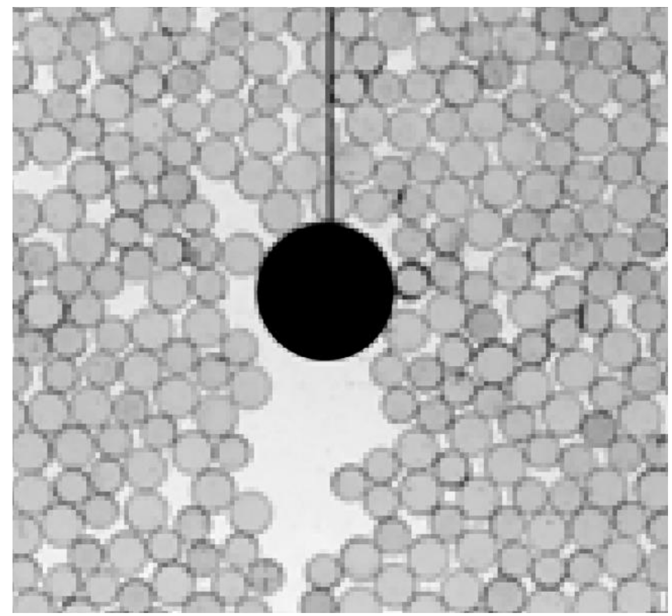
Pedestrians are not grains, but gamers !

[a Mean Field Games description of crowd dynamics]

Denis Ullmo (LPTMS)

[Nicolas et al. Scientific Reports ***9****, 105 (2019)]*



[Seguin et al. PRE ***93****, 012904 (2016)]*

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Introduction

The three levels of pedestrian dynamics

4.1. Pedestrian behavioral levels

In out approach we distinguish choices at the following three levels:

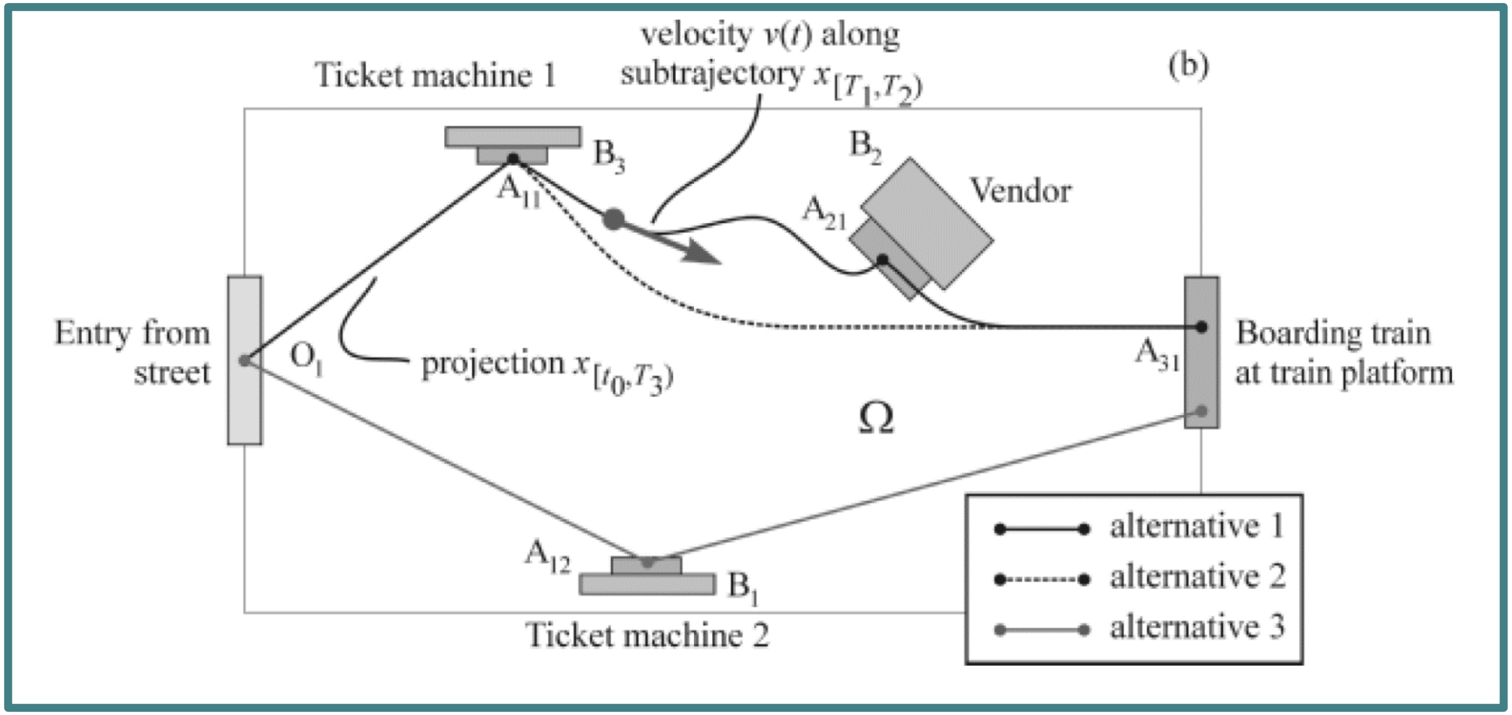
1. Departure time choice, and activity pattern choice (*strategic level*);
2. Activity scheduling, activity area choice, and route-choice to reach activity areas (*tactical level*);
3. Walking behavior (*operational level*).

In this hierarchy, expected utilities at lower levels influence choices at higher levels. Choices at higher levels condition choice sets at lower levels. This article focuses on pedestrian behavior at

*[Hoogendoorn and Bovy, Transportation Research Part B: **38**, 169 (2004)]*

- **Strategic level (where people are heading for)**
  - Departure time choice
  - Activity pattern choice

➢ Tactical level (what route they will take)



- Activity scheduling,
- Activity area choice,
- Route-choice to reach activity areas

- **Operational level** : how they will move along that route in response to interactions with other people

[Abhishek Atre · Jun 29, 2015 (https://www.youtube.com/watch?v=95wrgAvV474)]

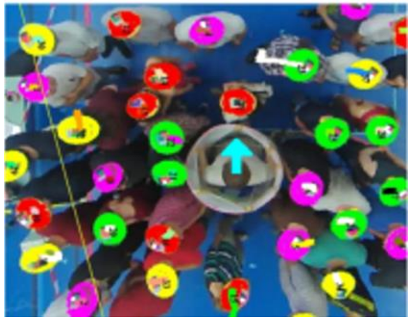
Crossing a static crowd

*[Nicolas et al. Scientific Reports **9**, 105 (2019)]*

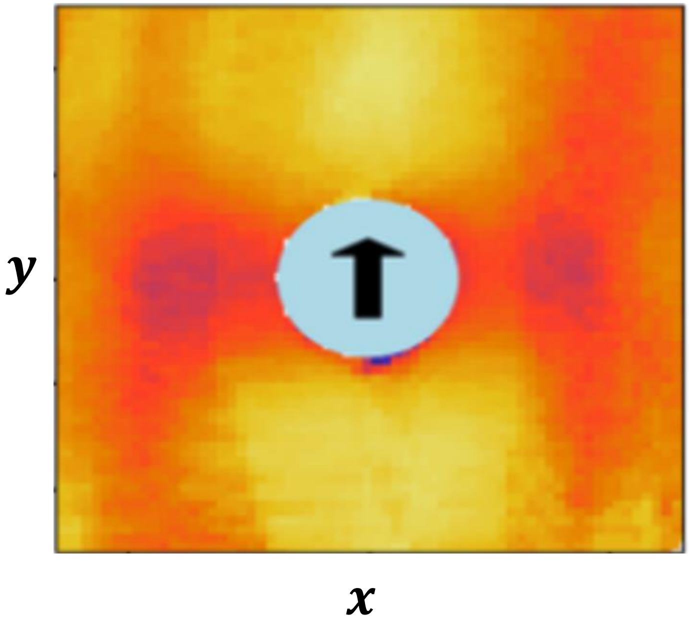
Experimental data

$m_0 = 2.5$ pedestrian/m²

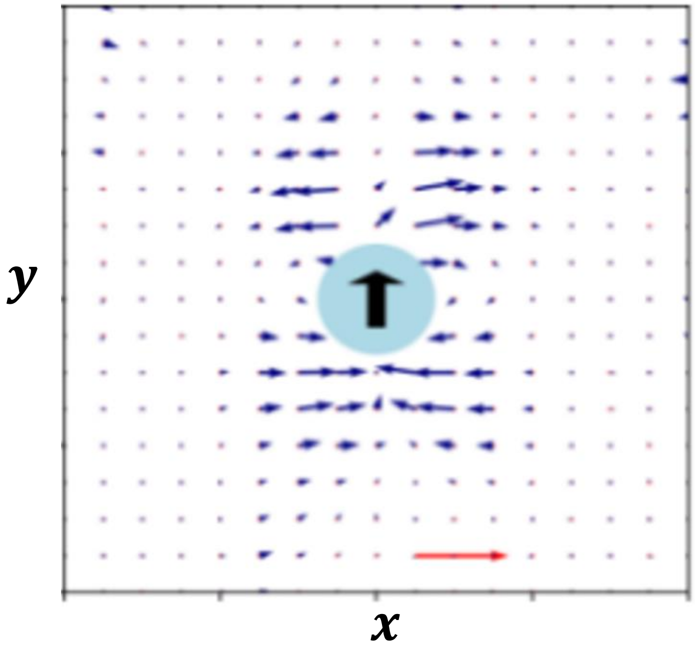
$v_y = 0.6$ m/s $\quad R = 0.32$m



density plot



velocity plot



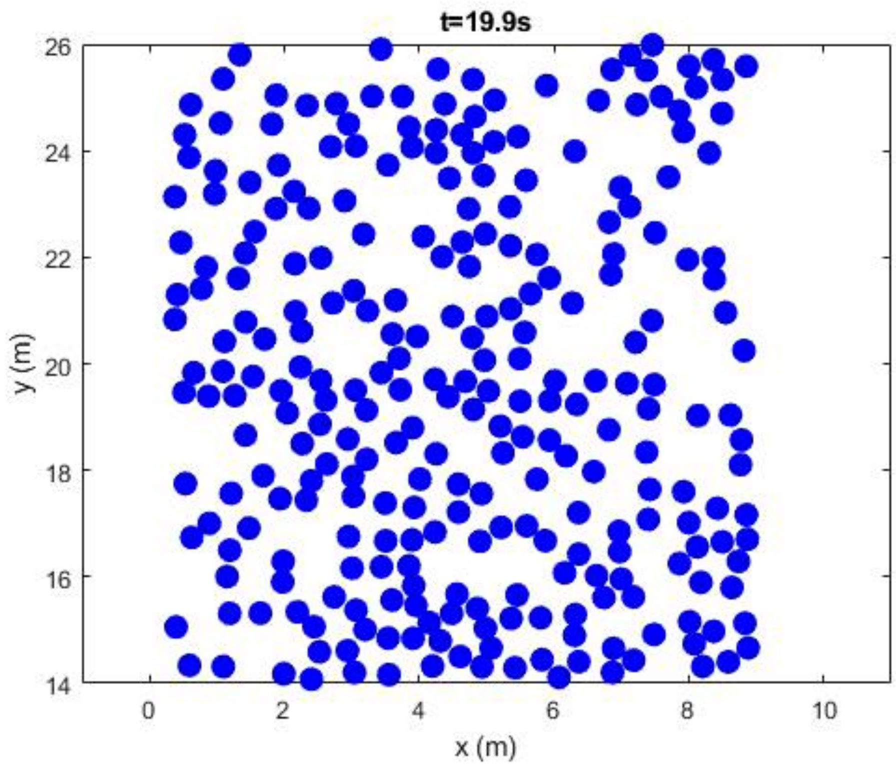
- Rather symmetric density plot :
  - Lower density in front an behind the cylinder
  - Higher density on the wing

- Motion is lateral

*[Nicolas et al. Scientific Reports **9**, 105 (2019)]*

Question : Can we interpret these data with “dynamical” models ?

