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# “Phase diagram” of a mean field game

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## HIGHLIGHTS

- We study a simple model of “mean field game”.
- We provide an exact solution of the associated system of coupled differential equations.
- We analyze the resulting self-consistent equation in various limiting regimes, resulting in the construction of a “phase diagram” of the considered mean field game.

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## ABSTRACT

Mean field games were introduced by J-M. Lasry and P-L. Lions in the mathematical community, and independently by M. Huang and co-workers in the engineering community, to deal with optimization problems when the number of agents becomes very large. In this article we study in detail a particular example called the “seminar problem” introduced by O. Guéant, J-M. Lasry, and P-L. Lions in 2010. This model contains the main ingredients of any mean field game but has the particular feature that all agents are coupled only through a simple random event (the seminar starting time) that they all contribute to form. In the mean field limit, this event becomes deterministic and its value can be fixed through a self consistent procedure. This allows for a rather thorough understanding of the solutions of the problem, through both exact results and a detailed analysis of various limiting regimes. For a sensible class of initial configurations, distinct behaviors can be associated to different domains in the parameter space. For this reason, the “seminar problem” appears to be an interesting toy model on which both intuition and technical approaches can be tested as a preliminary study toward more complex mean field game models.


## 1. Introduction

Many problems in different fields deal with a situation where many identical and interacting agents try to minimize a cost through the choice of a strategy. One can think of economic agents trying to maximize their profits, of people in a crowd trying to minimize their discomfort or to particles in a fluid “trying” to minimize their energy.

A general framework making possible to model a large class of such problems has been introduced in 2006 by Lasry and Lions [1,2] and Huang et al. [3] under the general terminology of “mean field game theory”. Largely inspired by statistical physics, this approach addresses the limit where the agents face a continuum of choices (states) in which they can evolve only locally, and the number of agents is large enough that self averaging processes are at work. This approach leads to a system of partial differential equations coupling the density of players and the optimization part of the problem.

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Mean field game theory has been intensively studied in the past few years, and in spite of its relative youth, a very large number of results have been obtained in the mathematical [4–8] and socio-economic communities [9–13]. A recent overview is given by Gomes and Saúde in Ref. [14]. Most of the focus however has been put either on the conditions required to prove rigorously the existence and unicity of the solutions of the equations of mean field game theory [8], or on the study of particular models based primarily on numerical treatments [7]. A more "qualitative" understanding of the behavior of the solutions, based on the identification of the relevant time and length scales, and on the analytical study of the solution in various limiting regime, has received significantly less attention.

Our goal in this paper is to perform this program for a simple model, introduced by Guéant et al. in 2010 [15], called the seminar problem to be described in more details below. The essential point here is that this "mean field game model" is in some sense very close to the everyday "physicists' mean field" since all agents are interacting only through a very simple "field" which is actually a simple number, the time $T$ at which the seminar actually starts. This particular feature allows for an analytical approach, similar in spirit to the physicists' one: For fixed $T$, the behavior of each agent becomes independent on the other, making the associated problem to be solvable to a large extent; then, for a given distribution of agents, the actual value of $T$ can be evaluated by a self-consistency procedure. The main interest in this model is to provide a fully understandable toy model on which one can develop its own intuition and tools before tackling the full complexity of mean field game models.

The paper is organized as follows: In Section 2 we introduce the seminar problem in detail and show that its resolution involves two essentially independent parts: a system of coupled (Hamilton–Jacobi–Bellman and Kolmogorov) differential equations on one hand, and a self-consistency problem on the other. Sections 3 and 4 address the Hamilton–Jacobi–Bellman and Kolmogorov equations, respectively. Various limiting regimes are studied in detail for both. Moreover, we show that an exact solution to these coupled differential equations can actually be given in a closed form. The self-consistency condition determining the effective beginning of the seminar $T$ is discussed in Section 5, eventually leading to the construction of a "phase diagram" for this toy model. Concluding remarks are gathered in Section 6. The paper is completed by three Appendices where technical computations are shown.

## 2. The seminar problem

*The model*

Consider a corridor at the end of which is a seminar room. A seminar is planned at time $\bar{t}$ but people know that in practice, it will only begin when a large enough proportion of the lab members $\theta$ (known), will be seated.

The members of the laboratory thus move according to the following considerations: They do not want to arrive too early in the seminar room because they do not particularly enjoy waiting idly as the room fills. On the other hand they are aware that the lab director and the seminar organizers will already be in the room at time $\bar{t}$, and will frown upon late comers. Furthermore they really want to understand the content of the seminar and are concerned that missing the actual beginning might not help in this respect.

For every agent, this is summarized by the following cost function associated with the arrival time $t$:

$$c(t) = \alpha[t-\bar{t}]_+ + \beta[t-T]_+ + \gamma[T-t]_+, \tag{1}$$

where $T$ is the effective beginning time of the seminar. In Eq. (1), $\alpha$, $\beta$ and $\gamma$ are positive real numbers and respectively quantify the sensitivity to social pressure, the desire not to miss the beginning of the seminar, and the reluctance to useless waiting. We assume these parameters to be the same for all members of the laboratory. We also assume ($\gamma < \alpha$) so that the cost $c(t)$ is actually minimal for the official starting time $\bar{t}$.

The corridor is represented by the negative half-line line $]-\infty, 0]$, and the seminar room is located at $x = 0$. At time $t = 0$, people leave their office to go to the seminar. Each member of the laboratory $i = 1 \dots N$, controls her drift $a_i(t)$ toward the seminar room but is subject to random perturbations (stopping to discuss with somebody, going back to take a pen and then giving up the idea, or speeding up to catch up a friend for example), modeled by a Gaussian white noise of variance $\sigma^2$. A given participant thus moves according to a noisy dynamics:

$$\mathrm{d}X_i = a_i(t)\,\mathrm{d}t + \sigma\,\mathrm{d}W_i(t) \tag{2}$$

where,

$X_i(t)$ is the agent position at time $t$,

$a_i(t)$ is her drift at the same time,

$\mathrm{d}W_i(t)$ is a normal white noise.

Again, except for their initial positions, all agents have the same characteristics.

In addition to the cost $c(t)$ associated to the arrival time (Eq. (1)), agents dislike having to rush on their way to the seminar room and the total cost function therefore includes a term quadratic in the (controlled) drift $a_i(t)$. An agent leaving her workplace $x_0$ at $t = 0$ has thus to adapt her drift in order to minimize the expected cost

$$J_T[a] = \mathbb{E}\left[c(\tilde{\tau}) + \frac{1}{2}\int_0^{\tilde{\tau}} a_i^2(\tau)\,\mathrm{d}\tau\right] \tag{3}$$

associated with Eq. (2) and the initial condition $X(t = 0) = x_0 < 0$. In Eq. (3) $\tilde{\tau}$ is the first passage time at $x = 0$

$$\tilde{\tau} = \inf\{t : X(t) = 0\}, \tag{4}$$

and $\mathbb{E}$ is the expectation with respect to the noise.

We define $N(t)$ as the cumulative distribution of arrival time (the percentage of people arrived before $t$). If the quorum is met before the official time $\bar{t}$ of the seminar, this latter starts exactly at $\bar{t}$. If on the other hand the quorum is met at a later time, $T$ is determined by the self-consistency relation $N(T) = \theta$ (more formally $T = \inf\{t \geq \bar{t} : N(t) \geq \theta\}$).[1]

Within the mean field approximation, the total number of researchers in the lab is assumed to be large enough that the individual choices of a given agent, and thus her arrival time, cannot have any significant impact on $T$. Each agent should thus solve the optimization problem Eqs. (2)–(3) for herself, assuming $T$ fixed. Introducing the value function

$$u(x, t) = \min_{a_i(t)} \left\{ \mathbb{E}\left[ c(\tilde{\tau}) + \frac{1}{2}\int_t^{\tilde{\tau}} a_i^2(\tau)\, d\tau \right] \right\} \tag{5}$$

subject to the initial condition $X(t) = x$, this optimization problem is equivalent to the Hamilton–Jacobi–Bellman (HJB) equation (see e.g. Ref. [16]):

$$\begin{cases} \dfrac{\partial u}{\partial t} - \dfrac{1}{2}\left(\dfrac{\partial u}{\partial x}\right)^2 + \dfrac{\sigma^2}{2}\dfrac{\partial^2 u}{\partial x^2} = 0 \\ u(x = 0, t) = c(t), \end{cases} \tag{6}$$

and the optimal drift is given by

$$a(x, t) = -\partial_x u(x, t). \tag{7}$$

The second hypothesis underlying the mean field approximation is that, beyond the total size of the agent population, the agent density itself is large enough that the distribution of agents is self-averaging: therefore everything happens as if at any given location and time, each realization of the noise was experienced by somebody. Assuming a (normalized) initial density of participants $m_0(.)$ at time $t = 0$, Eq. (2) thus implies that this density will evolve under the Kolmogorov equation (see e.g. Ref. [17]):

$$\begin{cases} \dfrac{\partial m}{\partial t} + \dfrac{\partial am}{\partial x} - \dfrac{\sigma^2}{2}\dfrac{\partial^2 m}{\partial x^2} = 0 \\ m(x = 0, t) = 0 \\ m(x, t = 0) = m_0(x). \end{cases} \tag{8}$$

Once this equation is solved, the quorum condition

$$N(T) = \left[1 - \int_{-\infty}^{0} m(x, T)\right] = \theta \quad (\text{if } T > \bar{t}) \tag{9}$$

(or $N(T) \geq \theta$ if $T = \bar{t}$) provides a self-consistent condition which has to be fulfilled if $T$ is indeed the actual starting time of the seminar.

*General strategy*

The toy model that we just described depends on a few parameters which play different roles. Some of these parameters are just "numbers". For instance the official time of the seminar $\bar{t}$, which mainly fixes a time scale. Or the parameters $\alpha$, $\beta$ and $\gamma$ of the cost function $c(t)$ Eq. (1) which, as we shall see, govern the typical amplitude of the drift velocity. In the same way the noise strength $\sigma$ will govern the diffusion velocity.

Another parameter of the problem is the initial distribution of agents $m_0(x)$. Being a function rather than just a number it is a little bit more difficult to characterize simply. It defines a mean initial position $\langle x \rangle_0$, but also moments of arbitrary order, which may introduce various length scales into the problem.

We are helped here by the linear character of the Kolmogorov equation. Indeed introducing the elementary solutions $G(x, t|x_0)$ which are the solutions of Eq. (8) with a Dirac mass $\delta(x - x_0)$ as initial condition, the solution for an arbitrary $m_0(x)$ is obtained through the convolution

$$m(x, t) = \int_{-\infty}^{0} dx_0\, G(x, t|x_0) m_0(x_0).$$

[1] Strictly speaking, the effective starting time $T$ is a random event (a random stopping time). Here we assume that in the mean field limit of the present model, this event becomes deterministic so that we can confuse it with its expectation for almost all realizations.

