Mean Field Games: An [imaginary time] Schrödinger approach

Denis Ullmo (LPTMS-Orsay)

Collaboration with
Thierry Gobron (LPTM-Cergy)  Igor Swiecicki (LPTM(S))

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Outline

1. Introduction to mean field games
   • Optimization problems
   • Game theory
   • Mean field games

1. Quadratic mean field games and the Non-Linear Schrödinger equation
   • Mapping to NLS
   • A case study: a quadratic mean field game in the strong positive coordination regime
Part I
Introduction to mean field games
Optimization problems

The lifeguard problem

Swimming speed: $v_s$

Running speed: $v_r$

Aatish Bhatia

$$\min_{\theta} [t = t_r + t_s] \Rightarrow \frac{\sin \theta_s}{v_s} = \frac{\sin \theta_r}{v_r}$$
Dynamics of a classical point particle \([\mathbf{r} = (x, y)]\)

Kinetic energy : \(T = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2)\)

Potential energy: \(V(x, y)\)

Lagrangian : \(L = T - V\)

\[
S(\mathbf{r}_f, \mathbf{r}_o, t) = \int_0^t L(\mathbf{r}(t'), \dot{\mathbf{r}}(t')) \, dt'
\]

(action)

minimize \(S(\mathbf{r}', \mathbf{r}''', t)\) \(\implies\) \(m\ddot{\mathbf{r}} = -\nabla V(\mathbf{r})\)

Hamilton Jacobi: \(\partial_t S = \frac{1}{2m} (\nabla S)^2 + V(\mathbf{r})\)
Control

- $X \equiv$ motor speed or position, chemical concentration, etc ...
- dynamics: $dX_t = a_t dt + \sigma d\omega_t$
- Cost function: $c[a(\cdot), w(\cdot), X_t, t] \equiv \int_t^T \ell(a_t, X_t) dt + c_T(X_T)$
  
  e.g. $\ell(a, X) \equiv \frac{\mu}{2}a^2 - V(X)$

Problem

choose control $a(.\cdot)$ to minimize expected cost $\langle c(X_{t_0}) \rangle_{\text{noise}}$
Linear programming

- value function: \[ u(x, t) \equiv \min_{a(.)} \langle c(X_t) \rangle_{\text{noise}} |_{x_t=x} \]

![Graph showing linear programming concepts with arrows and labels](image)

- boundary condition: \[ u(x, T) = c_T(x) \]

- Hamilton-Jacobi-Bellman equation:
  \[ \partial_t u = \frac{1}{2\mu} \left( \partial_x u \right)^2 + V(x) - \frac{\sigma^2}{2} \partial_x^2 u \]
  \[ a(x, t) = -\partial_x u(x, t) \]
Game “theory”

A simple game:
- 2 players
- 2 strategies

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Differential games:
- Each player $i$ characterized by a set of continuous “state variables” $x^i = (x^i_1, x^i_2, ..., x^i_n)$
- Each player solve an optimization problem
- The cost function of player $i$ depend on $x^i$ but also on all the $x^j \neq i$

E.g.:
- Three persons trying to sell ice creams on a beach
- Two competing companies trying to decide on the size of their marketing department.
- Air heaters in different houses when the price of energy depends on total consumption.
Games with a large number of players:

- As the number of players increases, the study of such games becomes quickly intractable.
- However, for a very large number of « small » players, one can recover some degree of simplification through the notion of “mean field”.

Weiss [mean field] theory of magnetization

\[ \hat{H} = -J \sum_{\langle i,j \rangle} s_i \cdot s_j \Rightarrow \hat{H}_i^{\text{mf}} = -J \nu \langle s \rangle \cdot s_{i_0} \]

Mean Field (differentiable) Games

[Lasry & Lions (2006)]
Mean Field Games

A mean field game paradigm: model of population dynamics

[Guéant, Lasry, Lions (2011)]

- $N$ agents $i = 1, 2, \cdots, N \quad (N \gg 1)$
- state of agent $i \rightarrow$ real vector $\mathbf{X}^i$ (here just physical space)

$$m(x, t) \equiv \frac{1}{N} \sum_{1}^{N} \delta(x - \mathbf{X}^i_t) \quad \text{density of agents}$$