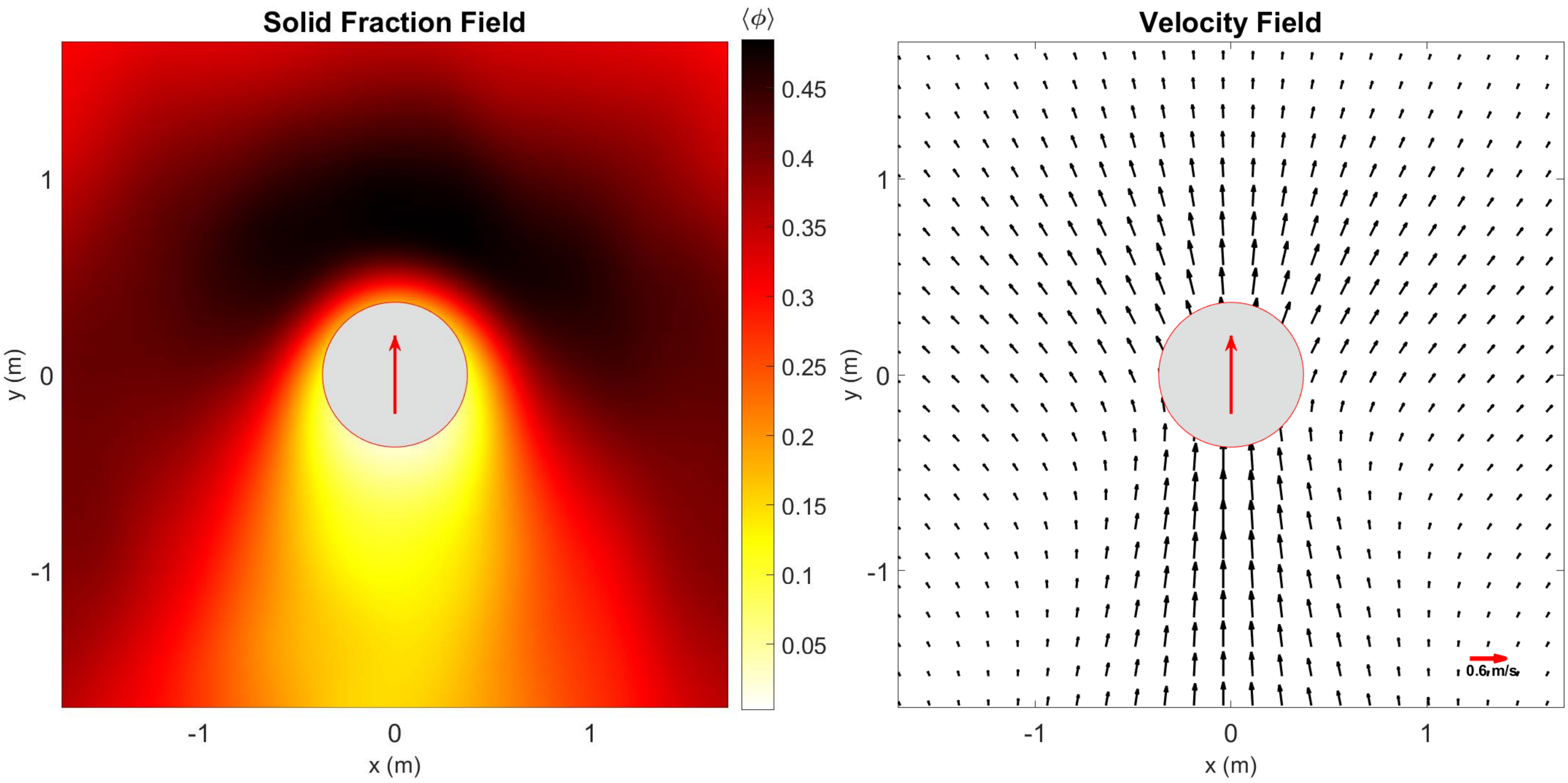
First try : Granular model



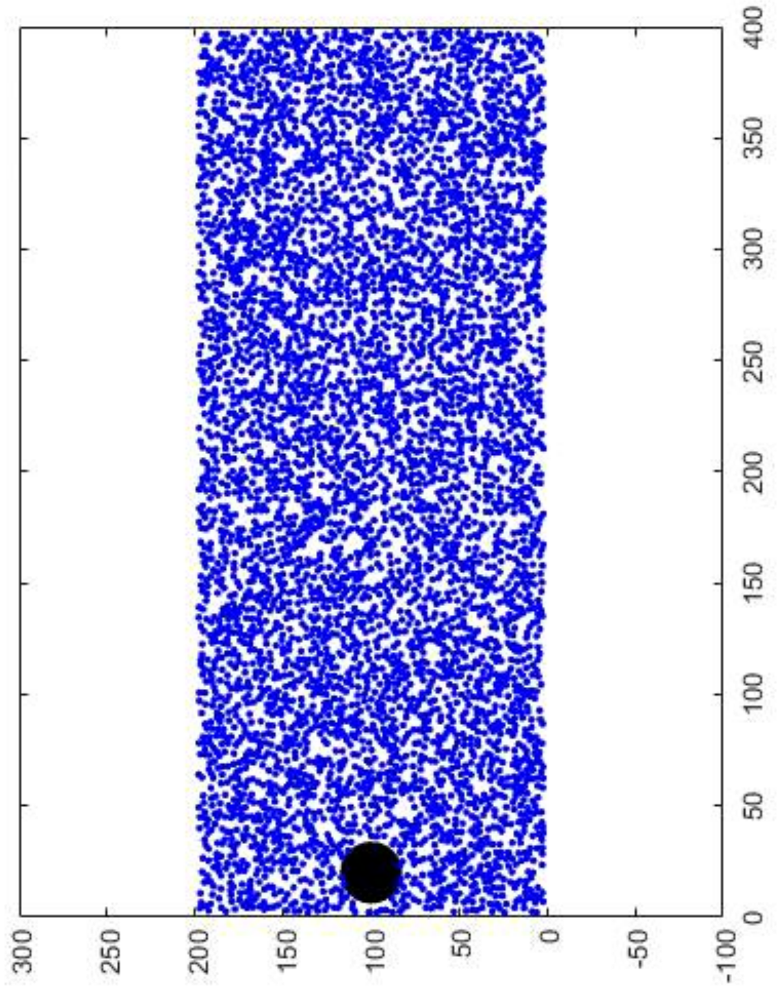
[Details of the calculations : Seguin et al. EPL (2009)]

$m_0 = 2.5$ •/m²
$v_y = 0.6$ m/s
$R = 0.32$m
Gaussian noise + Inelastic collisions
(restitution coef: $e = 0.5$)

Density and velocity fields for the granular model



NB : human crowds vs fish schools



Granular simulation without Gaussian noise

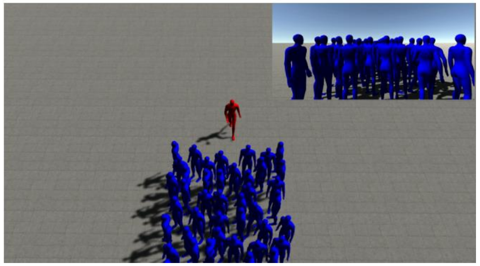
[Tovi Sonnenberg, Aug 21, 2017 (https://youtu.be/fukfYbmgEcs)]

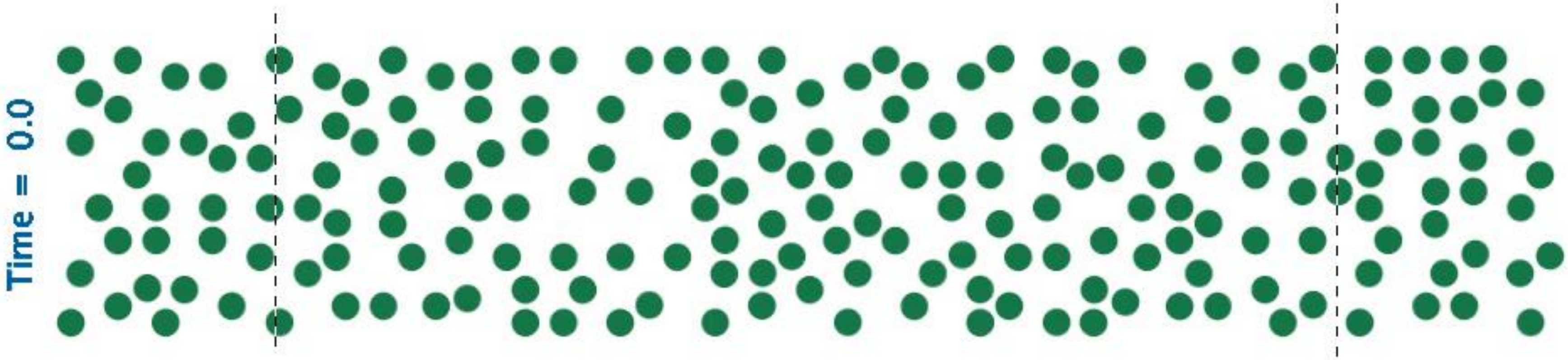
Question : Can we interpret these data with “dynamical” models ? (take two)