Therefore, introducing

$$\rho(x_0, t) \equiv \int_{-\infty}^{0} \mathrm{d}x\, G(x, t|x_0), \tag{10}$$

which thus measure the proportion of agents starting from $x_0$ who have not yet reached the seminar room at time $t$, the self-consistent condition Eq. (9) reads

$$\int_{-\infty}^{0} \mathrm{d}x_0\, \rho(x_0, T) m_0(x_0) = \bar{\theta} \quad (\text{if } T > \bar{t}), \tag{11}$$

with $\bar{\theta} = (1-\theta)$ the proportion of agents still in the corridor when the quorum is met.

The solution of the self consistent problem can therefore be split quite neatly in two distinct parts. The first part will be to analyze, and solve, the Hamilton–Jacobi–Bellman and Kolmogorov equations (6) and (8) assuming $T$ known. More specifically, the goal in this first part will be to compute the function $\rho(x_0, T)$ for arbitrary $x_0$ and $T$. This is what we shall do in the two following sections. For this part we obviously do not need to specify what is $m_0(x_0)$.

Once $\rho(x_0, T)$ is known, the self consistent problem reduces to Eq. (11). It then of course involves the initial density $m_0(x_0)$, as well as $\rho(x_0, T)$, but this latter quantity summaries all the required information, beyond $m_0(x_0)$, about the system. This second aspect of the problem will be addressed in Section 5.

## 3. Resolution of the Hamilton–Jacobi–Bellman equation

Except for its rather non-standard boundary conditions, the HJB equation (6) is closely related to a Burger's equation, and it can in the same way be solved exactly through a rather standard Cole–Hopf transformation. Before we do so however, we find it useful to consider first the limiting behaviors of very small and very large $\sigma$'s.

### 3.1. Small $\sigma$

To understand the regime of very weak noise, let us consider the noiseless limit of Eq. (6), which takes the form of the Hamilton–Jacobi equation

$$L(\partial_t u, \partial_x u) = 0, \tag{12}$$

associated with the free propagation Hamiltonian

$$L(E, p) \equiv E - \frac{p^2}{2} \tag{13}$$

complemented with the boundary conditions

$$u(x = 0, t) = c(t). \tag{14}$$

Introducing a fictitious time $\xi$ and noting $\dot{(\ )} = \mathrm{d}(\ )/\mathrm{d}\xi$ the corresponding time derivative, the Hamilton dynamics associated with $L$ is given by [18,19]

$$\begin{aligned} \dot{t} &= \frac{\partial L}{\partial E} = 1 \qquad \dot{E} = -\frac{\partial L}{\partial t} = 0 \\ \dot{x} &= \frac{\partial L}{\partial p} = -p \qquad \dot{p} = -\frac{\partial L}{\partial x} = 0. \end{aligned} \tag{15}$$

Solution of the Hamilton–Jacobi equation are typically obtained through the method of characteristics. Here, this amounts to build a one parameter family of rays $\mathbf{r}_{\tilde{\tau}}(\xi)$ ($\mathbf{r} \equiv (E, t, p, x)$) indexed by $\tilde{\tau}$, such that $\mathbf{r}_{\tilde{\tau}}(\xi)$ is solution of the Hamilton's equations (15), and with initial conditions $\mathbf{r}(\xi = 0) = (E_0, t_0, p_0, x_0)$ imposed by Eq. (14) as

$$\begin{aligned} t_0 &= \tilde{\tau} \\ x_0 &= 0 \\ E_0 &= \frac{\mathrm{d}c}{\mathrm{d}t}(\tilde{\tau}) \equiv c'(\tilde{\tau}) \\ L(E_0, p_0) &= 0. \end{aligned}$$

This last equation imposes

$$p_0 = -\sqrt{2c'(\tilde{\tau})}. \tag{16}$$

Once this one parameter family of rays is build, the solution of the Hamilton–Jacobi equation just reads (for $t < \tilde{\tau}$, and thus negative $\xi$)

$$u(x = x_{\tilde{\tau}}(\xi), t = t_{\tilde{\tau}}(\xi)) = c(\tilde{\tau}) + \int_0^{\xi} (E\dot{t} + p\dot{x})\mathrm{d}\xi$$
$$= c(\tilde{\tau}) - (\tilde{\tau} - t)c'(\tilde{\tau}) - x\sqrt{2c'(\tilde{\tau})}. \tag{17}$$

As illustrated on Fig. 1 the quarter plan $(x < 0, t > 0)$ has to be divided in four different regions,

$$\begin{cases} \text{Region (0)}: & x \leqslant -\sqrt{2(\alpha+\beta)}(T-t) \\ \text{Region (1)}: & -\sqrt{2(\alpha+\beta)}(T-t) \leqslant x \leqslant -\sqrt{2(\alpha-\gamma)}(T-t) \\ \text{Region (2)}: & \sqrt{2(\alpha-\gamma)}(T-t) \leqslant x \leqslant -\sqrt{2(\alpha-\gamma)}(\bar{t}-t) \\ \text{Region (3)}: & -\sqrt{2(\alpha-\gamma)}(\bar{t}-t) \leqslant x < 0, \end{cases} \tag{18}$$

for which the application of Eq. (17) is somewhat different. In region (0) for instance the relevant rays reach $x = 0$ at $\tilde{\tau} > T$ which corresponds to $c'(\tilde{\tau}) = (\alpha + \beta) \equiv c_0'$. In the same way for region (2) $\bar{t} < \tilde{\tau} < T$ and $c'(\tilde{\tau}) = (\alpha - \gamma) \equiv c_2'$.

Region (1) then corresponds to $\tilde{\tau} = T$, where $c'(\tilde{\tau})$ is discontinuous. It can be easily justified (e.g. by viewing $c(t)$ as the limit of a family of differentiable functions) that the correct procedure here is to use all the rays emerging from $(x = 0, t = T)$ with all possible values of $c'(\tilde{\tau})$ within the interval $](\alpha - \gamma), (\alpha + \beta)[$ (and thus all velocities $p_0$ within $]\sqrt{2(\alpha - \gamma)}, \sqrt{2(\alpha + \beta)}[$).

In the same way region (2) corresponds to $\tilde{\tau} = \bar{t}$, and one should use all the rays emerging from $(x = 0, t = \bar{t})$ with all possible values of $c'(\tilde{\tau})$ within $]0, (\alpha - \gamma)[$ (and thus all velocities $p_0$ within $]0, \sqrt{2(\alpha - \gamma)}[$).

Note that because $c'(\tilde{\tau})$ is negative for $\tilde{\tau} < \bar{t}$, Eq. (16) has no real solution for $p_0$ and it is not possible to fulfill the boundary conditions Eq. (14) in this time interval for the Hamilton–Jacobi equation. In a small layer near the line, $(x = 0, 0 < t < \bar{t})$, the fact the Hamilton–Jacobi equation is first order when the HJB equation is second order implies a qualitative difference between the limit of small $\sigma$'s and $\sigma = 0$.

Acknowledging this, and keeping in mind the procedure explained above to handle the discontinuities of $c'(\tilde{\tau})$, Eq. (17) gives an explicit solution for the Hamilton–Jacobi equation (12)–(14). What will be needed though as an input for the Kolmogorov equation (8) is not so much $u(x, t)$ itself than its spatial derivative $-\partial_x u$ which through Eq. (7) specifies the drift $a(x, t)$ in Eq. (8). From Eq. (17) we see that $-\partial_x u$ is just the velocity $-p_0$ of the free motion on the corresponding ray. We obtain therefore the following results,

$$a(x,t) = -\partial_x u(x,t) = \begin{cases} \sqrt{2(\alpha+\beta)} \equiv a_0 & \text{in Region (0)} \\ \dfrac{-x}{(T-t)} & \text{in Region (1)} \\ \sqrt{2(\alpha-\gamma)} \equiv a_2 & \text{in Region (2)} \\ \dfrac{-x}{(\bar{t}-t)} & \text{in Region (3)} \end{cases} \tag{19}$$

valid as $\sigma \to 0$. On Fig. 1, this velocity is shown as the inverse slope of the arrows.

In the deterministic limit considered in this subsection a finite fraction of the agents (namely all those starting in region (1)) arrive exactly at time $T$. The quorum condition Eq. (9) is therefore ill-defined in this limit.

### 3.2. Large $\sigma$

Let us consider now the HJB equation (6) in the limit of very large $\sigma$'s. This amounts here to neglect the nonlinear term $\frac{1}{2}(\partial_x u)^2$, yielding the backward diffusion equation

$$\begin{cases} \dfrac{\partial u}{\partial t} + \dfrac{\sigma^2}{2}\dfrac{\partial^2 u}{\partial x^2} = 0 \\ u(x = 0, t) = c(t). \end{cases} \tag{20}$$

In this subsection it will be convenient to use for the boundary conditions a slightly modified version $c_\Lambda(t)$ of the cost function Eq. (1),

$$\begin{cases} c_\Lambda(t) = c(t) & \text{for } t \leq \Lambda \\ c_\Lambda(t) = c(\Lambda) & \text{for } t \geq \Lambda \end{cases} \tag{21}$$

with $(\Lambda \gg \bar{t}, T)$ a very large time (one may imagine for instance that once the seminar is over, there is less marginal incentive to reach the seminar room).

There are many ways to derive a solution of Eq. (6), but a relatively transparent one consists in going back to the original optimization problem, i.e. to define $u(x, t)$ as Eq. (5). Indeed, in the limit of very large $\sigma$'s, this optimization is

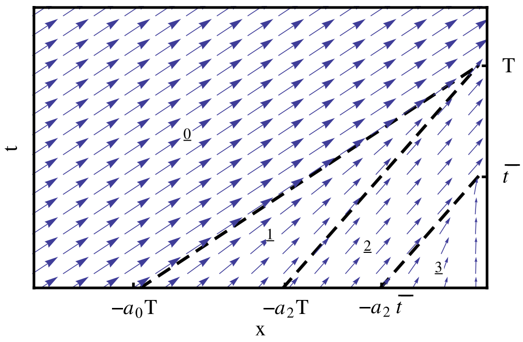


**Fig. 1.** Regions of the space $(x, t)$ $(x < 0, t > 0)$ and the corresponding drift fields for $\sigma \to 0$ (cf. Eq. (19)) $[a_0 = \sqrt{2(\alpha+\beta)}, a_2 = \sqrt{2(\alpha-\gamma)}]$.

straightforward: if the motion is overwhelmingly dominated by the noise, the best strategy for an agent is just to renounce paying the cost of the drift, and hope that the diffusive motion will bring her in time for the seminar.

