor of two. Furthermore, we have made the dependence on D explicit, which in particular means replacing $p(X)$ by $Dp(X/D)$ as well as replacing $\bar{\alpha}$ and $\bar{v}_t$ by their original expressions.

4.2. More structured ensembles

For the simple ensemble equation (2.5), the two main assumptions we made to replace the exact distribution by equation (4.2), namely to neglect correlation of chaotic states among a given symmetry class and to replace the interacting Brownian diffusion by a free one for small $\bar{\alpha}$ are under control. Indeed, section 3 gives full justification of the first assumption and the second can be seen as a simple consequence of standard perturbation theory. Actually, as shown in figure 3, one can see that our analytical findings perfectly agree with an 'exact' Monte Carlo evaluation of the splitting distribution generated by the ensemble equation (2.5). We stress that in this very simple case the parameter α that we are using is (for small α) simply related to the variance σ^2 of the non-diagonal matrix elements of $(\mathrm{GOE})_A$ (indeed $\alpha^2 = 2(2\sigma)^2$), and that therefore there are no adjustable parameters in this comparison.

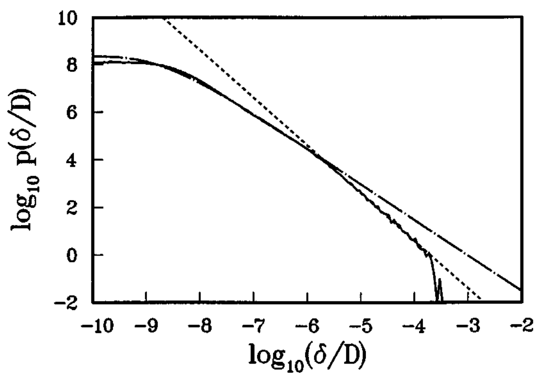

Figure 3. Comparison between a Monte Carlo calculated distribution of splittings δ for the simple ensemble equation (2.5) (full curve) and the the predicted form equations (2.8)–(2.10). The parameters of the Monte Carlo calculations are $\Lambda = 10^{-2}/8$ (imposing $\alpha/D = 0.1$ for the theoretical curve), $v_t/D = 10^{-4}$, number of matrices 3×10^5 and size of matrices 60×60. The three regimes, G-like behaviour (chained dot), Cauchy-like behaviour (dash) and truncation of the Cauchy law for splitting greater than v_t/D are clearly seen.

More structured ensembles deserve, however, some further discussion. Consider, for instance, the ensemble relevant to the quartic oscillator system used as illustration in section 2. Symbolically, this ensemble can be written as [10]

$$\mathbf{H}^{\pm}_{\mathrm{qo}} = \begin{pmatrix} E_R & \{v\} & 0 & 0 \\ \{v\} & (\mathrm{GOE})_1 & (\mathrm{GOE})^{\pm}(\Lambda_{12}) & (\mathrm{GOE})^{\pm}(\Lambda_{13}) \\ 0 & (\mathrm{GOE})^{\pm}(\Lambda_{12}) & (\mathrm{GOE})^{\pm}_2 & 0 \\ 0 & (\mathrm{GOE})^{\pm}(\Lambda_{13}) & 0 & (\mathrm{GOE})^{\pm}_3 \end{pmatrix} \tag{4.18}$$

where the subscript $\pm$ again indicates ensembles which are independent in the $+$ and $-$ symmetry class. Noting D_{tot} the total density of states (in a given symmetry class), $(\mathrm{GOE})_i$ stands for a Gaussian orthogonal ensemble such that the mean level density in the centre of the semicircle is $f_i D_{\mathrm{tot}}$, and $(\mathrm{GOE})^{\pm}(\Lambda_{jk})$ represents Gaussian distributed independent matrix elements of variance $\sigma_{jk}^2 = \Lambda_{jk} D_{\mathrm{tot}}^2$. (For the configuration of the quartic oscillators corresponding to figure 1, one has $f_1 = 0.5$, $f_2 = 0.2$, $f_3 = 0.3$ and $\Lambda_{12} = 0.14$, $\Lambda_{13} = 0.11$.) For such complicated ensembles the two assumptions concerning the irrelevance of intra-class correlations and essentially Gaussian distribution of the $(E_i^+ - E_i^-)$ are presumably equally well fulfilled as in the simple case of equation (2.5). What is lost, however, is the uniformity of the distribution of the tunnelling matrix elements and of the variance of the $(E_i^+ - E_i^-)$. Indeed, in the above example, a diagonalization of the chaotic part of the Hamiltonian is going to transfer some tunnelling matrix elements from the block connecting E_R to $(\mathrm{GOE})_1$ to the ones connecting $(\mathrm{GOE})_2$ and $(\mathrm{GOE})_3$. One may end in this way with three different scales for the variance of the tunnelling matrix elements as well as for the parameter α (one for each (GOE) block).