Beyond granular models

- *Helbing and Molnar, PRE **51**, 4282 (1995)*
  - combination of contact forces and pseudo-forces (“social” forces).
  - still at the heart of several commercial software products.
- *Zanlungo, Ikeda, and Kanda, EPL **93**, 68005 (2011)*
  *Karamouzas, B. Skinner, and S. J. Guy, PRL **113**, 238701 (2014)*
  *Karamouzas, Sohre, Narain, and Guy, ACM(TOG) **36**, 1 (2017)*
  - More complex pseudo-forces depending on future positions rather than current ones

Simulation with a "anticipated-time-to-first collision model" inspired from Karamouzas et al. (2017)





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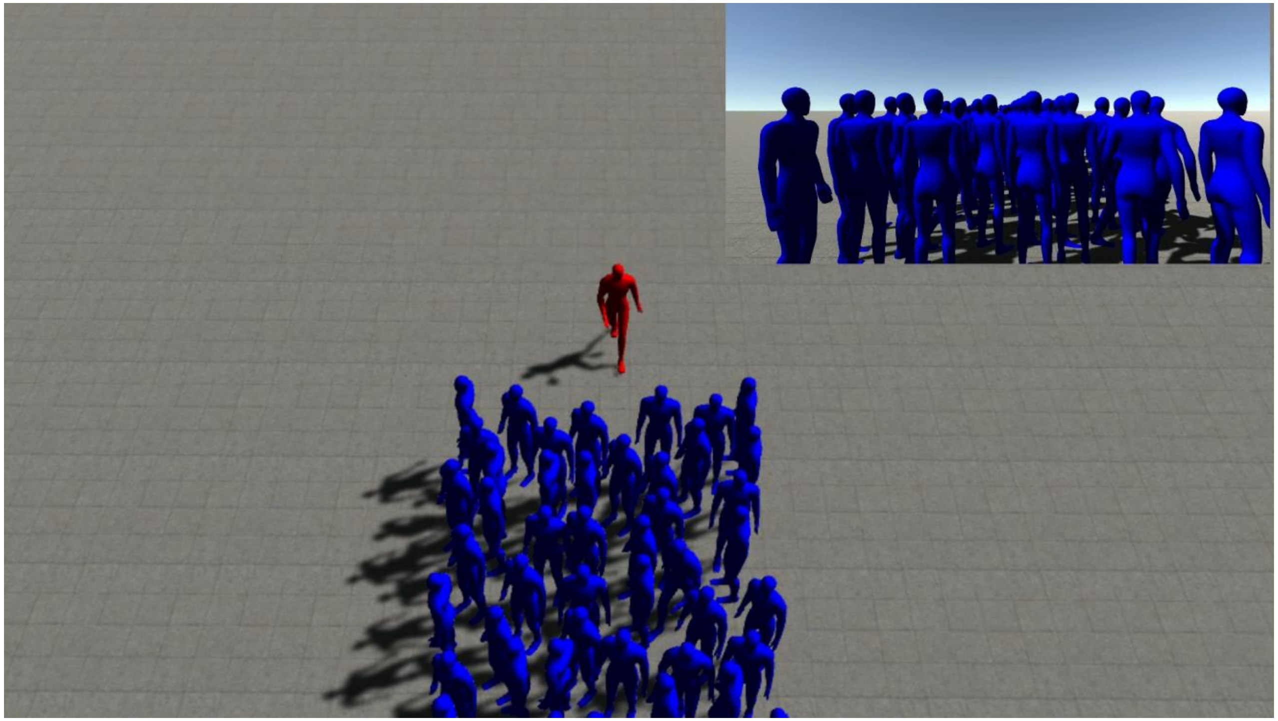
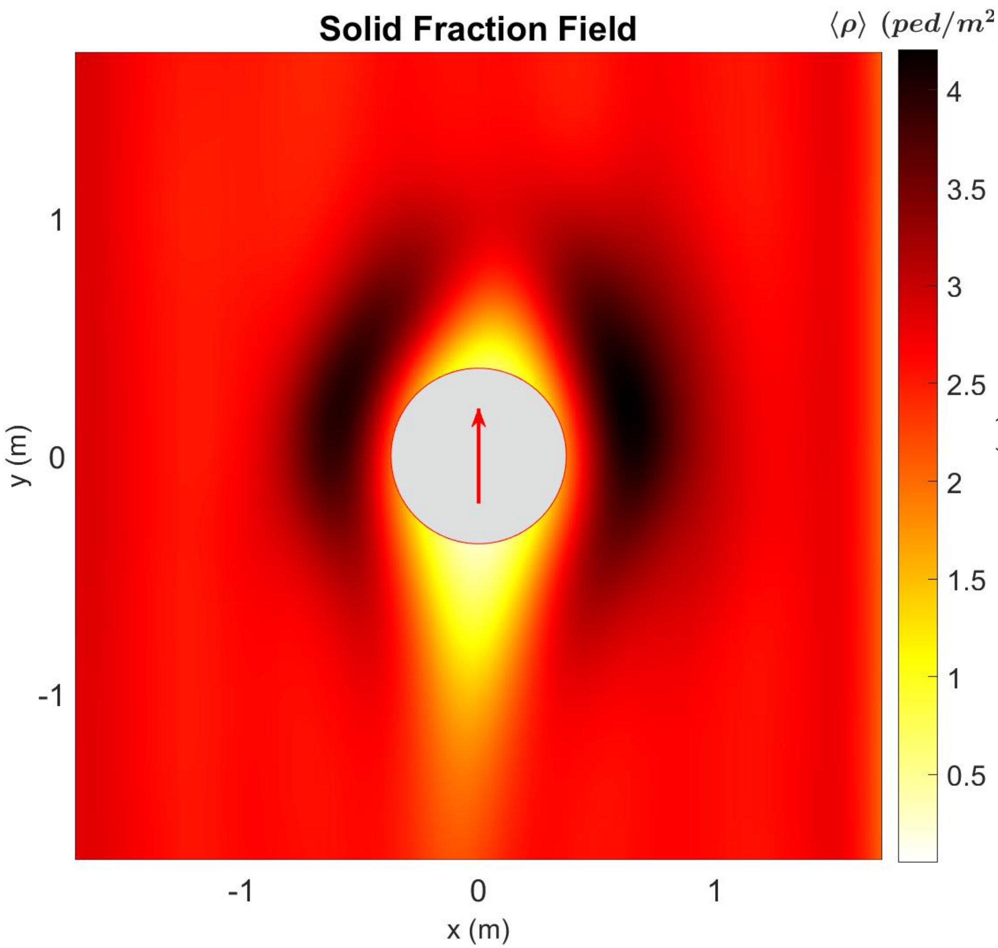
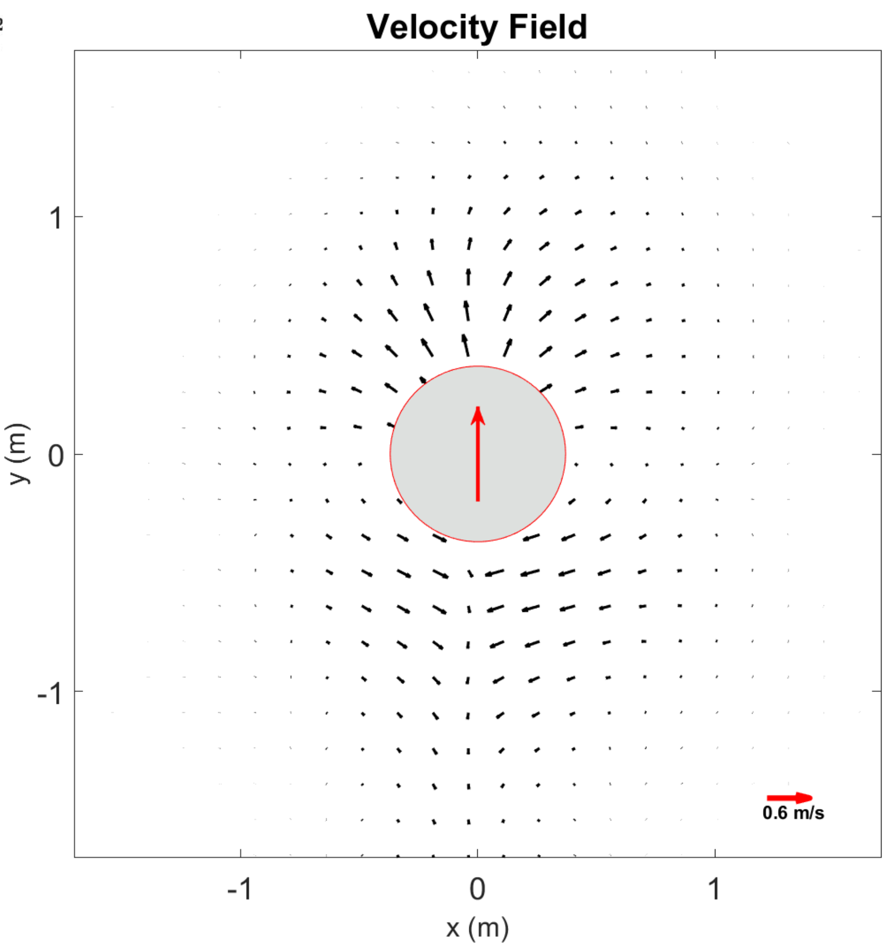


Density and velocity fields for the "Time to Collision" model





Conclusions from the simulations

- Simple dynamical models (granular, presumably original “social force”) fail drastically to reproduce the qualitative features of the experiments
- Even the more modern version of these models (“time to collision”) will struggle to do so.
- On the other hand the experimental observations are rather intuitive : pedestrians **anticipate** that it will cost them less effort to step aside and then resume their positions, even if it entails enduring high densities for some time, than to endlessly run away from an intruder that will not deviate from its course.