Let us note

$$G_0(x,t) \equiv \frac{1}{\sqrt{2\pi\sigma^2 t}} \exp\left(-\frac{x^2}{2\sigma^2 t}\right) \tag{22}$$

the elementary solution of the free diffusion problem. The distribution of time of first passage at $x = 0$ for free diffusion started at $t_0$ in $x_0 < 0$ is given by [16]

$$P(t) = -\frac{\mathrm{d}}{\mathrm{d}t}\int_{+x_0}^{-x_0} \mathrm{d}x\, G_0(x, t-t_0).$$

The value function $u(x,t)$ is just the average of the cost function $\tilde{c}(t) = c_\Lambda(t)$ with this first passage time distribution. It therefore reads

$$\lim_{\sigma\to\infty} u(x_0,t_0) = \int_{t_0}^{\infty} \mathrm{d}t\, \tilde{c}(t)P(t) \tag{23}$$

$$= -x_0 \int_0^\infty \mathrm{d}t\, \frac{\tilde{c}(t+t_0)}{t} G_0(x_0,t). \tag{24}$$

(The fact that $G_0(x,t)$ is the elementary solution of the diffusion equation has been used to transform (23) into (24).)

Note that Eq. (24) would be valid for any choice of the final cost function $\tilde{c}(t)$ as long as the integral converges in $+\infty$, i.e. as long as $\tilde{c}(t)$ grows less than linearly at infinity. Thus the need to modify the large $t$ behavior of $c(t)$ in this subsection.

### 3.3. *Arbitrary* $\sigma$

As was mentioned at the beginning of this section, the HJB equation can actually be solved for arbitrary values of $\sigma$. Indeed, using the Cole–Hopf transformation i.e. setting $u(x,t) = -\sigma^2 \ln\phi(x,t)$ yields a linear equation for $\phi(x,t)$:

$$\begin{cases} \dfrac{\partial\phi}{\partial t} + \dfrac{\sigma^2}{2}\dfrac{\partial^2\phi}{\partial x^2} = 0 \\ \phi(x=0,t) = \mathrm{e}^{-\frac{c(t)}{\sigma^2}}. \end{cases} \tag{25}$$

Its solution is thus the same as Eq. (24) with $\tilde{c}(t) \equiv \exp[-c(t)/\sigma^2]$ as cost function. We obtain in this way

$$\phi(x,t) = -x\int_0^\infty \frac{\mathrm{e}^{-\frac{c(t+\tau)}{\sigma^2}}}{\tau} G_0(x,\tau)\,\mathrm{d}\tau \tag{26}$$

$$u(x,t) = -\sigma^2 \ln\phi(x,t). \tag{27}$$

An explicit expression of $\phi(x,t)$ in terms of elementary functions is given in Appendix A (see Eq. (A.4)). We just stress here that, because for $t$ larger than $T$ the cost function Eq. (1) becomes linear ($c(t) = (\alpha+\beta)t - (\alpha\bar{t} + \beta T)$), $\phi(x,t)$ takes a particularly simple form (see Eq. (A.5)) from which the value function is deduced as

$$u(x, t > T) = -\sqrt{2(\alpha+\beta)}x - c(t). \tag{28}$$

As a consequence, for times beyond $T$ the drift $a(x,t)$ is just the constant

$$a(x, t>T) = \sqrt{2(\alpha+\beta)} = a_0. \tag{29}$$

From Eqs. (26)–(27), the limiting behaviors Eqs. (19)–(24) can be recovered. This is particularly simple, for instance, in the large $\sigma$ regime if one uses the regularized version $c_\Lambda(t)$ of the cost function. Indeed in that case we see that as soon as $\sigma^2 \gg c(\Lambda)$, we can expand the exponential in Eqs. (26) and the logarithm in Eqs. (27), and $u(x, t)$ reduces to (24) in lowest order in $1/\sigma^2$. Things are slightly trickier for the true (non regularized) cost function $c(t)$ since however large $\sigma$ maybe, the $+\infty$ limit of the integral in Eqs. (26) is such that $c(t + t_0) \gg \sigma^2$ (which actually simply provides an effective cutoff for the integral). We find in this case (see Appendix B) that

$$\phi(x, t) = \exp\left[-\frac{1}{\sigma^2}|x|\sqrt{2(\alpha+\beta)} - c(t)\right] + O(\sigma^{-2}), \tag{30}$$

implying $a(x, t) = \sqrt{2(\alpha+\beta)}$. (If one considers furthermore the diffusive regime $a_0|x| \gg \sigma^2$, where the above approximation is most useful, one can further show that Eq. (30) is valid up to $O(\sigma^{-3})$ corrections.)

When $\sigma^2 \to 0$, the integral in Eq. (26) can be approximated using the steepest descent approximation in regions (0) and (2); or noting that it is dominated by the boundary contributions at $t + t_0 = T$ in regions (2) or $t + t_0 = \bar{t}$ in region (3) (see Eq. (18) or Fig. 1). Details of the calculations and the precise condition under which the approximation applies are given in Appendix C. The drift velocity which can be expressed as $a(x, t) = \sigma^2 \partial_x \phi/\phi$ is computed along the same lines, and one recovers in this way exactly Eq. (19), except for a small region near $\{x = 0; 0 \leqslant t \leqslant \bar{t}\}$ scaling as $\sigma^2$ in regions (0) and (2) and as $\sigma$ in region (1) and (3).

To conclude this section, we note that the drift $a(x, t)$ obtained from the agents' optimization is exactly $\sqrt{2(\alpha+\beta)}$ for $t > T$, but is actually close to this value in the entire region (0) already for small $\sigma$'s. As $\sigma$ increases, the part of the domain for which $a(x, t) \simeq \sqrt{2(\alpha+\beta)}$ increases beyond region (0), and extends to essentially all positions and times for large $\sigma$'s. This evolution of the drift with $\sigma$ is illustrated in Fig. 2.

## 4. Resolution of the Kolmogorov equation

We turn now to the resolution of the Kolmogorov equation. As we shall see below, this equation can, in the particular case we consider here, also be solved exactly. Before proceeding to the description of this exact solution, we find it useful nevertheless once again to discuss briefly the two limiting cases.

### 4.1. Limiting cases

#### 4.1.1. Case $\sigma^2 \to 0$

If we just set $\sigma$ to 0, the Kolmogorov equation reduces to:

$$\begin{cases} \dfrac{\partial m}{\partial t} + \dfrac{\partial(a(x,t)m)}{\partial x} = 0 \\ m(x, t=0) = m_0(x), \end{cases} \tag{31}$$

with the velocity $a(x, t)$ given by Eq. (19).

Noting $\frac{\mathrm{D}}{\mathrm{D}t}$ the total derivative attached to the flow $a(x, t)$, Eq. (31) reads $\mathrm{D}m/\mathrm{D}t = 0$ in regions (0) and (2) of Fig. 1, $\mathrm{D}m/\mathrm{D}t = -x/(T-t)$ in region (1), and $\mathrm{D}m/\mathrm{D}t = -x/(\bar{t}-t)$ in region (3). This yields (see e.g. Ref. [18])

$$m(x,t) = \begin{cases} m_0\left(x - \sqrt{2(\alpha+\beta)}t\right) & \text{in Region (0)} \\ \dfrac{T}{T-t} m_0\left(x\dfrac{T}{T-t}\right) & \text{in Region (1)} \\ m_0\left(x - \sqrt{2(\alpha-\gamma)}t\right) & \text{in Region (2)} \\ \dfrac{\bar{t}}{\bar{t}-t} m_0\left(x\dfrac{\bar{t}}{\bar{t}-t}\right) & \text{in Region (3).} \end{cases} \tag{32}$$

In other words, all agents starting at $t = 0$ from a position $x_0 < -\sqrt{2(\alpha+\beta)}T$ will arrive after $T$, all agents starting at $t = 0$ from a position $x_0 > -\sqrt{2(\alpha-\gamma)}T$ will arrive before $T$, and all agents between $-\sqrt{2(\alpha+\beta)}T$ and $-\sqrt{2(\alpha-\gamma)}T$ will arrive *exactly at time* $T$. Therefore, at $\sigma = 0$, the function $\rho(x_0, t)$ needed to define the self consistent condition Eq. (11) become singular at $t = T$. We shall see below how this behavior is regularized for a small but finite $\sigma$.

#### 4.1.2. Case $\sigma^2 \to +\infty$

When $\sigma^2 \to +\infty$, we have seen that the drift velocity $a(x, t)$ tends toward the constant value $a_0 = \sqrt{2(\alpha+\beta)}$. Intuitively, this is due to the fact that when optimizing her drift, the agent does so not so much having in minds the median arrival time but rather try to ensure herself against possible late arrival due to the noise. In this limit the motion of the agents

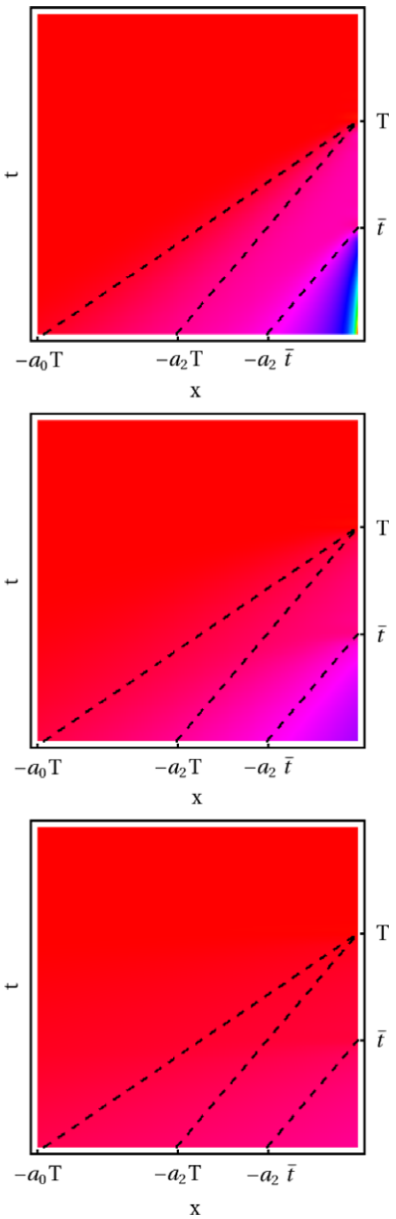


**Fig. 2.** Evolution of the drift with $\sigma$ in heat representation. Top : $\sigma = 0.5$, middle: $\sigma = 2$, bottom: $\sigma = 7[\alpha = 2, \beta = 1, \gamma = 1, \bar{t} = 1, T = 2]$.

becomes extremely simple, as the mass of participants is transported by an advection–diffusion equation with constant drift $a_0$ and diffusion coefficient $\frac{\sigma^2}{2}$. Forgetting for now the small technicalities existing for small $|x|$, this would imply that we should simply consider the Kolmogorov equation with constant drift

$$\begin{cases} \dfrac{\partial m}{\partial t} + a_0 \dfrac{\partial m}{\partial x} - \dfrac{\sigma^2}{2} \dfrac{\partial^2 m}{\partial x^2} = 0 \\ m(x = 0, t) = 0 \\ m(x, t = 0) = m_0(x). \end{cases} \tag{33}$$

Explicit solutions of Eq. (33) are well known [20], and in particular the elementary solution for an initial distribution $m_0(x) = \delta(x - x_0)$ is given by

$$G^{CD}(x, t|x_0) = \frac{1}{\sigma\sqrt{2\pi t}} \left\{ \exp\left( -\frac{(x - x_0 - a_0 t)^2}{2\sigma^2 t} \right) - \exp\left( \frac{2a_0 x}{\sigma^2} \right) \exp\left( \frac{(x + x_0 + a_0 t)^2}{2\sigma^2 t} \right) \right\}. \tag{34}$$

We shall see below that this expression indeed provides the leading large $\sigma$ asymptotic approximation of the true solution.