More generally, the typical situation will be that a (possibly large) number of (GOE) blocks, $(\mathrm{GOE})_1, (\mathrm{GOE})_2, \ldots, (\mathrm{GOE})_K$ are involved in the tunnelling process. After diagonalization of the chaotic part of the Hamiltonian, both the variance of the tunnelling matrix elements, and the degree of correlation between symmetry classes, will be block dependent. Each block $(\mathrm{GOE})_k$ $(k = 1, \ldots, K)$ would then have to be characterized by a tunnelling parameter v_k and a transport parameter α_k (α_k and v_k highly correlated), in addition to its dimension $N_k = f_k N$. Let us introduce the notation

$$I(\alpha, v_t; q) = -\frac{\alpha}{D\sqrt{2\pi}} \Psi(v_t^2 q/\alpha D) \tag{4.19}$$

where $\Psi(\tilde{q})$ is defined by equation (4.4). A straightforward modification† of the derivation of equation (4.4) gives that taking into account the block dependence of α_k and v_k merely amounts to replacing this equation (i.e. $F(q) = \exp(-I(\alpha, v_t; q))$) by

$$F(q) = \exp\left(-\sum_{k=1}^{K} f_k I(\alpha_k, v_k; q) \right). \tag{4.20}$$

Inspection of equations (4.11) and (4.12) then shows that they remain valid provided α and v_t are now defined as

$$v_t^2 \equiv \sum_k f_k v_k^2 \tag{4.21}$$

$$\alpha \equiv \frac{1}{v_t^2} \left(\sum_k f_k \alpha_k^{1/2} v_k \right)^2. \tag{4.22}$$

Multiplying equation (4.21) by N, Nv_t^2 appears as the (average) square norm of the projection of the quasi-mode $\Psi_R^{\pm}$ on the chaotic space. It is therefore independent of the chaotic phase space structure. This, for instance, allows computation of v_t from the variance of the tunnelling matrix elements *before* diagonalization of the chaotic part of the Hamiltonian. The parameter α and v_t have, moreover, a certain number of intuitively clear properties: if all v_k are multiplied by a constant factor, the effective tunnelling element v_t is multiplied by the same factor, whereas α is unaffected. Further, if all v_k are identical, then the effective tunnelling element is the same. On the other hand, the same is not true of α: if all α_k and all v_k are taken to be equal, the effective efficiency of the classical barrier now depends on the number of different components of phase space through which tunnelling can take place. Further, we see that any components with negligible values of v_k will contribute negligibly both to v_t and α. Thus we can identify a given part of phase space through which tunnelling actually occurs and limit ourselves to it.

With the definitions equations (4.21) and (4.22) of v_t and α, structured ensembles are therefore seen to behave in essentially the same way as the simple ensemble equation (2.5). The only difference is that the condition of validity of equations (4.11) and (4.12), that is

$$q \gg (\alpha_k D)/v_k^2 \tag{4.23}$$

† In equation (A4), $\lim_{N\to\infty}(1 - I(q)/N)^N$ has to be replaced by

$$\lim_{N\to\infty} \left[\prod_{k=1}^{K} (1 - I(\alpha_k, v_k; q)/N)^{f_k N} \right].$$

and

$$v_k^{-1} \ll q \ll (\alpha_k D)/v_k^2 \tag{4.24}$$

respectively, must now be fulfilled for all k's. The transition between the different regimes of the distributions may therefore be less sharp than for the ensemble equation (2.5).

If the partial barriers structures were to become highly developed, say to the point that the ensemble could meaningfully be described in terms of band matrices, then obviously the issue of localization would have to be considered. In this case, the orders of magnitude of the α_k might become comparable to those of the v_k and most of the splitting distribution might be actually in a transition-like regime. These are exactly the sort of problems we pointed out in our earlier discussion of the physical situation. However, as long as the α_k's are clearly larger than the v_k's, the transition from one regime to another should still take place on a short scale as compared to the range spanned by the distribution. Physically speaking, this condition amounts to saying that classically forbidden processes are always much slower than classically allowed ones. In that case, the form of the result should not be visibly affected. For example, as seen is figure 4, the distribution resulting from the ensemble described in equation (4.18) still perfectly follows the predicted form equations (2.8)–(2.10) (note, however, that α is now a tunable parameter).