This requires a change of paradigm :

- anticipation ➔ competitive optimization ➔ Game theory
- Large crowd ➔ Many Body Problem ➔ Mean Field

Mean Field Game model

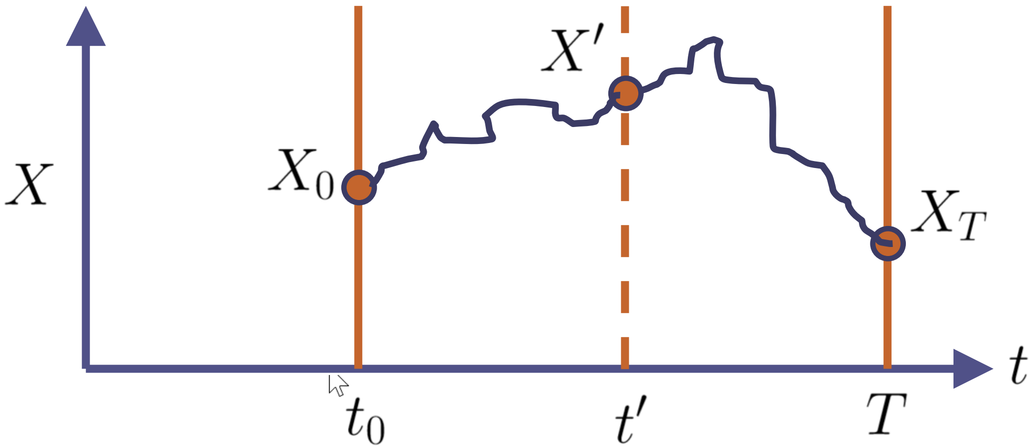
[Lasry & Lions (2006), Huang et al (2006)

- Players $\Rightarrow$ Pedestrians ($i = 1, \cdots, N$), characterized by their spatial position (state variable) ($\mathbf{X}_t^i = (x_t, y_t)$)
- Dynamics $\Rightarrow$ Langevin $\quad d\mathbf{X}_t^i = \underbrace{\mathbf{a}_t^i}_{\text{control}} dt + \sigma \underbrace{d\xi_t^i}_{\text{white noise}}$
- Cost function

$$c[\mathbf{a}^i](t, \mathbf{x}_t^i) = \left\langle \int_t^T \left[\frac{\mu a_\tau^2}{2} - \overbrace{\left(g m_t(\mathbf{x}_\tau^i) + U_0(\mathbf{x}_\tau^i - \mathbf{v}t) \right)}^{\boxed{V[m(\cdot)](x,t)}} \right] d\tau \right\rangle_{\text{noise}}$$

$$\boxed{m_t(\mathbf{x}) = \frac{1}{N} \sum_i \delta(\mathbf{x} - \mathbf{X}_t^i)} \qquad \boxed{U_0(\mathbf{x}) = V_0 \Theta(\|\mathbf{x}\| - R)}$$

Solving the optimization problem : Linear programming and Bellman Equation



Optimization ($t_0 \rightarrow T$) = Optimization ($t_0 \rightarrow t'$) + Optimization ($t' \rightarrow T$)
+ Optimization at t'

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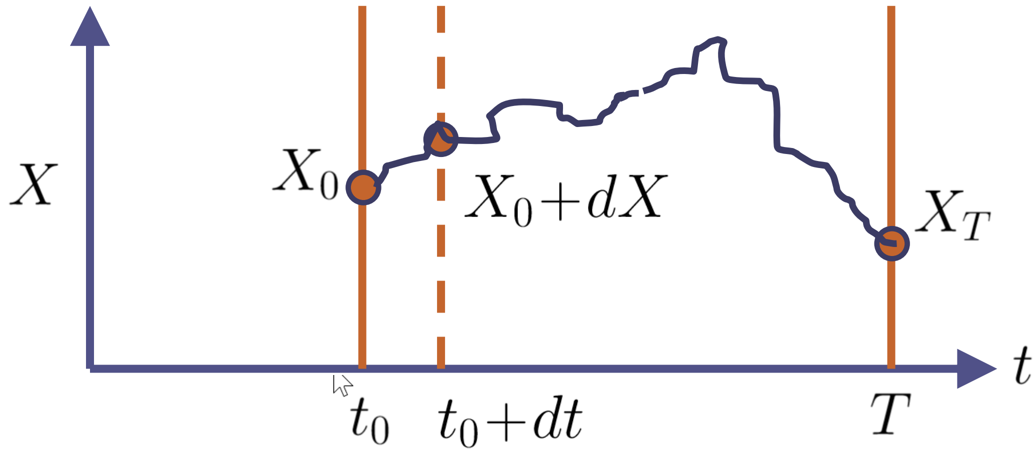

Optimization (t_0 → T) = Optimization (t_0 → t') + Optimization (t' → T) + Optimization at t'

t' = t_0 + dt ➔ differential equation

More formally :

- Introduce de value function $u_i(t, \mathbf{x}) = \min_{\mathbf{a}_i(\cdot)} c[\mathbf{a}_i(\cdot)](t, \mathbf{x})$
- Apply Bellman

$$\partial_t u_i(t, \mathbf{x}) = \frac{1}{2\mu} \left[\nabla u_i(t, \mathbf{x})\right]^2 - \frac{\sigma^2}{2} \Delta u_i(t, \mathbf{x}) + g m(t, \mathbf{x}) + U_0(\mathbf{x} - \mathbf{v}t)$$

- Boundary condition : $u_i(T, \mathbf{x}) \equiv 0 \qquad (\text{backward})$
- Optimal control : $a_i^*(t, \mathbf{x}) = -\nabla u_i(t, \mathbf{x})/\mu$

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→2N coupled differential equations = **Many Body Game Theory**

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➔2N coupled differential equations = **Many Body Game Theory**

➔**Mean Field approximation**

$$\boxed{m(t,\mathbf{x}) \simeq \langle m(t,\mathbf{x})\rangle_{\text{noise}}}$$

Mean Field Game [Lasry & Lions (2006), Huang et al (2006)]**=** coupling between a (collective) stochastic motion and an (individual) optimization problem through a mean field

- Langevin dynamic $d\mathbf{X}_t^i = \mathbf{a}_t^i dt + \sigma d\mathbf{w}_t^i$ leads to a *(forward)* diffusion equation for the density $m(x,t)$

$$\begin{cases} \partial_t m + \nabla_{\mathbf{x}}(am) - \dfrac{\sigma^2}{2}\Delta_{\mathbf{x}} m = 0 \\ m(x, t=0) = m_0(x) \end{cases} \qquad \text{(Kolmogorov)} .$$

- Optimization problem, through linear programming, leads to a *(backward)* Hamilton-Jacobi-Bellman equation for the value function $u(\mathbf{x}, t)$

$gm(t,\mathbf{x}) + U_0(\mathbf{x} - \mathbf{v}t)$

$$\begin{cases} \partial_t u + \dfrac{1}{2\mu}(\nabla_{\mathbf{x}} u)^2 + \dfrac{\sigma^2}{2}\Delta_{\mathbf{x}} u = V[m](x,t) \\ u(x, t=T) = c_T(x) \end{cases} \qquad \text{(HJB)} .$$

- Kolmogorov coupled to HJB through the drift $a(x,t) = -\partial_x u(x,t)$
- HJB coupled to Kolmogorov through the mean field $V[m](x,t)$

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Transformation to NLS

- Cole-Hopf transform: $\Phi(\mathbf{x},t) = \exp\left(-\frac{1}{\mu\sigma^2}u(\mathbf{x},t)\right)$

$$\Longrightarrow \qquad -\mu\sigma^2\partial_t\Phi = \frac{\mu\sigma^4}{2}\Delta_{\mathbf{x}}\Phi + V[\mathbf{x},m]\Phi$$

- "Hermitization" of Kolmogorov: $\Gamma(\mathbf{x},t) \equiv m(\mathbf{x},t)\exp\left(u(\mathbf{x},t)/(\mu\sigma^2)\right)$ (i.e. $m(\mathbf{x},t) = \Gamma(\mathbf{x},t)\Phi(\mathbf{x},t)$)

$$\sigma^2\partial_t\Gamma - \frac{\sigma^4}{2}\Delta_{\mathbf{x}}\Gamma = \frac{\Gamma}{\mu}\underbrace{\left(\frac{\partial u}{\partial t} - \frac{1}{2\mu}\left(\nabla_{\mathbf{x}}u\right)^2 + \frac{\sigma^2}{2}\Delta_{\mathbf{x}}u\right)}_{V[\mathbf{x},m] \quad !!!}$$

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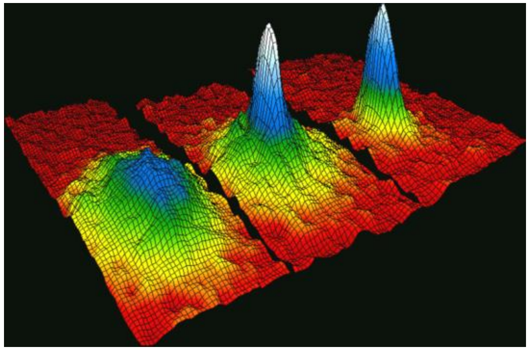
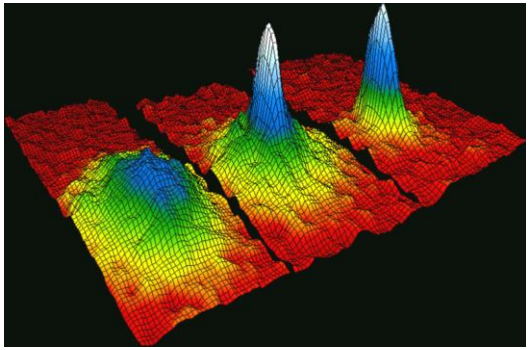


$$\boxed{i\hbar\partial_t\Psi = -\frac{\hbar^2}{2\mu}\Delta_{\mathbf{x}}\Psi + U_0(\mathbf{x})\Psi + g|\Psi|^2\Psi}$$

<u>Non-Linear Schrödinger</u>



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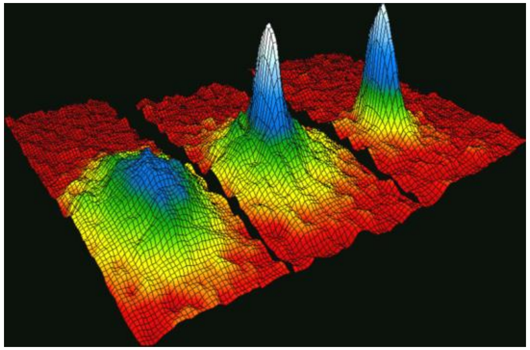
Non-Linear Schrödinger

- MFG equations, specifying to $V[m](\mathbf{x}) \equiv U_0(\mathbf{x}) + gm(\mathbf{x},t)$

$$\begin{cases} \mu\sigma^2\partial_t\Gamma = \dfrac{\mu\sigma^4}{2}\Delta_{\mathbf{x}}\Gamma + U_0(\mathbf{x})\Gamma + g\,m\Gamma \\ -\mu\sigma^2\partial_t\Phi = \dfrac{\mu\sigma^4}{2}\Delta_{\mathbf{x}}\Phi + U_0(\mathbf{x})\Phi + g\,m\Phi \end{cases}$$

$m = \Gamma\Phi$



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$m = \Gamma\Phi$

Formal change $(\Psi, \Psi^*, \hbar) \to (\Phi, \Gamma, i\mu\sigma^2)$ maps NLS to MFG !!!