### 4.2. Full resolution of the coupled problem

We turn now to the solution of the Kolmogorov equation for an arbitrary $\sigma$. For this purpose, let us write the agent density as [21]

$$m(x,t) = \mathrm{e}^{-u(x,t)/\sigma^2}\boldsymbol{\Gamma}(x,t), \tag{35}$$

with $u(x,t)$ the solution of the (HJB) equation (6), which is thus such that $-\partial_x u = a(x,t)$. Inserting Eq. (35) into Eq. (8), we find that:

$$\sigma^2\partial_t\boldsymbol{\Gamma} - \frac{\sigma^4}{2}\partial^2_{xx}\boldsymbol{\Gamma} = \boldsymbol{\Gamma}\left(\frac{\partial u}{\partial t} - \frac{1}{2}\left(\frac{\partial u}{\partial x}\right)^2 + \frac{\sigma^2}{2}\frac{\partial^2 u}{\partial x^2}\right). \tag{36}$$

But $u(x,t)$ is a solution of the (HJB) equation (6). The right hand side of Eq. (36) is therefore uniformly zero, and this equation can be written as a simple diffusion equation without drift

$$\begin{cases} \dfrac{\partial\boldsymbol{\Gamma}}{\partial t} - \dfrac{\sigma^2}{2}\dfrac{\partial^2\boldsymbol{\Gamma}}{\partial x^2} = 0 \\ \boldsymbol{\Gamma}(x=0,t) = 0 \quad \text{and} \quad \boldsymbol{\Gamma}(x,t=0) = \mathrm{e}^{\frac{u(x,t=0)}{\sigma^2}} m_0(x). \end{cases}$$

Noting

$$G_0^{\mathrm{abs}}(x,t|x_0) = (G_0(x,t|x_0) - G_0(x,t|-x_0)) \tag{37}$$

the elementary solution (Green's function) for the diffusion without drift but with absorbing boundary in zero, obtained straightforwardly using the method of image from the elementary solution of the free diffusion problem Eq. (22), we find

$$\boldsymbol{\Gamma}(x,t) = \int_{-\infty}^{0} G_0^{\mathrm{abs}}(x,t|x_0)\mathrm{e}^{u(x_0,t=0)/\sigma^2} m_0(x_0)\,\mathrm{d}x_0.$$

Inserting the expression Eq. (27) of $u(x,t)$ yields

$$m(x,t) = \phi(x,t)\int_{-\infty}^{0}\frac{G_0^{\mathrm{abs}}(x,t|x_0)}{\phi(x_0,t=0)} m_0(x_0)\,\mathrm{d}x_0, \tag{38}$$

and in particular, setting the initial distribution as a Dirac mass located in $x_0$, we get for the elementary solution

$$G(x,t|x_0) = \frac{\phi(x,t)}{\phi(x_0,t=0)} \times G_0^{\mathrm{abs}}(x,t|x_0). \tag{39}$$

The self consistency Eq. (11) only involves $\rho(x_0,t=T)$, and thus we need to compute $G(x,t|x_0)$ at $t=T$, which is in the range for which $\phi$ can be expressed through Eq. (A.5). After integration over the final position $x$ we obtain

$$\rho(x_0,T) = \frac{\mathrm{e}^{-c_0(T)/\sigma^2}}{\phi(x_0,t=0)}\int_{-\infty}^{0}\mathrm{d}x\,\mathrm{e}^{xa_0/\sigma^2}G_0^{\mathrm{abs}}(x,T|x_0). \tag{40}$$

With Eq. (40) we actually obtain an *exact solution* of the first part of the program defined in Section 2. Indeed, both $\phi(x_0,t=0)$ (cf. Appendix A) and the integral on the r.h.s. of Eq. (40) can be written explicitly in terms of the complementary error function erfc and elementary functions. We thus have an *exact* and *explicit* expression for the quantity $\rho(x_0,T)$ required to address self consistency (cf. Eq. (11) and the discussion below). Before we do so however, we will consider the large and small $\sigma$ asymptotics of this exact result; and relate them to the expressions obtained in Section 4.1.

### 4.3. Asymptotic regimes

For large $\sigma$'s, and more specifically when the condition (B.4) is met, the approximation (30) can be used in Eq. (39) and we get

$$\begin{aligned} G(x,T|x_0) &= \mathrm{e}^{-\frac{1}{\sigma^2}(c(T)-c(0))}\mathrm{e}^{-\frac{1}{\sigma^2}(x_0-x)a_0}\,G_0^{\mathrm{abs}}(x,T|x_0) \\ &= \mathrm{e}^{-\frac{1}{2\sigma^2}(c(0)-c_0(0))}G^{\mathrm{CD}}(x,T|x_0). \end{aligned} \tag{41}$$

Up to the factor $\mathrm{e}^{-\frac{1}{2\sigma^2}(c(0)-c_0(0))} = 1 + O(\sigma^{-2})$, one thus recovers the elementary solution of the convection–diffusion equation (34) so that

$$\begin{aligned} \rho(x_0,T) &\simeq \int_{-\infty}^{0} G^{\mathrm{CD}}(x,T|x_0)\,\mathrm{d}x \\ &= \frac{1}{2}\left[\mathrm{erfc}\left(\frac{x_0+a_0T}{\sqrt{2\sigma^2T}}\right) - \mathrm{e}^{-2a_0x_0/\sigma^2}\mathrm{erfc}\left(-\frac{x_0-a_0T}{\sqrt{2\sigma^2T}}\right)\right]. \end{aligned} \tag{42}$$

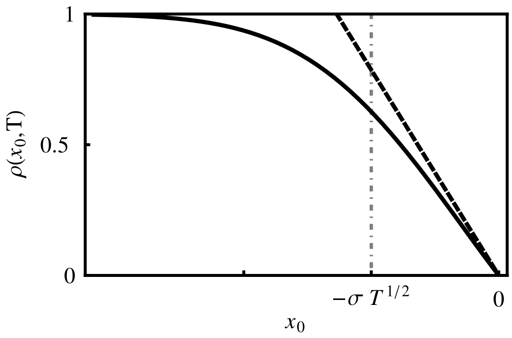


**Fig. 3.** Solid line: asymptotic form Eq. (42) of $\rho(x_0, T)$ valid for large $\sigma$'s (i.e. under the condition (B.4)) and for $T$ such that ($a_0T \ll \sqrt{\sigma^2 T}$); Dashed: linear behavior Eq. (43) corresponding to the diffusion regime. [$\alpha = 2$, $\beta = 1$, $T = 1.1$ and $\sigma^2 = 49$.]

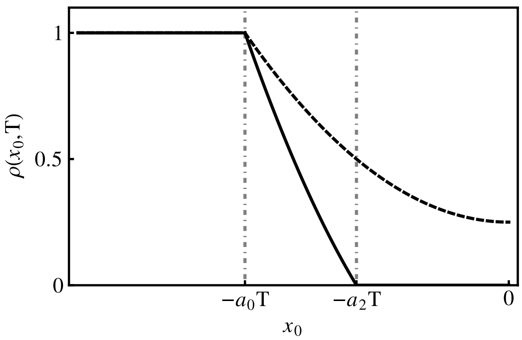


**Fig. 4.** Asymptotic form of $\rho(x_0, T)$ for $\sigma$ small. Full line: $T > \bar{t}$; dashed: $T = \bar{t}$. [$\alpha = 2,\ \beta = 1,\ \gamma = 1,\ T = 2$.]

As we shall see in Section 5, this equation will be mainly useful in the diffusion regime where, beyond the condition (B.4) one may assume $x_0^2 \ll \sigma^2 T$. In this case Eq. (42) simplifies to (see Fig. 3)

$$\rho(x_0, T) = -\frac{2x_0}{\sqrt{2\pi\sigma^2 T}} \exp\left(-\frac{a_0^2 T}{2\sigma^2}\right). \tag{43}$$

Another asymptotic regime is obtained when the diffusion time $t_\sigma = x_0^2/\sigma^2$ is much larger than the drift time $t_d(x_0)$ (defined as the arrival time for the participant located at $x_0$ when $\sigma = 0$; thus here $t_d = |x_0|/a_0$ in region (0), $t_d = |x_0|/a_2$ in region (2), $t_d = T$ in region (1), and $t_d = \bar{t}$ in region (3)). For small $\sigma$'s this condition applies for most of the $x_0$ axis, except a boundary layer in the region $x_l \le x \le 0$, where $x_l$ is defined by $\sigma^2 \gg \frac{x_l^2}{\bar{t}}$.

In this asymptotic regime the Laplace method [22] can be used to evaluate the integrals occurring in Eq. (40) (we detail the calculation of $\phi(x_0, 0)$, in Appendix C, and the evaluation of the numerator is done along the same lines). We get

$$\rho(x_0, T) = \begin{cases} 1, & \text{if } x_0 \le -Ta_0 \\ \dfrac{\frac{x_0^2}{T^2} - a_2^2}{a_0^2 - a_2^2} & \text{if } -Ta_0 \le x_0 \le -Ta_2 \\ 0, & \text{if } x_0 \ge -Ta_2. \end{cases} \tag{44}$$

We see in this way how the singular behavior of the strict $\sigma = 0$ limit is regularized for small but non-zero sigma (cf. Eq. (32) and the discussion below). An illustration of this function is given in Fig. 4.