- agent’s dynamic

$$d\mathbf{X}^i_t = a^i_t dt + \sigma d\mathbf{w}^i_t$$

$d\mathbf{w}^i_t \equiv$ white noise

drift $a^i_t \equiv$ control parameter

- agent tries to optimize (by the proper choice of $a^i_t$) the cost function

$$\int_{t}^{T} d\tau \left[ \frac{\mu}{2}(a^i_\tau)^2 - V[m](\mathbf{X}^i_\tau, \tau) \right] + c_T(\mathbf{X}^i_T)$$
**Mean Field Game** = coupling between a (collective) stochastic motion and an (individual) optimization problem through a mean field $V[m](x, t)$

- Langevin dynamic $dX_t^i = a_t^i dt + \sigma dw_t^i$ leads to a *(forward)* diffusion equation for the density $m(x, t)$

\[
\begin{cases}
\partial_t m + \nabla_x (am) - \frac{\sigma^2}{2} \Delta_x m = 0 \\
m(x, t=0) = m_0(x)
\end{cases}
\quad \text{(Kolmogorov)}.
\]

- Optimization problem, through linear programming, leads to a *(backward)* Hamilton-Jacobi-Bellman equation for the value function $u(x, t)$

\[
\begin{cases}
\partial_t u + \frac{1}{2\mu} (\nabla_x u)^2 + \frac{\sigma^2}{2} \Delta_x u = V[m](x, t) \\
u(x, t=T) = c_T(x)
\end{cases}
\quad \text{(HJB)}.
\]

- Kolmogorov coupled to HJB through the drift $a(x, t) = -\partial_x u(x, t)$
- HJB coupled to Kolmogorov through the mean field $V[m](x, t)$
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\begin{aligned}
\partial_t u + \frac{1}{2\mu} (\nabla_x u)^2 + \frac{\sigma^2}{2} \Delta_x u &= V[m](x, t) \\
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Long time limite and the « ergodic » state

**Theorem**  [Cardaliaguet, Lasry, Lions, Porretta (2013)]

- No explicit time dependence: $V[m](x,t)$
- Long time limit for the optimization: $T \to \infty$
- ... + other (technical) conditions ....

\[ \exists \text{ an ergodic state } (\bar{m}(x), \bar{u}(x)) \text{ such that, for } 0 \ll t \ll T \]
\[ m(x,t) \simeq \bar{m}(x) \]
\[ u(x,t) \simeq \bar{u}(x) \]

\[ (\bar{m}, \bar{u}, \lambda) \text{ such that } \]
\[ \begin{cases} 
\lambda + \frac{1}{2\bar{\mu}} (\nabla_x \bar{u})^2 + \frac{\sigma^2}{2} \Delta_x \bar{u} = V[\bar{m}](x,t) \\
- \nabla_x (\bar{m}(\nabla_x \bar{u})) - \frac{\sigma^2}{2} \Delta_x \bar{m} = 0 
\end{cases} \]
Recent, applications oriented, mean field game models

- Price formation process in the presence of high frequency participant [Lachapelle, Lasry, Lehalle, Lions (2015)]
- Load shaping via grid wide coordination of heating-cooling electric loads [Kizilkale and Malhamé, (2015)]
Part II
Quadratic Mean Field Games and the Non-Linear Schrödinger equation
The two main avenues of research for MFG

- Proof of the internal consistency of the theory, and of the existence and uniqueness of solutions to the MFG equations [cf Cardaliaguet’s notes from Lions collège de France lectures]
- Numerical schemes to compute exact solutions of the problem [eg: Achdou & Cappuzzo-Dolcetta (2010), Lachapelle & Wolfram (2011), etc ...]