Why is this useful ?

- Man Field Games exist since 2005-2006, the Non-Linear Schrödinger equation since at least the work of Landau and Ginzburg on superconductivity in 1950.
- NSL applies to many field of physics : superconductivity, non-linear optic, gravity waves in inviscid fluids, Bose-Einstein condensates, etc..

  → huge literature on the subject
- We feel we have a good qualitative understanding of the "physics" of NLS, together with a large variety of technical tools to study its solutions.

[NB : Change of variable giving NLS known by Guéant, (2011)]

Intermezzo : a few different ways to use the connection with NLS

Tool #1 : Heisenberg representation & Ehrenfest relations

Quantum mechanics

- State of the system $\equiv$ wave function $\Psi(x,t)$
- Observables $\equiv$ operators: $\hat{O} = f(\hat{p}, \hat{x})$
- Average $\langle \hat{O} \rangle \equiv \int dx \Psi^*(x) \hat{O} \Psi(x)$
- Hamiltonian $\equiv$ $\hat{H} = \frac{\hat{p}^2}{2\mu} + V(x) = -\frac{\hbar^2}{2\mu}\Delta_x + V(x)$

$$\hat{x} \equiv x \times$$
$$\hat{p} \equiv i\hbar\partial_x$$

$$i\hbar\partial_t \Psi = \hat{H}\Psi \qquad \Rightarrow \qquad i\hbar\frac{d}{dt}\langle \hat{O}\rangle = \langle [\hat{H}, \hat{O}] \rangle$$

$$\Rightarrow \quad \begin{cases} \frac{d}{dt}\langle \hat{x} \rangle = \frac{1}{\mu}\langle \hat{p} \rangle \\ \frac{d}{dt}\langle \hat{p} \rangle = -\langle \nabla_x V(\hat{x}) \rangle \end{cases} \qquad \text{(Ehrenfest)}$$

Quadratic Mean Field Games

- Operators: $\hat{X} \equiv x\times \qquad \hat{\Pi} \equiv \mu\sigma^2\partial_x \qquad \hat{O} = f(\hat{\Pi}, \hat{X})$

- Average: $\langle \hat{O} \rangle(t) \equiv \int dx \Gamma(x,t) \hat{O} \Phi(x,t)$

$m = \Gamma\Phi$

$$\Rightarrow \qquad \text{if } \hat{O} = O(\cancel{\hat{\Pi}}, \hat{X}) \qquad \langle \hat{O} \rangle \equiv \int dx\, m(x) O(x)$$

$$\left(\langle \hat{1} \rangle \equiv \int dx\, m(x) = 1 \qquad \langle \hat{X} \rangle \equiv \int dx\, x m(x) \right)$$

- Hamiltonian $\equiv \qquad \hat{H} = -\left(\frac{\hat{\Pi}^2}{2\mu} + V[m](x) \right)$

$$\begin{cases} -\mu\sigma^2 \partial_t \Gamma = \hat{H}\Gamma \\ +\mu\sigma^2 \partial_t \Phi = \hat{H}\Phi \end{cases} \quad \Rightarrow \quad \mu\sigma^2 \frac{d}{dt} \langle \hat{O} \rangle = \langle [\hat{O}, \hat{H}] \rangle$$

Exact relations

$$\text{Force operator : } \hat{F}[m_t] \equiv -\nabla_x V[m_t](\hat{X})$$

$$\Sigma^2 \equiv \langle(\hat{X}^2\rangle - \langle\hat{X}\rangle^2) \qquad \Lambda \equiv (\langle\hat{X}\hat{\Pi} + \hat{\Pi}\hat{X}\rangle - 2\langle\hat{\Pi}\rangle\langle\hat{X}\rangle)$$

$$\begin{cases} \dfrac{d}{dt}\langle\hat{X}\rangle = \dfrac{1}{\mu}\langle\hat{\Pi}\rangle \\ \dfrac{d}{dt}\langle\hat{\Pi}\rangle = \langle F[m_t]\rangle \end{cases} \qquad \begin{cases} \dfrac{d}{dt}\Sigma^2 = \dfrac{1}{\mu}\Big(\langle\hat{X}\hat{\Pi} + \hat{\Pi}\hat{X}\rangle - 2\langle\hat{\Pi}\rangle\langle\hat{X}\rangle\Big) \\ \dfrac{d}{dt}\Lambda = -2\langle\hat{X}\hat{F}[m_t]\rangle + 2\langle\hat{\Pi}^2\rangle \end{cases}$$

$\underline{\text{Local interactions}}$ $\quad V[m_t](\mathbf{x}) = U_0(\mathbf{x}) + f(m_t(\mathbf{x}))$

$$\rightarrow \hat{F}[m_t] \equiv \underbrace{\hat{F}_0}_{-\nabla_x U_0} - g\,\nabla_x m_t f'(m_t))$$

$$\langle\hat{F}\rangle = \langle\hat{F}_0\rangle$$

$$\langle\hat{X}\hat{F}\rangle = \langle\hat{X}F_0\rangle - \int d\mathbf{x}\,\mathbf{x} m_t(\mathbf{x}) f'[m_t(\mathbf{x})]$$

Tool #2 : action and variational approach

Action

$$S[\Gamma(x,t),\Phi(x,t)] \equiv \int dt\, dx \left[\frac{\mu\sigma^2}{2}(\partial_t\Phi\,\Gamma - \Phi\partial_t\Gamma) - \frac{\mu\sigma^4}{2}\nabla\Phi.\nabla\Gamma + U_0(x)\Phi\Gamma + \frac{g}{2}\Phi^2\Gamma^2\right]$$

$$\left[\frac{\delta S}{\delta\Gamma}=0\right] \Leftrightarrow -\mu\sigma^2\partial_t\Phi = \frac{\mu\sigma^4}{2}\Delta_{\mathbf{x}}\Phi + V[\mathbf{x},m]\Phi$$

$$\left[\frac{\delta S}{\delta\Phi}=0\right] \Leftrightarrow +\mu\sigma^2\partial_t\Gamma = \frac{\mu\sigma^4}{2}\Delta_{\mathbf{x}}\Gamma + V[\mathbf{x},m]\Gamma$$

- Conserved quantity: $\mathcal{E}_{\mathrm{tot}}\cancel{(t)} \equiv \frac{1}{2\mu}\langle\hat{\Pi}^2\rangle + \langle U_0(\hat{X})\rangle + \langle\hat{H}_{\mathrm{int}}\rangle$
- Variational anzatz $\Longrightarrow$ Ordinary Differential Equations

$$\langle\hat{H}_{\mathrm{int}}\rangle \equiv \frac{g}{2}\int dx\, m_t(x)^2$$

Tool #3 : solitons and integrability

For *d=1* and *V[m](x) = g m + ~~U₀~~ (x)* ➔ NLS, and thus MFG, integrable

➔ Infinite number of conserved quantities *[Bonnemain et al. 2021]*

$$Q_1 = \frac{1}{2}\int_{\mathbb{R}} (w_1\Phi + \tilde{w}_1\Gamma)dx = \frac{\mu\sigma^4}{2g}\int_{\mathbb{R}} (\Gamma\partial_x\Phi - \Phi\partial_x\Gamma)dx \propto P$$

$$Q_2 = \frac{1}{2}\int_{\mathbb{R}} (w_2\Phi + \tilde{w}_2\Gamma)dx = \frac{\mu\sigma^4}{2g^2}\int_{\mathbb{R}} dx\left[-\frac{\mu\sigma^4}{2}\nabla\Gamma.\nabla\Phi + \frac{g}{2}(\Gamma\Phi)^2\right] \propto E_{\mathrm{tot}}$$

$$Q_3 = \frac{1}{2}\int_{\mathbb{R}} (w_3\Phi + \tilde{w}_3\Gamma)dx = \frac{\mu^2\sigma^8}{2g^2}\int_{\mathbb{R}} dx\left[\frac{\mu\sigma^4}{g}(\Gamma\partial_{xxx}\Phi - \Phi\partial_{xxx}\Gamma) + \frac{3}{2}\left(\Gamma^2\partial_x\Phi^2 - \Phi^2\partial_x\Gamma^2\right)\right]$$

$$Q_4 = \frac{1}{2}\int_{\mathbb{R}} (w_4\Phi + \tilde{w}_4\Gamma)dx = \frac{\mu^2\sigma^8}{2g^2}\int_{\mathbb{R}} dx\left[\frac{\mu^2\sigma^8}{g^2}\partial_{xx}\Phi\partial_{xx}\Gamma + \frac{3\mu\sigma^4}{g}\partial_x\Phi^2\partial_x\Gamma^2 + (\Phi\Gamma)^3\right]$$

➔ Scaling solutions (Thomas-Fermi regime)