As discussed in Appendix C, Eq. (44) requires to be valid that $T$ is sufficiently distant from $\bar{t}$ (cf. Eq. (C.6) for the precise condition). If for instance $T = \bar{t}$ one would have instead

$$\rho(x_0, T = \bar{t}) = \begin{cases} 1, & \text{if } x_0 \le -\bar{t}a_0 \\ \dfrac{\frac{x_0^2}{2\bar{t}^2} - c_3'}{c_0' - c_3'} & \text{if } -\bar{t}a_0 \le x_0 \le 0 \end{cases} \tag{45}$$

with $c_0' = (\alpha + \beta)$ and $c_3' = -\gamma$ the slopes of $c(\tau)$ for $\tau > T$ and for $\tau < \bar{t}$ respectively. The shape of this function is illustrated as a dashed line in Fig. 4. Note that Eq. (45) is valid only for $|x_0|$ large enough that the motion is dominated by convection, and in particular does not apply at $x_0 = 0$, where in any case one should have $\rho(x = 0, t) \equiv 0$. The condition of validity of Eq. (45) can be shown for small $\sigma$'s to read $|x_0| \gg (a_0^2\bar{t}^{3/2}/\sigma)\exp(-\gamma\bar{t}/\sigma^2)$.

## 5. Self-consistent condition

It is time now to answer the question: "when does the meeting start?". Answering this question implies taking into account the coupling between agents mediated by the mean-field condition, and means in practice solving the self-consistent Eq. (11) with the form of the function $\rho(x_0, T)$ given by Eq. (40).

One thing worth noticing already is that having an explicit expression for $\rho(x_0, T)$ could provide an alternative route – to the one given by Guéant et al. [15] – for the proof of the existence of the solution for $T$, which is associated with the continuity of $\rho(x_0, T)$. This route is of course restricted to particular models such as the present one. In this section however, we are not so much interested in this "proof of existence" than into a qualitative description, and whenever possible a more quantitative one, of the behavior of $T$ as a function of the various parameters of the problem.

As stressed in Section 2, among these parameters the initial density of agents $m_0(x_0)$ plays a specific role. Indeed, the other parameters, namely $(\alpha, \beta, \gamma)$ characterizing the cost function $c(t)$, the official time of seminar $\bar{t}$, and the intensity of the noise $\sigma$, enter through the function $\rho(x_0, T)$, and their specific role has been discussed at length when analyzing the property of this function. The initial density $m_0(x_0)$ on the other hand enters only now in the discussion since the behavior of the agents is coupled only through $T$. Furthermore, $m_0(x_0)$ being a function, it may have a infinite variety of shape, and it is clearly not realistic to discuss the more esoteric among them. In the following, we shall therefore restrict our study to initial distributions $m_0(x_0)$ that can be characterized by their mean value $\langle x_0 \rangle$ and their variance $\Sigma^2$, and thus implicitly assume that $\Sigma$ sets a scale below which the variations of $m_0(x_0)$ are small.

The mean value $\langle x_0 \rangle$ will mainly determine how much $T$ is influenced, or not, by the official time of the seminar. Clearly, if $\langle x_0 \rangle$ is close enough to zero, almost all the mass will be close to the origin of the negative semiaxis and there is a point where the noise $\sigma$ will be sufficient to fill the seminar room, and the quorum will be met before the official beginning time, giving $T = \bar{t}$. For larger, but not too large $|\langle x_0 \rangle|$, agents have a real possibility to arrive in the seminar room near, or even a bit before, $\bar{t}$, which will influence their optimization choices, and eventually lead to a self consistent $T$ which depend on $\bar{t}$, although $T > \bar{t}$. For very negative $\langle x_0 \rangle$ on the other hand, there is very little chance for an agent to arrive before $\bar{t}$. Indeed, as we have seen in Section 3, it is never optimal for an agent to choose a drift velocity higher than $a_0 = \sqrt{2c'_0}$, where $c'_0$ is the slope of the cost function $c(t)$ for time $t \geq T$. As a consequence, if $|\langle x_0 \rangle| \gg a_0 \bar{t}$, the agents determining the quorum condition (i.e. the last ones to arrive before the quorum is met) will never consider the possibility to arrive before $\bar{t}$, and thus the official starting time of the seminar will play no role in setting $T$.

The parameter $\Sigma$ on the other hand will balance the effect of the Kolmogorov diffusion term in the determination of $T$. Indeed, a set of agents starting from an identical initial location will have spread on a distance $\sigma\sqrt{T}$ at time $T$. So for

$$\Sigma \ll \sigma\sqrt{T} \tag{46}$$

the diffusion will essentially erase any of the initial features of $m_0(x_0)$, while for

$$\Sigma \gg \sigma\sqrt{T} \tag{47}$$

diffusion plays little role for the transport of $m(x, t)$. Keeping in mind this general picture, we turn now to a more detailed description of the various limiting cases.

### 5.1. Self consistent condition in the diffusion regime (large $\sigma$'s)

We characterize the diffusion regime by the fact that $\sigma$ is large enough (condition (B.4)) and that, for the relevant positions $x_0$, the time of drift is much larger than the diffusion time, i.e. here

$$|x_0| a_0 \ll \sigma^2. \tag{48}$$

We consider successively narrow initial distributions and wider ones.

#### 5.1.1. Narrow initial distributions

A narrow initial distribution corresponds to a configuration where the initial width $\Sigma$ is significantly smaller than the spreading $\sigma\sqrt{T}$ acquired because of the noise during the transport—the notion of narrow initial distribution is thus $\sigma$-dependent, and this configuration will typically be met when the noise is rather large. In that case the details of the initial distribution become irrelevant and $m_0(x_0)$ can be approximated by a Dirac function $\delta(x_0 - \langle x_0 \rangle)$. The integral Eq. (11) therefore just becomes the simple standard equation

$$\rho(\langle x_0 \rangle, T) = \bar{\theta}, \tag{49}$$

in which $m_0(x_0)$ is entirely characterized by its mean value $\langle x_0 \rangle$. In the large noise regime the conditions (B.4) and (48) hold. Furthermore, we will see that for the self consistent value of $T$ obtained at the end of the process one gets

$$\langle x_0 \rangle^2 \ll \sigma^2 T. \tag{50}$$

Under these conditions, we can use the approximation Eq. (43) for $\rho(x_0, T)$, and Eq. (49) reads $e^{-u}/\sqrt{\pi u} = \bar{\theta}\sigma^2/(a_0\langle x_0\rangle)^2$, with $u \equiv a_0^2 T/2\sigma^2$.

If $\bar{\theta} \gg a_0|\langle x_0\rangle|/\sigma^2$, which deep in the diffusive regime will usually hold except for very small $\bar{\theta}$, we get in leading $1/\sigma$ order, $T = \sup(T^*, \bar{t})$, with

$$T^* \simeq \frac{2}{\pi}\frac{\langle x_0\rangle^2}{\sigma^2\bar{\theta}^2}. \tag{51}$$

(If $T = T^*$, the condition (50) then just amounts to have $\bar{\theta} \ll 1$, which we assume. If $T = \bar{t}$, the condition (50) is even more easily fulfilled.) $T^*$ is proportional to $\langle x_0\rangle^2$, and we recover the intuitive result that if the initial distribution is located too close from the seminar room, the noise fills this latter before the official starting time, giving $T = \bar{t}$.

For very small $\bar{\theta}$, there is a range of $\sigma^2$ for which even in the diffusive regime (48) one has $\bar{\theta} \ll a_0|\langle x_0\rangle|/\sigma^2$. In that case

$$T \simeq \frac{2\sigma^2}{a_0^2}\log\left(\frac{a_0|\langle x_0\rangle|}{\sqrt{\pi}\sigma^2\bar{\theta}}\right), \tag{52}$$

and one can check that (50) holds.

#### 5.1.2. Wide initial distributions

When $\Sigma \gg \sigma\sqrt{T}$ – which since we assume here $\sigma^2 \gg (c(0) - c_0(0))$ implies fairly large $\Sigma$'s – the convolution with a Gaussian of width $\sigma\sqrt{T}$ barely changes the distribution. Everything appears then as if the Kolmogorov approximation was dominated by convection.

Let us introduce $x_\theta$ such that

$$\int_{-\infty}^{x_\theta} m_0(x_0)\mathrm{d}x_0 = \bar{\theta}, \tag{53}$$

which is thus the position of the participant such that a fraction $\bar{\theta}$ of the agents is more distant from the origin. The beginning of the seminar is entirely determined by the time at which the agent starting from this location and evolving in a deterministic way under the influence of the drift $a(x, t)$ (i.e. ignoring the effect of the noise) will arrive.

In the large noise limit that we consider here, the drift is constant and equal to $a_0$, and this just gives

$$T = -\frac{x_\theta}{a_0}.$$

(The fact that $x_\theta$ is necessarily of the order of or larger than $\Sigma$, together with (B.4), implies $T > \bar{t}$.)

### 5.2. Self consistent condition in the convection regime

In the convection regime, and more precisely under the conditions (C.4)–(C.6), the function $\rho(x, T)$ is well approximated by Eq. (44). We consider below how Eq. (11) can be solved for this form of $\rho(x, T)$ for different ranges of the initial distribution's width $\Sigma$.

#### 5.2.1. Narrow initial distributions

For very narrow initial conditions, Eq. (11) can be as before replaced by Eq. (49) which, with Eq. (44), is solved as

$$\begin{aligned} T &= \frac{|\langle x_0\rangle|}{\bar{a}(\bar{\theta})} \\ \bar{a}(\bar{\theta}) &\equiv \sqrt{a_2^2 + (a_0^2 - a_2^2)\bar{\theta}}. \end{aligned} \tag{54}$$

For small $\bar{\theta}$, Eq. (54) corresponds to $\bar{a}(\bar{\theta}) \simeq a_2$, i.e. to a $\langle x_0\rangle$ near the lower border of region(1), for which the condition (C.5) might not be fulfilled. Re-inserting Eq. (54) into (C.5) we indeed see that Eq. (54) applies only if

$$\sigma^2 \ll a_0|\langle x_0\rangle|\bar{\theta}^2. \tag{55}$$

For larger $\sigma$'s – or smaller $\bar{\theta}$ – we need to use for $\phi(x, 0)$ in Eq. (40) the uniform approximation Eq. (C.7) valid for $x$ near $-a_2 T$. Writing $T = T_0 + \delta T$ with $T_0 = |\langle x_0\rangle|/a_2$, we find then in linear order

$$\frac{\delta T}{T_0} = \frac{\sqrt{\pi}}{2}\frac{a_0^2 - a_2^2}{a_2^2}(\bar{\theta}_0 - \bar{\theta}), \tag{56}$$

with $\bar{\theta}_0 \equiv \sqrt{8\sigma^2/\pi|\langle x_0\rangle|a_2}(a_2^2/(a_0^2 - a_2^2))$.

The expressions (54)–(56) are clearly independent of the official beginning time of the seminar $\bar{t}$. The condition (C.6), which is actually required for Eq. (44) to apply, indeed implies that $T$ is sufficiently above $\bar{t}$ to become independent of this latter.