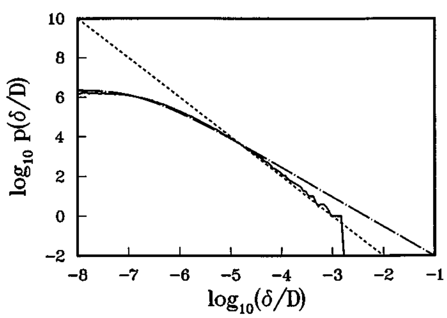

Figure 4. Comparison between a Monte Carlo calculated distribution of splittings δ for the ensemble equation (4.18) with $f_1 = 0.5$, $f_2 = 0.2$, $f_3 = 0.3$ and $\Lambda_{12} = 0.14$, $\Lambda_{13} = 0.11$ (full curve), and the predicted form equations (2.8)–(2.10). The Monte Carlo calculations have been performed with 10^5 matrices of size 100×100, using as tunnelling parameter $(v_1)^2/D = 10^{-3}$. For the theoretical curves, namely the G-like (chained dot curve) and Cauchy-like (dash curve) behaviours, the tunnelling parameter is determined by equation (4.21) as $(v_t)^2 = (v_1)^2/2$. The transport parameter, however, is here a tunable parameter, which has been taken equal to $\alpha/D = 0.1$.

As a final comment, let us remark that the construction of ensembles such as equation (4.18) was made in [10] under the assumption that the transport across partial barriers can be modeled classically by a Markovian process involving only one parameter (time scale). When a set of dense cantori exists either within or between chaotic regions, it is possible, however, that the transport properties are better described by a power law lacking identifiable characteristic times. Although this is not *a priori* the configuration we want to describe, it should be borne in mind that the structure of the matrix ensembles modelling the statistical properties of the associated quantum system is essentially affected by the features of the classical dynamics for times of the order of the Heisenberg time. Events occurring on a much shorter time scale are in any case treated as random and the effect of times much longer is presumably strongly suppressed by localization effects. Therefore the difference between power law and exponential decay may be of less importance quantum mechanically than it is classically, and it is plausible that the discussion given above for a large number of Markovian partial barriers apply as well for instance to the case of chaotic region separated by a dense set of cantori (at least provided these latter do not occupy a significant fraction of the phase space). In particular the criterion of applicability of the theory should be that the matrix elements associated with transport across the barrier (related to the classical flux crossing the barrier within Heisenberg time) is much larger than the tunnelling matrix elements, rather than whether a single time scale can classically characterize the barrier. Further studies are, however, required to confirm, or infirm, this handwaving argument.

5. Conclusion

In conclusion, we have provided in this paper an analytical study of the splitting distributions generated by ensembles of random matrices constructed in [10] to model a tunnelling process in the chaos-assisted regime. The original ensembles may contain such a complicated structure that a general answer to this problem may seem *a priori* out of reach. Nevertheless, it turns out that only the average size of the tunnelling matrix elements and the degree of correlation between the chaotic spectra in the different symmetry class affect the distribution and that therefore the problem can be reduced to a simpler formulation which is tractable.

The basic reason for the considerable simplifications encountered was in essence already pointed out in [10]. It is due to the fact that for large splittings only the situation of near-resonance to a given state of the chaotic sea is of relevance. To this obvious remark we only need add that, for the case in which efficient barriers are at work, the tunnelling operates not through single states, but through quasi-degenerate doublets of states of opposite parity. These are of course less efficient in promoting tunnelling, since the particle requires a time of the order of the width of the doublet to reach the symmetrical torus. In either case, the behaviour is determined by rather natural probabilistic considerations. It turns out to be sufficient to consider only the probability of one single eigenvalue being near the tunnelling state, so that correlations between eigenvalues could be safely ignored. Further, the very simplicity of the physical picture given here results in it being fairly robust to changes in minor details of the model. Thus it does not appear necessary that all states in the chaotic sea should participate equally in the tunnelling process, nor that the couplings should be uniform. In fact, the main limitations of our result seem to be the ones related to localization phenomena. If the structure of the barriers in phase space is sufficiently complicated, it is possible that localization effects, associated with the presence of a large number of partial barriers, become as effective in limiting tunnelling from one quasi-mode to its symmetric partner as the initial classically forbidden process. In this case, the splitting distributions we have obtained would no longer be relevant. However, this should not be too severe a limitation, and it should generally be possible to determine for any given system whether this takes place or not. When it does not, the picture of tunnelling in the presence as well as in the absence of barriers to transport is indeed the one we gave. This is substantiated by the numerical work done. In particular, we showed that not only the simplest model of a barrier gives results in good agreement with theoretical predictions, but also a highly specific random matrix ensemble constructed explicitly in order to model chaos-assisted tunnelling in a system of coupled quartic oscillators was well fitted by the theoretical predictions, as were also the actual splitting distribution for this system.