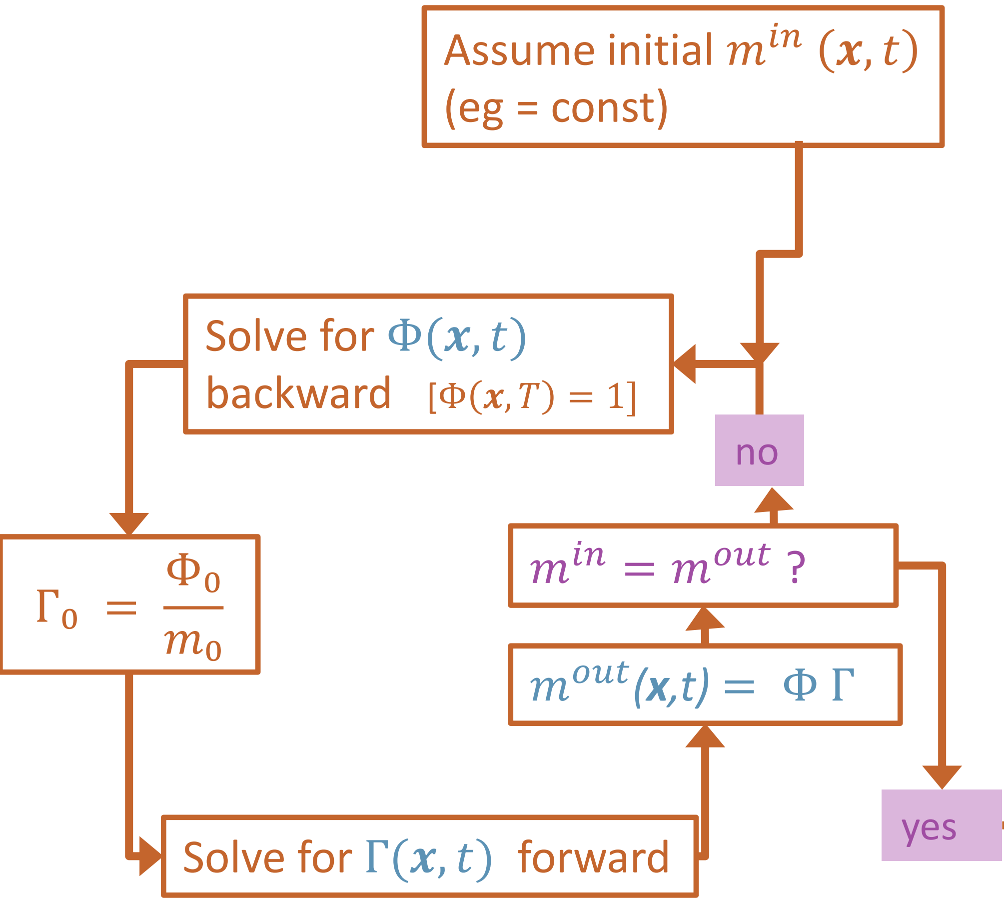
[Bonnemain et al. 2020]

$$\begin{cases} m(t,x) = \dfrac{3\left(z(t)^2 - x^2\right)}{4z(t)^3} \\[2ex] v(t,x) = -\dfrac{z'(t)}{z(t)}x \end{cases}$$

$$z(t) = 3\left(\frac{|g|}{4\mu}\right)^{1/3} t^{2/3}$$

Back to pedestrian motion

Numerical implementation



Propagation :

$$\begin{cases} -\mu\sigma^2\partial_t\Phi = \dfrac{\mu\sigma^4}{2}\Delta\Phi + (U_0 + gm^{\text{in}})\Phi \\ +\mu\sigma^2\partial_t\Gamma = \dfrac{\mu\sigma^4}{2}\Delta\Gamma + (U_0 + gm^{\text{in}})\Gamma \end{cases}$$

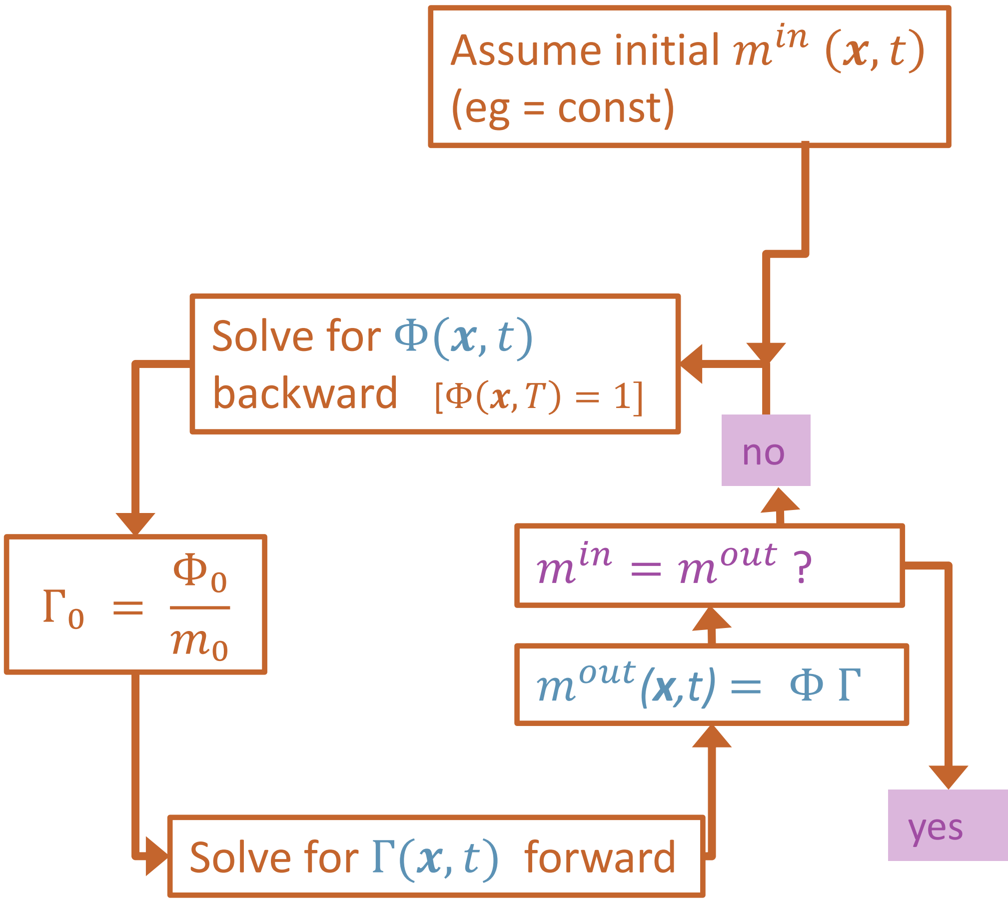
$[\Phi(T,\cdot) = 1]$ $\qquad$ $[\Gamma(0,\cdot) = m_0(\cdot)/\Phi(0,\cdot)]$

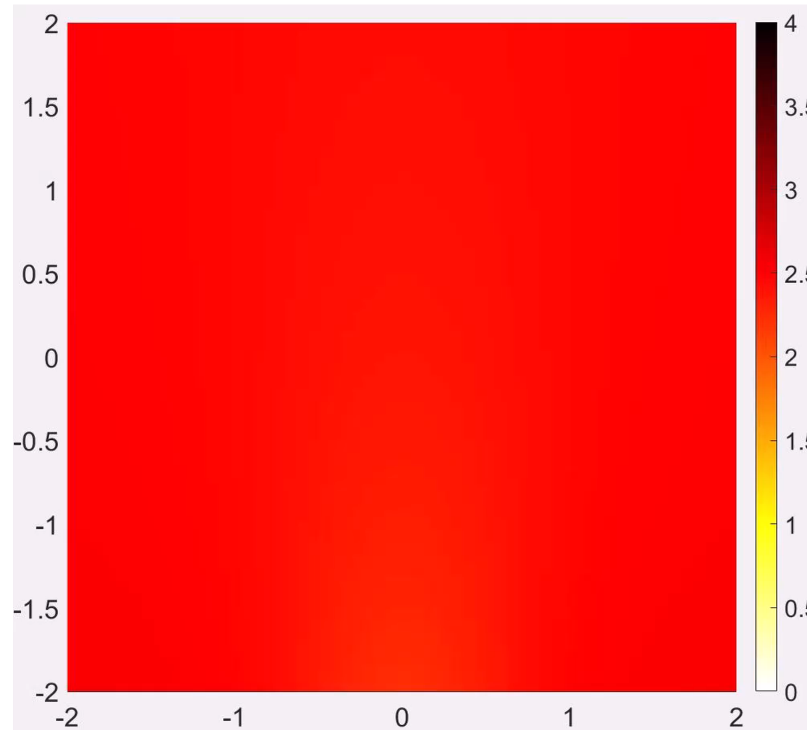
Self consistent equation :

$$m^{\text{out}}(\mathbf{x}, t) \equiv \Gamma(\mathbf{x}, t)\Phi(\mathbf{x}, t) = m^{\text{in}}(\mathbf{x}, t)$$

Back to pedestrian motion

Numerical implementation





However :

- We are not really interested in the full dynamics (transient regime, etc…
- Self consistence is expensive (full time dependent calculation → 4 days).

⇒ Permanent regime (a.k.a ergodic)

[cf Cardialaguet et al. (2013)]

NB $\Gamma(t,\mathbf{x}) = \exp\left[-\lambda t/\mu\sigma^2\right]\Gamma_{\rm er}(\mathbf{x}) \quad \Phi(t,\mathbf{x}) = \exp\left[\lambda t/\mu\sigma^2\right]\Phi_{\rm er}(\mathbf{x})$ (time dependent)

$m_{\rm er}(\mathbf{x}) = \Phi_{\rm er}(\mathbf{x})\Gamma_{\rm er}(\mathbf{x}) \qquad j_{\rm er}(\mathbf{x}) = \sigma^2\Gamma_{\rm er}(\mathbf{x})\nabla\Phi_{\rm er}(\mathbf{x})$ (observable are time in dependent)