Once $|\langle x_0\rangle|$ diminishes, and more specifically when it reaches a value close to $a_2\bar{t}$ or smaller, $T$ will on the other hand approach $\bar{t}$. It may be interesting then to determine under which condition one has exactly $T = \bar{t}$, i.e. when the quorum is met *before* the official beginning time $\bar{t}$.

In the convection regime, $\rho(x, T = \bar{t})$ is described by the expression Eq. (45), and the self consistent condition to obtain $T = \bar{t}$ is that

$$\rho(\langle x_0\rangle, T = \bar{t}) < \bar{\theta}. \tag{57}$$

The approximation Eq. (45) is however bounded from below by its value at zero, $\gamma/(\alpha+\beta+\gamma)$, which is typically of order one. If, as we assume, $\bar{\theta}$ is small, Eq. (57) will thus not have any solution for $\langle x_0\rangle$ in the convection regime. As long as the motion of the agents is convective, they will manage to fill the seminar room after, though possibly barely, the official start of the seminar.

If $|\langle x_0\rangle|$ becomes so small that the motion at such distance is dominated by diffusion, then again one can eventually reach a point where the quorum is met before $\bar{t}$, implying $T = \bar{t}$. For small $\sigma$'s it can be shown that this happens when $|\langle x_0\rangle| \simeq \sqrt{\pi/2}(a_0^2\bar{t}^{3/2}/\sigma)(\bar{\theta}/(1-\bar{\theta}))\exp(-\gamma\bar{t}/\sigma^2)$.

### 5.2.2. *Wider initial distributions*

If the width of the initial distribution is non negligible, we need to distinguish two cases. For intermediate values of $\Sigma$, namely for $\Sigma$'s such that once self-consistency is obtained most of the initial distribution is in the range $]-a_0T, -a_2T[$, we can use that in this range the function $\rho(x, T) \simeq (x^2/T^2 - a_2^2)(a_0^2 - a_2^2)$ is a simple polynomial. The convolution with $m_0(x)$ thus simply leads to

$$\int_{-\infty}^{0} \mathrm{d}x_0\, \rho(x, T) m_0(x) = \frac{(\langle x_0\rangle^2 + \Sigma^2)/T^2 - a_2^2}{a_0^2 - a_2^2},$$

and Eq. (54) has just to be replaced by

$$T = \sqrt{\frac{\langle x_0\rangle^2 + \Sigma^2}{a_2^2 + (a_0^2 - a_2^2)\bar{\theta}}}. \tag{58}$$

The constraint that the initial distribution fits within $]-a_0T, -a_2T[$ implies that Eq. (58) applies only if $\Sigma \ll \bar{\theta}|\langle x_0\rangle|$, i.e. for not too small $\bar{\theta}$. For smaller $\bar{\theta}$, explicit (but less transparent) expressions can be written down under the less restrictive condition $\Sigma < (a_0 - a_2)T \simeq ((a_0 - a_2)/a_2)|\langle x_0\rangle|$ for specific forms of the initial distribution (e.g. Gaussian).

If now $\Sigma$ is large not only on the scale $\sigma\sqrt{T}$ but also on the scale $(a_0 - a_2)T$, another approach can be used. Subtracting Eq. (53) to Eq. (11) and neglecting the variation of $m_0(x_0)$ near $x_\theta$ in the whole region (1), we can write that $\int_{-a_0T}^{-a_2T} \frac{x_0^2/T^2 - a_2^2}{a_0^2 - a_2^2}\mathrm{d}x_0 = \int_{-a_0T}^{x_\theta} \mathrm{d}x_0$, which implies

$$T = -\frac{3}{2}|x_\theta|\left(\frac{a_0^2 - a_2^2}{a_0^3 - a_2^3}\right).$$

As before, this results apply only if $T - \bar{t}$ is large enough for Eq. (C.6) to be fulfilled.

## 5.3. *"Phase diagram" of the seminar problem*

The results of the previous subsections can be summarized into "phase diagrams" such as the one shown in Fig. 5 for narrow initial distributions (equivalent phase diagrams can be constructed in the same way for wider initial distributions). Keeping in mind that, except for the transition between $T = \bar{t}$ and $T \neq \bar{t}$, there is of course no true phase transition here, and that the lines representing the limits between various regimes should be thought as crossover regions (thus with a finite extension), we can distinguish the following "phases":

**Region I** corresponds to a motion dominated by convection, and such that the initial distribution is far enough from the seminar room that the initial time of the seminar becomes irrelevant. This region is split into two subregions. In the first one, $\mathbf{I_a}$, $T = |\langle x_0\rangle|/\bar{a}(\bar{\theta})$ with $\bar{a}(\bar{\theta})$ defined by Eq. (54). In the second one, $\mathbf{I_b}$, the fact that $\bar{\theta} \ll 1$ and thus that $\bar{a}(\bar{\theta}) \simeq a_2$ makes it necessary to use the uniform approximation Eq. (C.7) for $\phi(\langle x_0\rangle, T)$. In that case $T = |\langle x_0\rangle|/a_2 + \delta T$ where $\delta T$ is given by Eq. (56).

**Region II** corresponds to a motion dominated by diffusion, and such again that the initial distribution is far enough from the seminar room that the initial time of the seminar is irrelevant. Region **II**, too, has to be divided in two subregions. In $\mathbf{II_a}$, $T = T^*$ with $T^*$ given by Eq. (51). In $\mathbf{II_b}$ the smallness of $\bar{\theta}$ should be taken into account, leading to Eq. (52).

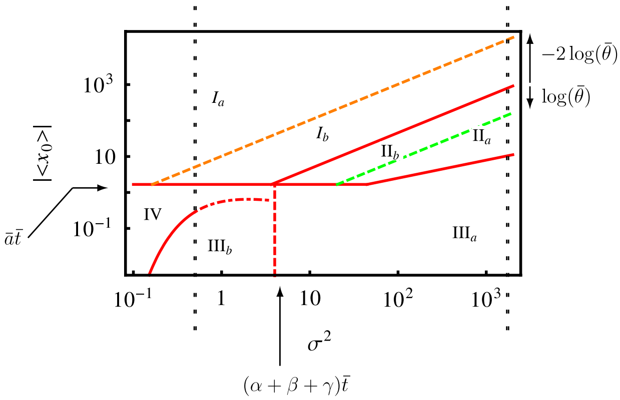


**Fig. 5.** Phase diagram of the seminar problem, in the $(\sigma^2, |\langle x_0\rangle|)$ plane, for narrow initial distributions (see text for the detailed description of the various regimes). The vertical dashed–dotted line correspond to the vertical cuts used in Figs. 6 and 7. For this illustration, the parameters of the problems have been taken as $[\alpha = 2, \beta = 1, \gamma = 1, \bar t = 1, \bar\theta = 0.2]$.

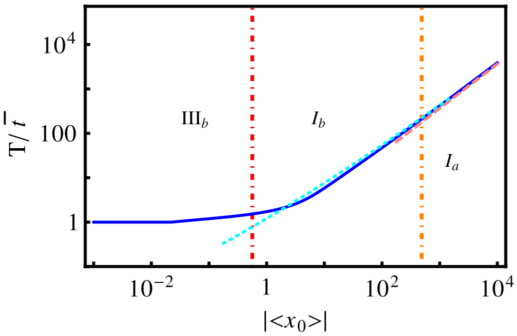


**Fig. 6.** $|\langle x_0\rangle|$ dependence of the time $T$ for narrow initial distributions and a small value of the noise parameter ($\sigma = 0.6$); which corresponds to the left vertical dashed–dotted line in the phase diagram Fig. 5. The numerical labels correspond to those of the different regimes in Fig. 5. Full line: numerical value obtained from the exact expression Eq. (40). Dashed: asymptotic expressions in the corresponding regime (see text). The parameters of the model are the same as in Fig. 5, except for $\bar\theta = 0.1$ which, to enhance readability, has been slightly decreased.

**Region III** corresponds to a motion which can be dominated either by diffusion (region $\mathbf{III_a}$) or by convection (region $\mathbf{III_b}$), but such that in any case the quorum is met before the official beginning time of the seminar. This region thus correspond to the phase $T = \bar t$.

**Region IV** corresponds finally to a configuration such that the quorum is met slightly after the official beginning time of the seminar, so that $T$ is different from, but close to, $\bar t$.

As an illustration, we show in Figs. 6 and 7 two vertical cuts in this phase diagram, in which are displayed the variations of the self-consistent time $T$ as a function of $\langle x_0\rangle$ for two (fixed) values of the noise $\sigma$, one "small" and one "large". For the small diffusion coefficient case Fig. 6 we observe, as expected from the phase diagram, a transition between a domain where $T = \bar t$ and a domain where $T = |\langle x_0\rangle|/\bar a(\bar\theta)$. For the large $\sigma$ case Fig. 7, we observe, again as predicted from the phase diagram, a richer behavior, with the same limiting behaviors for very large and very small $|\langle x_0\rangle|$, but a larger number of intermediate regimes.

## 6. Conclusion

In this article, we have considered a simple toy-model, sharing many of the characteristic features of generic mean field games, but with the essential simplification that the "mean field" actually reduces to a simple number (the actual starting time of the seminar $T$). The study of this problem can then be divided in two essentially independent parts: on one hand the resolution, for arbitrary $T$, of the system of partial differential equations (6)–(8) describing the coupling between the agents optimization decisions and their motion; and on the other hand the self-consistent condition Eq. (11) determining the value of $T$.

The first part of this program can be performed essentially completely. Indeed an explicit expression Eq. (40) can be obtained on a very general basis for the function $\rho(x_0, T)$ required to discuss self-consistency. From this general result transparent asymptotic expressions are derived in the relevant limiting regimes (cf. e.g. Eqs. (42), (44) and (45)). From these and the self-consistent condition Eq. (11) the qualitative behavior of $T$, and explicit expressions in many limiting regimes of interest, can be obtained.

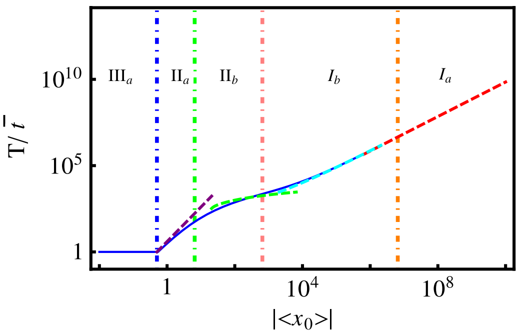


**Fig. 7.** $|\langle x_0\rangle|$ dependence of the time $T$ for narrow initial distributions and a large value of the noise parameter ($\sigma = 40$); which corresponds to the right vertical dashed–dotted line in the phase diagram Fig. 5. The numerical labels correspond to those of the different regimes in Fig. 5. Full line: numerical value obtained from the exact expression Eq. (40). Dashed: asymptotic expressions in the corresponding regime (see text). The parameters of the model are the same as in Fig. 5, except for $\bar{\theta} = 0.01$.