This might possibly open up a way to identify chaos-assisted tunnelling in experimental systems. In such systems, the exhaustive study of the classical mechanics necessary to produce a satisfactory random matrix ensemble would probably not be feasible. Nevertheless the above remarks strongly suggest that if chaos-assisted tunnelling is present, the splitting distribution will reflect this fact by showing a highly specific and well characterized behaviour. Indeed, as discussed throughout this paper, only the scale of the distribution and the position of the transitions between the different regimes are system dependent, but the shape of the distribution is essentially universal. In particular, the experimental detection of a transition from a $\delta^{-3/2}$ behaviour, characteristic of the G-like regime, to a δ^{-2} behaviour, characteristic of the Cauchy-like regime, would be a powerful argument in favour of the presence of chaos-assisted tunnelling.

Acknowledgments

We acknowledge many helpful discussions with Oriol Bohigas, as well as with Steven Tomsovic who we thank in addition for providing us with the numerical data of the quartic oscillators system used in figure 1 and for a careful reading of the manuscript. Denis Ullmo wants to thank the Institute for Nuclear Theory, Seattle WA, where part of this work has been done. François Leyvraz would like to thank the Division de Physique Théorique for its kind hospitality during his sabbatical stay there where most of this work was done, as well as acknowledge financial support by DGAPA during that same period. The Division de Physique Théorique is 'Unité de Recherche des Universités Paris 11 et Paris 6 Associée au CNRS'.

Appendix A. Computation of the distribution function

Denote by brackets the integration over e_i^+ and e_i^- with the weight function $P(e_i^+, e_i^-)$ (see equation (4.1)). We define

$$p(X) = \left\langle \delta\left(x - \bar{v}_t^2 \sum_{i=1}^{N} \left(\frac{1}{e_i^+} - \frac{1}{e_i^-}\right)\right)\right\rangle \tag{A1}$$

$$\begin{aligned} F(q) &= \int_{-\infty}^{\infty} F(X) \mathrm{e}^{\mathrm{i}qX}\, \mathrm{d}x \\ &= \left\langle \exp\left(\mathrm{i}q\bar{v}_t^2 \sum_{i=1}^{N} \left(\frac{1}{e_i^+} - \frac{1}{e_i^-}\right)\right)\right\rangle. \end{aligned} \tag{A2}$$

This last expression factorizes in N factors, each of which is a double integral. Denoting the corresponding average over e^+ and e^- also by brackets, one obtains

$$\begin{aligned} F(q) &= \langle \exp(\mathrm{i}q\bar{v}_t^2(1/e^+ - 1/e^-))\rangle^N \\ &= (1 - \langle 1 - \exp(\mathrm{i}q\bar{v}_t^2(1/e^+ - 1/e^-))\rangle)^N \\ &= \left(1 - \frac{I_N(q)}{N}\right)^N \end{aligned} \tag{A3}$$

where the last line defines $I_N(q)$. The reason for this manipulation is that in this way I_N goes to a finite limit I as $N \to \infty$ and therefore

$$\lim_{N\to\infty} F(q) = \lim_{N\to\infty} (1 - I(q)/N)^N = \exp(-I(q)). \tag{A4}$$

Note, moreover, that taking into account the fact that the tunnelling matrix elements are random variable of variance v_t^2 instead of being constant just amounts to interpreting $\langle\cdot\rangle$ as containing a further integral over the tunnelling matrix elements' distribution. This introduces no further difficulties in the calculation of I, except for more complicated notation. We shall therefore not consider it in this appendix and just modify the final result in the appropriate way at the end of section 4.1.