In the cylinder frame

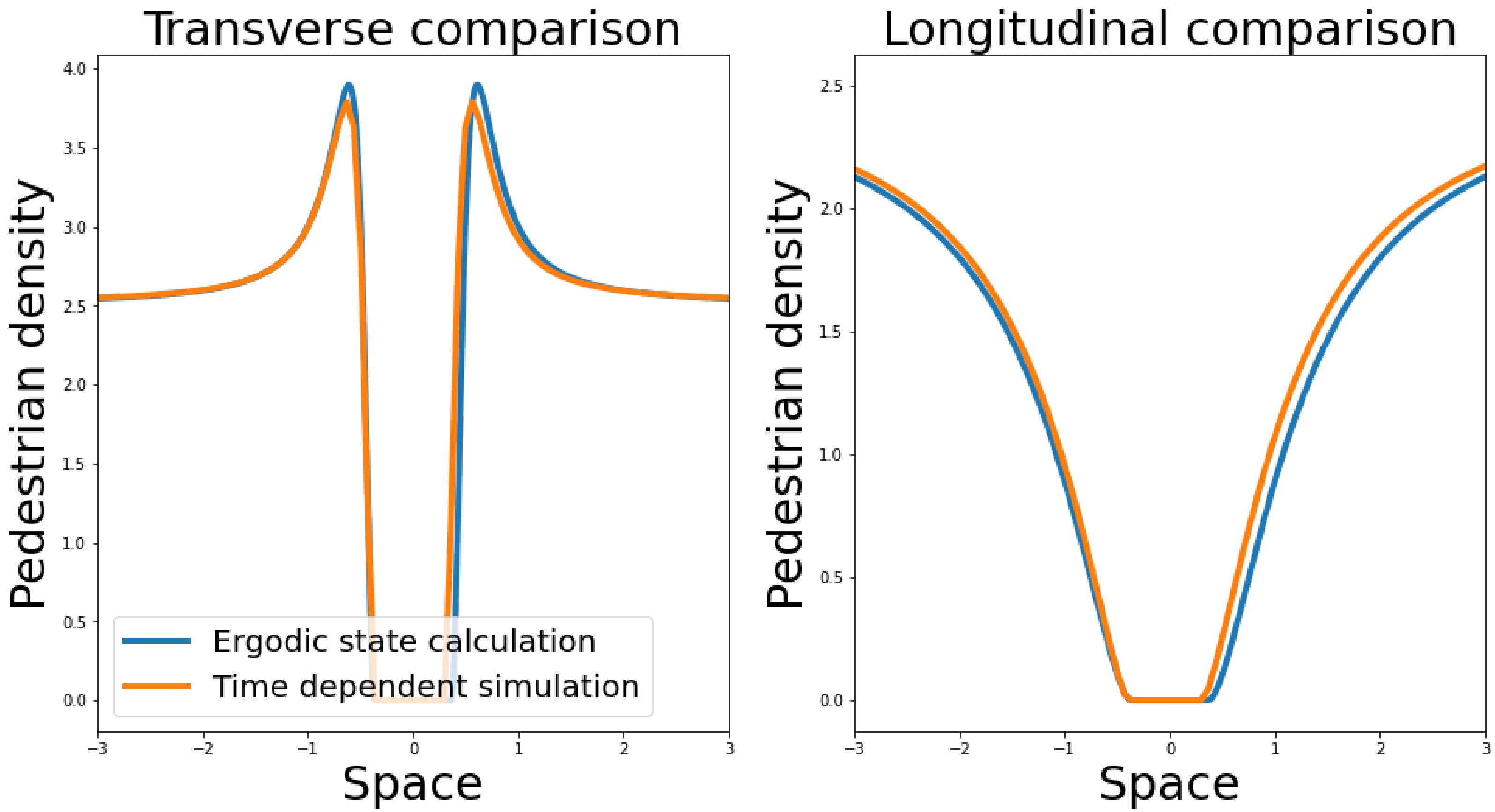
$$\begin{cases} \dfrac{\mu\sigma^4}{2}\Delta\Phi_{\rm er} - \mu\sigma^2\mathbf{v}\cdot\vec{\nabla}\Phi_{\rm er} + [U_0(\mathbf{x}) + gm_{\rm er}]\Phi_{\rm er} = -\lambda\Phi_{\rm er}, \\ \dfrac{\mu\sigma^4}{2}\Delta\Gamma_{\rm er} + \mu\sigma^2\mathbf{v}\cdot\vec{\nabla}\Gamma_{\rm er} + [U_0(\mathbf{x}) + gm_{\rm er}]\Gamma_{\rm er} = -\lambda\Gamma_{\rm er} \end{cases}$$

cylinder velocity

$$\boxed{\lambda = -gm_0 - \tfrac{3}{2}\mu v^2.}$$

And indeed

NB : $(\Gamma \leftrightarrow \phi)$ $[\Longleftrightarrow TR]$ + $(y \leftrightarrow -y)$
= exact symmetry for ergodic case



Comparison between time dependent and ergodic simulations along the transverse (x) and parallel (y) directions [$\nu = 0.22, \xi = 0.26$]

Characteristic length and velocity scales

Cylinder :

Radius : R

Velocity : v

Pedestrians :

Healing length : $\nu = \sqrt{|\mu\sigma^4/2gm_0|}$

“Healing” velocity : $\xi = \sqrt{|gm_0/2\mu|}$

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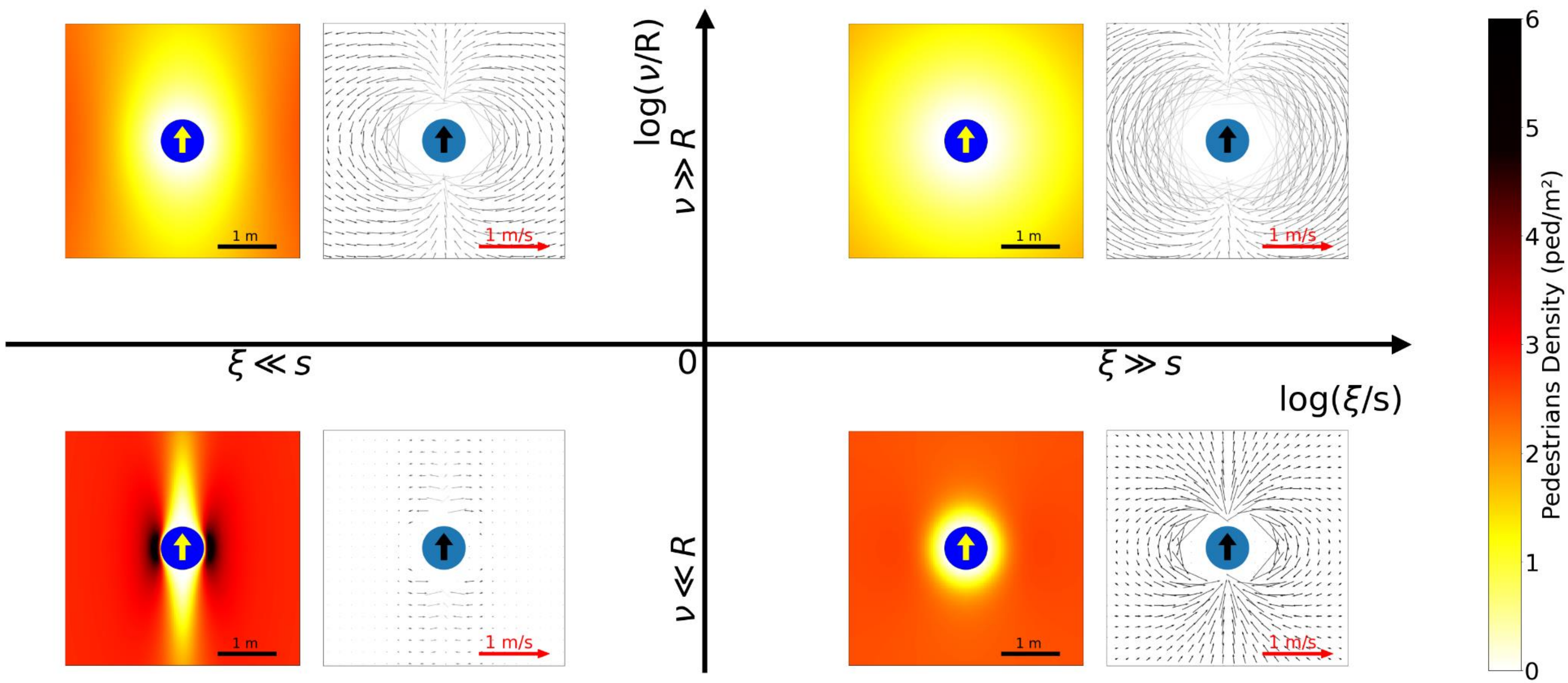
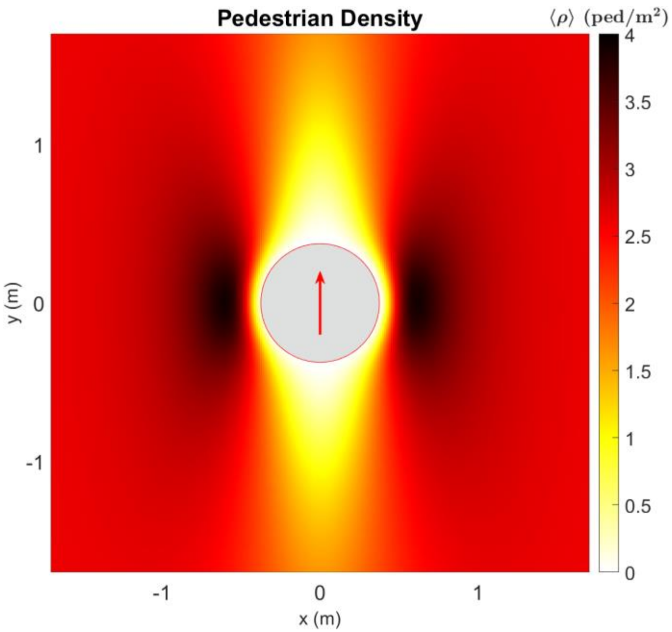
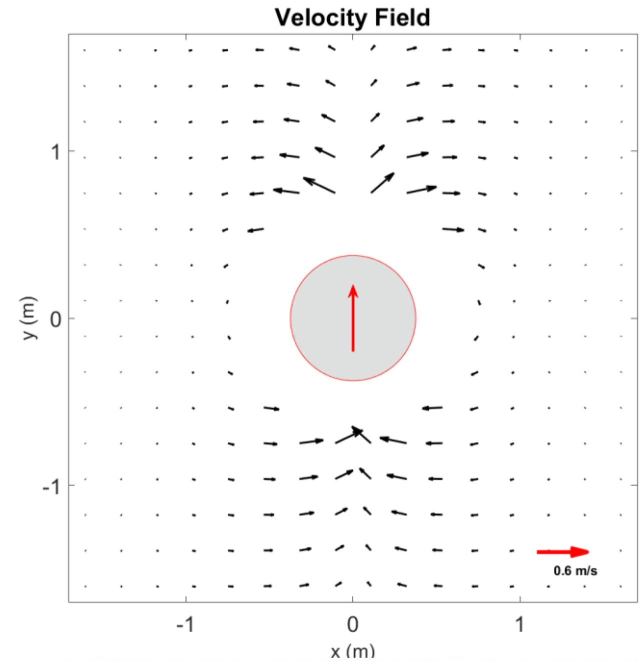


Comparison between experimental and Mean Field Game simulation

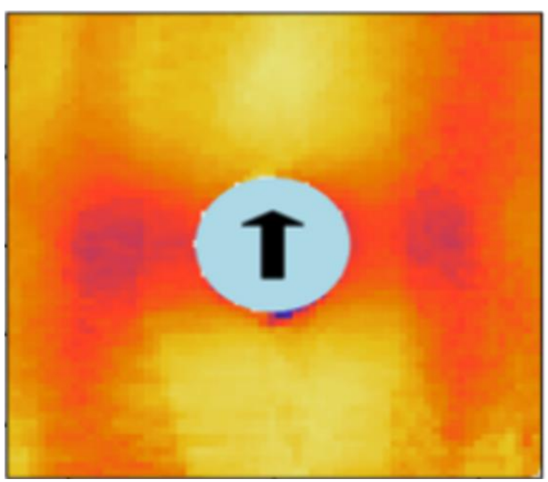
$[\nu=0.22,\ \xi=0.26,\ v=0.6,\ R=0.32\text{m}]$