We will not re-list here the various results derived for $T$ in Section 5, but the main features may be summarized as follows. An important point is that the slopes $c'_0 = (\alpha + \beta)$ and $c'_2 = (\alpha - \gamma)$ of the cost function $c(t)$ (cf. Eq. (1)) can be associated with drift velocities $a_0 = \sqrt{2c'_0}$ and $a_2 = \sqrt{2c'_2}$ which fix the scale of the drift velocities of the problem. Together with the characteristic length $l_0$ associated with the initial distribution of agent $m_0(x_0)$ (namely the center of mass, i.e. $l_0 = |\langle x_0\rangle|$ for narrow distributions, or $l_0 = |x_\theta|$ (cf. Eq. (53)) for wide distributions) and the value of the noise parameter $\sigma$, they organize the system "phase diagram" of the problem, and define the relevant limiting regimes. For instance if $|l_0| \gg a_0\bar{t}$ and $|l_0| \gg \sigma\sqrt{\bar{t}}$ the system is completely dominated by convection, and $T \simeq l_0/a$, with $a$ equal to $a_0$ for large noise and to some weighted average between $a_0$ or $a_2$ for smaller noise. Or if $l_0 \ll a_2\bar{t}$ one is essentially guaranteed that the quorum will be met before the official starting time $\bar{t}$ of the seminar, and $T = \bar{t}$, etc.

One point worth being stressed however is that in most circumstances, either the agents start from a location close from the seminar room ($l_0 \ll a_{0,1}\bar{t}$) and $T = \bar{t}$, or they are initially far from the seminar room ($l_0 \gg a_{0,1}\bar{t}$) and $T$ becomes relatively quickly independent of $\bar{t}$. The transition region for which $T > \bar{t}$ but keep some $\bar{t}$ dependence is actually rather restricted. This notion of an effective starting time of the seminar which is independent of its official starting time is clearly a bit disturbing, especially from the viewpoint of the seminar organizer.

This feature can be tracked back to the fact that there exists an initial time $t = 0$ at which all agents start their optimization process and motion, and what we see is that this initial time plays a role which is at least as important as $\bar{t}$ in the determination of $T$. One can of course imagine that this initial time has some physical meaning (e.g. time at which the organizers ring a bell, etc.). One could also modify slightly the model to remove the reference to a uniform initial time and have the time $\tau_0$ at which a given agent leaves her office taken as a parameter entering in the optimization decision. Assuming the marginal cost of not being in ones office is described by some parameter $\eta$, this would amount to replacing the cost function Eq. (3) by

$$J_T[a] = \mathbb{E}\left[c(\tilde{\tau}) - \eta\tau_0 + \frac{1}{2}\int_{\tau_0}^{\tilde{\tau}} a_i^2(\tau)\,\mathrm{d}\tau\right].$$

The problem could be analyzed along the lines of what we have done in this paper, and would lead to a stronger dependence of $T$ in $\bar{t}$.

More generally, various variations of the problem can be easily studied with the approach followed in this paper. In particular, Eqs. (39)–(26) are valid for essentially arbitrary cost functions $c(t)$, and could be studied for instance in circumstances for which the self consistent condition Eq. (11) has more than one solution. The seminar problem is therefore a very versatile model, and the very thorough understanding of its behavior obtained in this work should help develop the intuition on the properties of more generic mean field game models.

## Acknowledgments

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## Appendix A. Exact solution of HJB equation

In this appendix we derive exact expressions for $\phi(x, t)$ (see Eq. (26)).

$$\phi(x, t) = \frac{|x|}{\sqrt{2\pi\sigma^2}}\int_0^\infty \frac{\mathrm{d}\tau}{\tau^{3/2}} \mathrm{e}^{-\frac{1}{\sigma^2}\left(c(t+\tau)+\frac{x^2}{2\tau}\right)} \tag{A.1}$$

where $c(t)$ is piece-wise linear.

$$c(t) = \alpha[t-\bar{t}]_+ + \beta[t-T]_+ + \gamma[T-t]_+. \tag{A.2}$$

Using the explicit expression for $c(t)$ and the fact that $\bar{t} \le T$, one may write

$$\sqrt{2\pi\sigma^2}\phi(x,t) = |x|e^{-\frac{\gamma(T-t)}{\sigma^2}} \int_0^{[\bar{t}-t]^+} \frac{d\tau}{\tau^{3/2}} e^{-\frac{1}{\sigma^2}\left(-\gamma\tau+\frac{x^2}{2\tau}\right)} + |x|e^{-\frac{\alpha(t-\bar{t})+\gamma(T-t)}{\sigma^2}} \int_{[\bar{t}-t]^+}^{[T-t]^+} \frac{d\tau}{\tau^{3/2}} e^{-\frac{1}{\sigma^2}\left((\alpha-\gamma)\tau+\frac{x^2}{2\tau}\right)}$$
$$+ |x|e^{-\frac{\alpha(t-\bar{t})+\beta(t-T)}{\sigma^2}} \int_{[T-t]^+}^{\infty} \frac{d\tau}{\tau^{3/2}} e^{-\frac{1}{\sigma^2}\left((\alpha+\beta)\tau+\frac{x^2}{2\tau}\right)}.$$

We define the following integral [23]

$$\mathcal{I}(a,b,t) = 2\sqrt{\frac{b}{\pi}} \int_0^t \frac{d\tau}{\tau^{3/2}} e^{-a\tau-\frac{b}{\tau}}$$
$$= e^{-2\sqrt{ab}}\text{erfc}\left(\sqrt{\frac{b}{t}} - \sqrt{at}\right) + e^{2\sqrt{ab}}\text{erfc}\left(\sqrt{\frac{b}{t}} + \sqrt{at}\right) \tag{A.3}$$

where the function $\text{erfc}(z)$ is defined on the complex plane, $\lim_{x\to+\infty} \text{erfc}(x) = 0$, $\lim_{x\to-\infty} \text{erfc}(x) = 2$. Accordingly, $\lim_{t\to 0} \mathcal{I}(a,b,t) = 0$ and $\lim_{t\to+\infty} \mathcal{I}(a,b,t) = 2e^{-2\sqrt{ab}}$ for $a > 0$ and $b > 0$.

Thus the function $\phi(x,t)$ reads

$$\phi(x,t) = \frac{1}{2}e^{-\frac{\gamma(T-t)}{\sigma^2}} \mathcal{I}\left(-\frac{\gamma}{\sigma^2}, \frac{x^2}{2\sigma^2}, [\bar{t}-t]^+\right) + \frac{1}{2}e^{-\frac{\alpha(t-\bar{t})+\gamma(T-t)}{\sigma^2}} \left(\mathcal{I}\left(\frac{a_2^2}{2\sigma^2}, \frac{x^2}{2\sigma^2}, [T-t]^+\right)\right.$$
$$\left. - \mathcal{I}\left(\frac{a_2^2}{2\sigma^2}, \frac{x^2}{2\sigma^2}, [\bar{t}-t]^+\right)\right) + \frac{1}{2}e^{-\frac{\alpha(t-\bar{t})+\beta(t-T)}{\sigma^2}} \left(2e^{-\frac{a_0|x|}{\sigma^2}} - \mathcal{I}\left(\frac{a_0^2}{2\sigma^2}, \frac{x^2}{2\sigma^2}, [T-t]^+\right)\right), \tag{A.4}$$

(where $a_0 = \sqrt{2(\alpha+\beta)}$, $a_2 = \sqrt{2(\alpha-\gamma)}$).

Eq. (A.4) applies for arbitrary values of time and position. It takes however a simpler form in some time intervals:

- For $t \ge T$ the expression reduces to

$$\phi(x,t) = e^{-\frac{c(t)}{\sigma^2}} \exp\left(-\frac{1}{\sigma^2}\sqrt{2(\alpha+\beta)}|x|\right)$$
$$= e^{-\frac{\alpha(T-\bar{t})}{\sigma^2}} \exp\left(\frac{a_0^2(T-t) - a_0|x|}{2\sigma^2}\right). \tag{A.5}$$

- For $\bar{t} \le t \le T$ Eq. (A.4) can be written as

$$\phi(x,t) = \frac{e^{-\alpha(T-\bar{t})/\sigma^2}}{2}\left[\exp\left(\frac{a_2^2(T-t)+2a_2x}{2\sigma^2}\right)\text{erfc}\left(\frac{-x-a_2(T-t)}{\sqrt{2\sigma^2(T-t)}}\right)\right.$$
$$+ \exp\left(\frac{a_2^2(T-t)-2a_2x}{2\sigma^2}\right)\text{erfc}\left(\frac{-x+a_2(T-t)}{\sqrt{2\sigma^2(T-t)}}\right)$$
$$+ \exp\left(\frac{a_0^2(T-t)+2a_0x}{2\sigma^2}\right)\text{erfc}\left(\frac{x+a_0(T-t)}{\sqrt{2\sigma^2(T-t)}}\right)$$
$$\left. - \exp\left(\frac{a_0^2(T-t)-2a_0x}{2\sigma^2}\right)\text{erfc}\left(\frac{-x+a_0(T-t)}{\sqrt{2\sigma^2(T-t)}}\right)\right]. \tag{A.6}$$

- At $t = 0$ Eq. (A.4) reads

$$\phi(x,0) = \frac{1}{2}e^{-\frac{\gamma T}{\sigma^2}} \mathcal{I}\left(-\frac{\gamma}{\sigma^2}, \frac{x^2}{2\sigma^2}, \bar{t}\right) + \frac{1}{2}e^{\frac{\alpha\bar{t}-\gamma T}{\sigma^2}}\left(\mathcal{I}\left(\frac{\alpha-\gamma}{\sigma^2}, \frac{x^2}{2\sigma^2}, T\right) - \mathcal{I}\left(\frac{\alpha-\gamma}{\sigma^2}, \frac{x^2}{2\sigma^2}, \bar{t}\right)\right)$$
$$+ \frac{1}{2}e^{\frac{\alpha\bar{t}+\beta T}{\sigma^2}}\left(2e^{-\frac{\sqrt{2(\alpha+\beta)}|x|}{\sigma^2}} - \mathcal{I}\left(\frac{\alpha+\beta}{\sigma^2}, \frac{x^2}{2\sigma^2}, T\right)\right). \tag{A.7}$$

Finally, we stress that for large $\sigma$'s, and more precisely under the condition $(\alpha,\beta,\gamma)\bar{t} \ll \sigma^2$ (but irrespective of the value of $T$ and $x$) Eq. (A.6) *provides a good approximation of the exact* $\phi(x,0)$ *for* $(t \le \bar{t})$ ($t$ and $\bar{t}$ can then be set to zero in this equation).