One finds

$$I(q) = \frac{1}{\sqrt{2\pi}\bar{\alpha}} \int \mathrm{d}e^+ \, \mathrm{d}e^- \, (1 - \exp(\mathrm{i}\bar{v}_\mathrm{t}^2 q(1/e^+ - 1/e^-))) \exp\left(-\frac{(e^+ - e^-)^2}{2\bar{\alpha}^2}\right) \tag{A5}$$

since the above integral being convergent, $\lim_{N\to\infty} I_N$ is just obtained by dropping the term $-\pi(e^+/N)^2$ in the exponent of $P(e^+, e^-)$ (see equation (4.1)). Making the successive transformations $y = (1/e^+ + 1/e^-)/(1/e^+ - 1/e^-)$, $w = (1/e^+ - 1/e^-)$, followed by $t = w(\bar{\alpha}(1-u^2)/\sqrt{8})$, $I(q)$ can be expressed as

$$\begin{aligned} I(q) &= \frac{8}{\sqrt{2\pi}\bar{\alpha}} \int \frac{\mathrm{d}y}{(1-y^2)^2} \int \frac{\mathrm{d}w}{|w|^3} (\mathrm{e}^{\mathrm{i}w\bar{v}_\mathrm{t}^2 q} - 1) \exp\left(-\frac{8}{\bar{\alpha}^2(1-y^2)^2 w^2}\right) \\ &= \frac{\bar{\alpha}}{\sqrt{2\pi}} \int \mathrm{d}y \int \frac{\mathrm{d}t}{|t|^3} \left(1 - \cos\left(\frac{\sqrt{8} q \bar{v}_\mathrm{t}^2}{\bar{\alpha}(1-y^2)}\right)\right) \mathrm{e}^{-1/t^2}. \end{aligned} \tag{A6}$$

If we now introduce $\Phi(z)$ as in the text,

$$\Phi(z) = 2\int_0^\infty \frac{\mathrm{d}t}{t^3}(1 - \cos zt)\mathrm{e}^{-1/t^2} \tag{A7}$$

one easily obtains

$$I(q) = \frac{\bar{\alpha}}{\sqrt{2\pi}} \int \mathrm{d}y \, \Phi\left(\frac{\sqrt{8}\bar{v}_\mathrm{t}^2 q}{\bar{\alpha}|1-y^2|}\right). \tag{A8}$$

From this follows the formula given in the text,

$$F(q) = \exp\left(-\frac{\bar{\alpha}}{\sqrt{2\pi}} \Psi(\bar{v}_\mathrm{t}^2 q/\bar{\alpha})\right) \tag{A9}$$

where $\Psi(\tilde{q})$ is given by the expression

$$\Psi(\tilde{q}) = \int_{-\infty}^{\infty} \mathrm{d}y \, \Phi\left(\frac{\sqrt{8}\tilde{q}}{|1-y^2|}\right). \tag{A10}$$

Appendix B. Some useful integrals

We first give another expression for $\Phi(y)$,

$$\Phi(y) = \int_0^\infty \mathrm{d}t \, \left(1 - \cos\frac{y}{\sqrt{t}}\right) \mathrm{e}^{-t} \tag{B1}$$

which is obtained from the original definition by substituting $1/t^2$ by t. From this follows

$$\begin{aligned} \int_{-\infty}^{\infty} \Phi(1/y)\,\mathrm{d}y &= \int_{-\infty}^{\infty} \Phi(y) \frac{\mathrm{d}y}{y^2} \\ &= \int_0^\infty \mathrm{d}t \, \mathrm{e}^{-t} \int_{-\infty}^{\infty} \left(1 - \cos\frac{y}{\sqrt{t}}\right) \frac{\mathrm{d}y}{y^2} \\ &= \int_0^\infty \frac{\mathrm{d}t}{\sqrt{t}} \mathrm{e}^{-t} \int_{-\infty}^{\infty} \frac{1-\cos y}{y^2}\,\mathrm{d}y \\ &= \Gamma(1/2) \int_{-\infty}^{\infty} \frac{\sin y}{y}\,\mathrm{d}y = \pi^{3/2}. \end{aligned} \tag{B2}$$

The other integral is handled similarly:

$$\begin{aligned}\int_{-\infty}^{\infty} \Phi(1/y^2)\,\mathrm{d}y &= \int_{-\infty}^{\infty} \Phi(y^2)\frac{\mathrm{d}y}{y^2} \\ &= \int_0^{\infty} \frac{\mathrm{d}t}{t^{1/4}}\mathrm{e}^{-t}\int_{-\infty}^{\infty}\frac{1-\cos y^2}{y^2}\,\mathrm{d}y \\ &= 2\Gamma(3/4)\int_{-\infty}^{\infty}\sin y^2\,\mathrm{d}y = \sqrt{2\pi}\,\Gamma(3/4). \qquad \text{(B3)}\end{aligned}$$

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