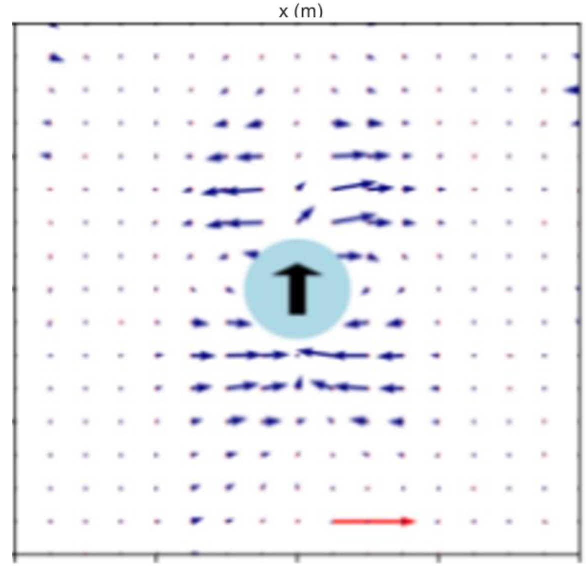
MFG





Exp





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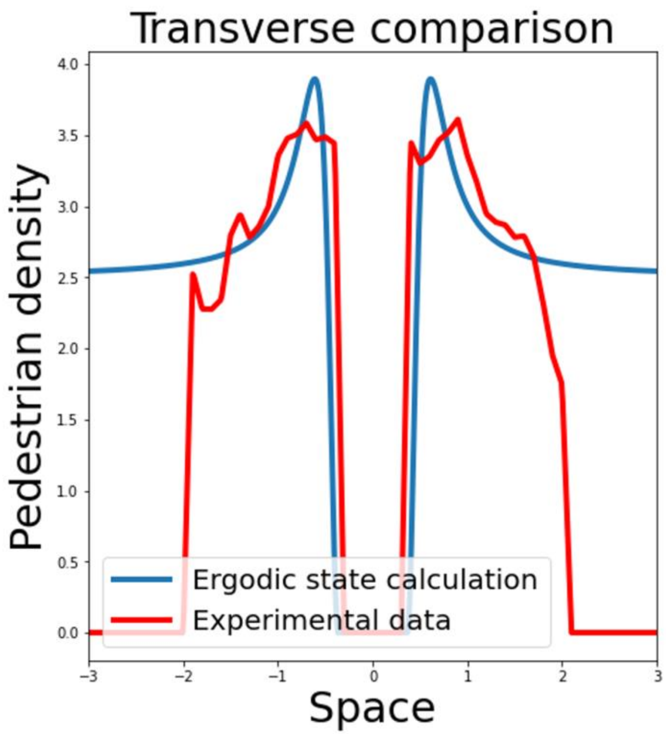

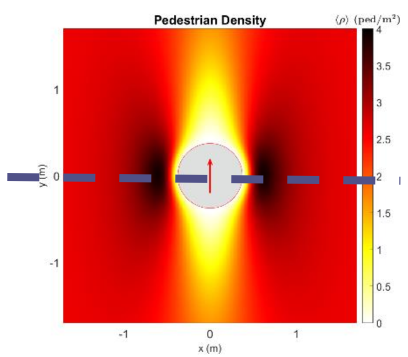

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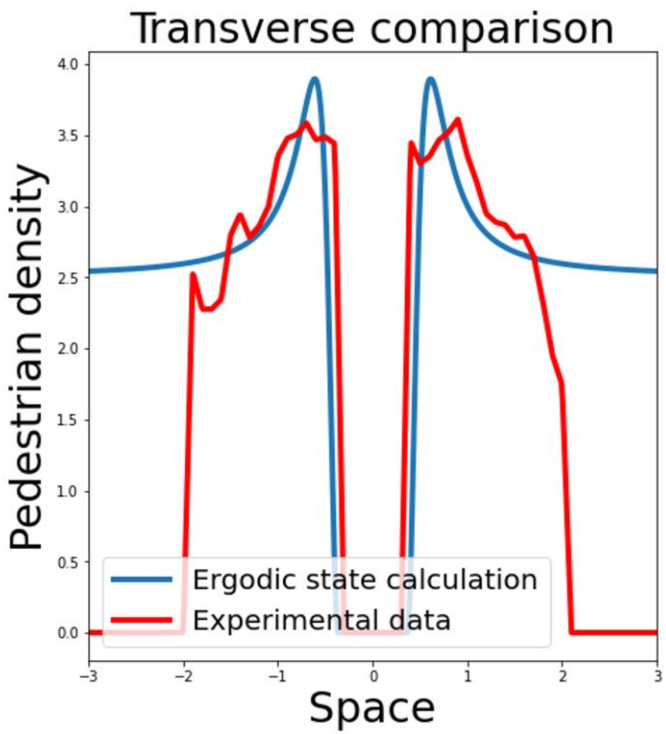

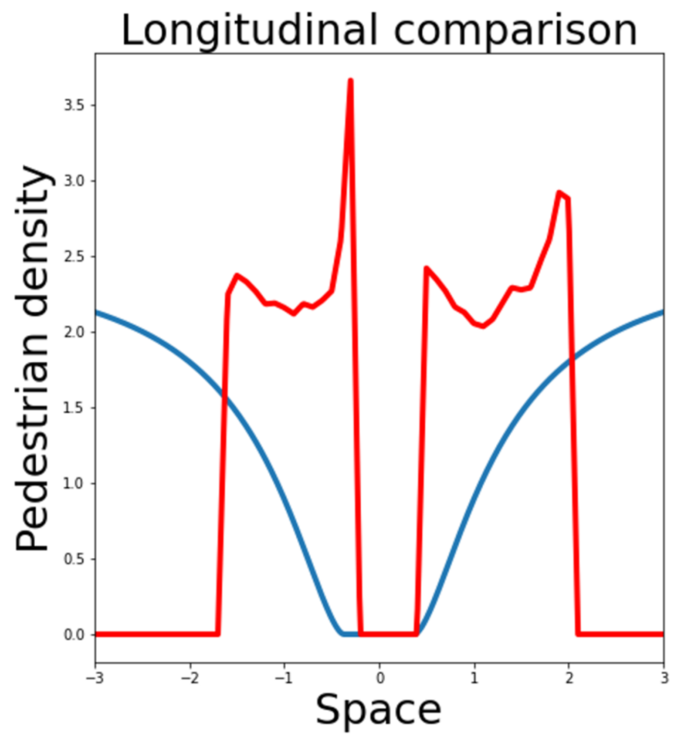

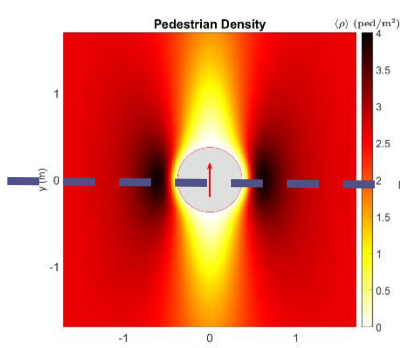

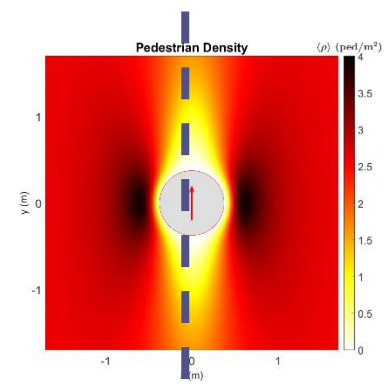

Conclusion

- The Mean Field Game approach reproduces naturally the important qualitative feature of the “crossing cylinder” experiment .
- It does significantly better than the “dynamical” approaches which are generally used to describe crowds behavior. This is especially true for the version found in most commercial software, but also for their more modern/research oriented version.
- We cannot pretend to quantitative accuracy.
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Conclusion

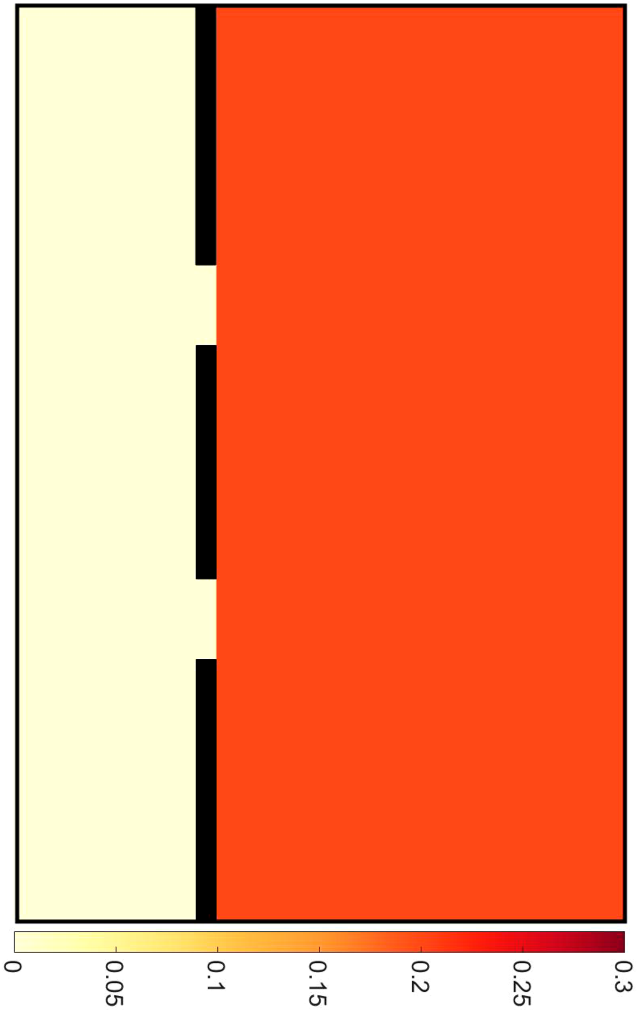
Keep in mind however that :

- we have used the "simplest" MFG model :
  - no congestion effect *[cf eg : Lachapelle and Wolfram (2011)]*
  - no discount ratio
  - linear ($\sim gm$) interaction
- There is no point in trying to exceed the level of accuracy of existing experiments.

And two last remarks:

- $\nu \approx 22\text{cm}$ ➔ this is about the effective size of a pedestrian
- The configurations where a MFG would capture a behavior out of reach of "dynamical approaches" are not limited to the one of a cylinder crossing a static crowd.

Eg : boarding an overcrowded metro wagon



Metro boarding in Lyon