## Appendix B. Evaluation of $\phi(x, t)$ for large $\sigma$'s (diffusion regime)

In this appendix, we evaluate the large $\sigma$ asymptotic of the function $\phi(x, t)$ defined by Eq. (26). For this purpose, let us introduce $c_0(t) = \alpha(t - \bar{t}) + \beta(t - T)$, the linear function such that $c(t) = c_0(t)$ for $t \geq T$, and

$$\phi_0(x, t) \equiv -x \int_0^\infty \frac{d\tau}{\tau} \exp\left(-\frac{c_0(t+\tau)}{\sigma^2}\right) G_0(x, \tau) \tag{B.1}$$

$$= \exp\left(-\frac{1}{\sigma^2}[c_0(t) + a_0|x|]\right) \tag{B.2}$$

(this last expression exactly corresponds to Eq. (A.5), and is obtained in the same way).

The difference between $\phi(x, t)$ and its approximation $\phi_0(x, t)$ can be expressed as

$$|\phi(x, t) - \phi_0(x, t)| = -x \int_0^\infty \frac{d\tau}{\tau} \exp\left(-\frac{c_0(t+\tau)}{\sigma^2}\right) G_0(x, \tau) K(t+\tau) \tag{B.3}$$

where

$$K(\tau) \equiv 1 - e^{-\frac{1}{\sigma^2}(c(\tau) - c_0(\tau))}$$

is a positive (since $c(\tau) \geq c_0(\tau)$) decreasing continuous function which is uniformly zero for $\tau$ larger than $T$. We thus have $|\phi(x, t) - \phi_0(x, t)| \leq \phi_0(x, t)K(t)$.

As soon as

$$\sigma^2 \gg (c(t) - c_0(t)), \tag{B.4}$$

$K(t)$ is $O(\sigma^{-2})$, and therefore

$$\phi(x, t) = \phi_0(x, t)(1 + O(\sigma^{-2})). \tag{B.5}$$

Eq. (B.5) applies whenever the condition (B.4) is fulfilled. In practice however, it is mainly useful if the diffusive regime, i.e. when

$$a_0|x| \ll \sigma^2 \tag{B.6}$$

($a_0 = \sqrt{2(\alpha + \beta)}$). In that case, since Eq. (B.4) morally implies $\sigma^2 \gg |c_0(\tau)|$ (this is clear as soon as $t \leq (\alpha\bar{t} + \beta T)/(\alpha + \beta)$ since then $c_0(t) < 0$, but remains generally true unless $\tau \simeq T$), one has $\phi_0(x, t) = (1 + O(\sigma^{-2}))$, and Eq. (B.5) provides little information on the variations of $\phi(x, t)$.

It may be therefore interesting in this case to compute the $O(\sigma^{-2})$ corrections. Noting that $K(\tau) = 0$ for $\tau > T$, we have

$$\phi(x, t) - \phi_0(x, t) = -x \int_0^{T-t} \frac{d\tau}{\tau} \exp\left(-\frac{c_0(t+\tau)}{\sigma^2}\right) G_0(x, \tau)(K(t) + \delta K(\tau)), \tag{B.7}$$

where $\delta K(\tau) \equiv (K(t+\tau) - K(t))$. The term involving $\delta K$, which is linear in $\tau$ near 0 (and thus do not benefit from the $\tau^{-1/2}$ divergence) can be shown to be $O(\sigma^{-3})$ relative to $\phi_0(x, t)$, and we get

$$\phi(x, t) = (\phi_0(x, t) + K(t))(1 + O(\sigma^{-3})) \tag{B.8}$$

$$= \exp\left(-\frac{1}{\sigma^2}[c(t) + a_0|x|]\right)(1 + O(\sigma^{-3})), \tag{B.9}$$

valid therefore when both conditions (B.4) and (B.6) apply.

## Appendix C. Evaluation of $\phi(x, t)$ for small $\sigma$'s (convection regime)

In this appendix we evaluate the small $\sigma$ asymptotics of $\phi(x, t)$ using the saddle point approximation (in regions (0) and (2)), and more generally the Laplace method.

Introducing

$$\Phi(\tau) = c(t+\tau) + \frac{x^2}{2\tau},$$

the integral we want to compute is of the form

$$\phi(x, t) = \frac{1}{\sqrt{2\pi\sigma^2}} \int_0^\infty d\tau f(\tau) \exp\left(-\frac{\Phi(\tau)}{\sigma^2}\right),$$

with $f(\tau) = -x/\tau^{3/2}$, and will therefore be dominated for small $\sigma$'s by the minima of $\Phi(\tau)$.

The condition $\Phi'(\tau^*) = 0$ leads to the equation

$$\tau^* = -x/\sqrt{2c'(t+\tau^*)} \tag{C.1}$$

which admits a solution in regions (0) and (2) but not in regions (1) and (3). One therefore has to use the saddle point approximation in regions (0) and (2), and boundary contributions in regions (1) and (3).

*C.1. Regions (0) and (2)*

In regions (0) and (2), the stationary point is

$$\tau^* = -x/a_0 \quad \text{[in region (0)]}$$
$$\tau^* = -x/a_2 \quad \text{[in region (2)]},$$

which, with $\Phi''(\tau) = x^2/\tau^3$, gives within the saddle point approximation

$$\phi(x,t) \simeq \exp\left(-\frac{\Phi(\tau^*)}{\sigma^2}\right) = \exp\left(-\frac{\alpha(T-\bar{t})}{\sigma^2}\right)\exp\left(\frac{a_{0,2}^2(T-t)+2a_{0,2}x}{2\sigma^2}\right). \tag{C.2}$$

This approximation is valid as long as $\sigma^2 \ll |x|a_{0,2}$, or in other words as long as the ratio between the drift time $t_d \equiv |x|/a_{0,2}$ and the diffusion time $t_\sigma \equiv x^2/\sigma^2$ is small.

*C.2. Region (1)*

In region (1), there are no solution to Eq. (C.1) as the minima of $\Phi(\tau)$ correspond to a discontinuity of the cost functions $c(t+\tau)$ (at $t+\tau = T$). Linearizing $\Phi(\tau)$ on both side of this discontinuity and neglecting the variation of $f(\tau)$ we get

$$\phi(x,t) \simeq -x\sigma\sqrt{\frac{2(T-t)}{\pi}}\exp\left(-\frac{1}{\sigma^2}\left[c(T)+\frac{x^2}{2(T-t)}\right]\right)\left[\frac{1}{a_0^2(T-t)^2-x^2}-\frac{1}{a_2^2(T-t)^2-x^2}\right]. \tag{C.3}$$

Noting $\hat{a}(x,t) \equiv x/(T-t)$ the drift velocity of an agent within region (1) in the $\sigma \to 0$ limit ($\hat{a}(x,t) \in [a_2, a_0]$), $t_d = (T-t)$ the drift time and $t_\sigma = x^2/\sigma^2$ the diffusion times, Eq. (C.3) applies under the condition that:

$$\frac{t_d}{t_\sigma} = \frac{\sigma^2(T-t)}{x^2} \ll \left(1-\frac{a_0^2}{\hat{a}^2(x,t)}\right)^2 \tag{C.4}$$

$$\frac{t_d}{t_\sigma} = \frac{\sigma^2(T-t)}{x^2} \ll \left(1-\frac{a_2^2}{\hat{a}^2(x,t)}\right)^2 \tag{C.5}$$

$$\sigma^2 \ll \frac{(T-\bar{t})}{2}\left(\hat{a}^2(x,t)-a_2^2\right). \tag{C.6}$$

The two first conditions express that if generally speaking Eq. (C.3) requires that the time of drift is much shorter than the diffusion time, the requirement becomes more and more stringent as $(x,t)$ get closer from the boundaries of region (1) where $\hat{a}(x,t) \to a_0$ or $\hat{a}(x,t) \to a_2$. The last condition signals that Eq. (C.3) is valid only if $T$ differs significantly from $\bar{t}$.

The calculation of $\phi(x,t)$ in region (3) in the small $\sigma$ limit proceeds essentially along the same lines.

*C.3. Uniform approximations*

The conditions (C.4)–(C.5) express that the transition between regions (0) and (1) (i.e. $x \simeq -a_0(T-t)$) as well as the transition between regions (1) and (2) (i.e. $x \simeq -a_2(T-t)$) need to be treated a bit more carefully and require the use of uniform approximations.

For $(\alpha, \beta, \gamma)\bar{t} \ll \sigma^2$, one way to derive these uniform approximations is just to select the dominating contribution of Eq. (A.6). Indeed, deep in region (0) (respectively deep in region(2)), one can check that the third term (respectively the first term) of Eq. (A.6) in which erfc is replaced by its asymptotic value 2 recovers exactly Eq. (C.2). Near $x = -a_0(T-t)$ or $x = -a_2(T-t)$ the uniform approximation amounts to keep the full dependence of the erfc.

For instance near $x = -a_2 T$

$$\phi(x,t) \simeq \frac{e^{-\alpha(T-\bar{t})/\sigma^2}}{2}\exp\left(\frac{a_2^2(T-t)+2a_2x}{2\sigma^2}\right)\operatorname{erfc}\left(\frac{-x-a_2(T-t)}{\sqrt{2\sigma^2(T-t)}}\right). \tag{C.7}$$

This expression will interpolate smoothly between Eqs. (C.2) and (C.3). (This latter can be seen as being obtained using the large $x$ asymptotic $\operatorname{erfc}(x) \simeq \exp(-x^2)/\sqrt{\pi x^2}$ for the four terms of Eq. (A.6)).

### C.4. Drift velocity

With the knowledge of $\phi(x, t)$, the drift velocities in the small $\sigma$ regime can be obtained from the spatial derivative of $u(x, t) = -\sigma^2 \log \phi(x, t)$ (cf. Eqs. (7) and (27)). In leading $\sigma$ order Eqs. (C.2) and (C.3) yield

$$u(x, t) \simeq \begin{cases} \alpha(T - \bar{t}) - \dfrac{a_0^2}{2}(T - t) - a_0 x & \text{in Region } (0) \\ c(T) + \dfrac{x^2}{2(T - t)} & \text{in Region } (1) \\ \alpha(T - \bar{t}) - \dfrac{a_2^2}{2}(T - t) - a_2 x & \text{in Region } (2), \end{cases} \tag{C.8}$$

(the expression of $u(x, t)$ for region (3) can be obtained in the same way). Taking the spatial derivative of these expressions yields the velocities Eq. (19).

Alternatively, one can compute the drift velocity $a(x, t) = \sigma^2 \partial_x \phi(x, t)/\phi(x, t)$ from the spatial derivative $\partial_x \phi(x, t)$, which can be evaluated following exactly the same steps as for $\phi(x, t)$. This of course gives the same result.

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