- Our (physicist) approach: develop a more “qualitative” understanding of the MFG (extract characteristic scales, find explicit solutions in limiting regimes, etc.).
- Facilitated for “quadratic” MFG thanks to the connection with Non-linear Schrödinger equation.
Quadratic mean field games

- $N$ agents, state $X^i \in \mathbb{R}^n$ with Langevin dynamics $dX^i_t = a^i_t dt + \sigma dw^i_t$

- Cost function $\int_t^T d\tau \left[ \frac{\mu}{2} (a^i_{\tau})^2 - V[m](X^i_{\tau}, \tau) \right] + c_T(X^i_T)$

- System of coupled PDE's $[a(x, t) = -\nabla_x u(x, t), m(x, t) \equiv \text{density of agents}]$

$$\begin{cases} 
\partial_t m + \nabla_x (am) - \frac{\sigma^2}{2} \Delta_x m = 0 & \text{(Kolmogorov)} \\
m(x, t=0) = m_0(x) 
\end{cases}$$

$$\begin{cases} 
\partial_t u - \frac{1}{2\mu} (\nabla_x u)^2 + \frac{\sigma^2}{2} \Delta_x u = -\nabla_x V[m](x, t) & \text{(HJB)} \\
u(x, t=T) = c_T(x) 
\end{cases}$$
Mapping of quadratic mean field games to the non-linear Schrödinger equation

**Quadratic mean field games**

- $N$ agents, state $\mathbf{X}^i \in \mathbb{R}^n$ with Langevin dynamics $d\mathbf{X}^i_t = \mathbf{a}^i_t dt + \sigma dw^i_t$

- Cost function $\int_t^T d\tau \left[ \frac{\mu}{2} (\mathbf{a}^i_{\tau})^2 + V[m](\mathbf{X}^i_{\tau}, \tau) \right] + c_T(\mathbf{X}^i_T)$

- System of coupled pde’s $[a(x, t) = -\nabla_x u(x, t), m(x, t) \equiv \text{density of agents}]$

\[
\begin{align*}
\frac{\partial_t m + \nabla_x (am) - \frac{\sigma^2}{2} \Delta_x m = 0}{m(x, t=0) = m_0(x)} & \quad \text{(Kolmogorov).} \\
\frac{\partial_t u - \frac{1}{2\mu} (\nabla_x u)^2 + \frac{\sigma^2}{2} \Delta_x u = -\nabla_x V[m](x, t)}{u(x, t=T) = c_T(x)} & \quad \text{(HJB).}
\end{align*}
\]
Transformation to NLS

- Cole-Hopf transform: \( \Phi(x, t) = \exp \left( -\frac{1}{\mu \sigma^2} u(x, t) \right) \)

\[-\mu \sigma^2 \partial_t \Phi = \frac{\mu \sigma^4}{2} \Delta_x \Phi + V[x, m] \Phi \]

- “Hermitization” of Kolmogorov: \( \Gamma(x, t) \equiv m(x, t) \exp \left( u(x, t)/(\mu \sigma^2) \right) \)
  (i.e. \( m(x, t) = \Gamma(x, t) \Phi(x, t) \))

\[\sigma^2 \partial_t \Gamma - \frac{\sigma^4}{2} \Delta_x \Gamma = \frac{\Gamma}{\mu} \left( \frac{\partial u}{\partial t} - \frac{1}{2\mu} (\nabla_x u)^2 + \frac{\sigma^2}{2} \Delta_x u \right) \]

\[\mu \sigma^2 \partial_t \Gamma = \frac{\mu \sigma^4}{2} \Delta_x \Gamma + V[x, m] \Gamma \]
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\[
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\]
Bose-Einstein condensates

At very low temperature (and sufficiently high density), systems of bosons "condensates" → all particles (the rubidium atoms here) are in the same "quantum state" \( \Psi(x, t) \)

- quantum mechanics of a particle of mass \( \mu \) in potential \( U_0(x) \)
  \[
  i\hbar \partial_t \Psi = -\frac{\hbar^2}{2\mu} \Delta_x \Psi + U_0(x)\Psi \quad \text{(Schrödinger)}
  \]

- Many particles with local interaction \( V(x - x') = g \delta(x - x') \)
  Mean field \( \Rightarrow \) \( U_0 \rightarrow U_0(x) + g|\Psi|^2 \)

Rubidium atoms (170 nK)
\[ i\hbar \partial_t \Psi = -\frac{\hbar^2}{2\mu} \Delta_x \Psi + U_0(x) \Psi + g|\Psi|^2 \Psi \]

(Non-linear Schrödinger (or Gross-Pitaevskii) equation)
\[ i\hbar \partial_t \Psi = -\frac{\hbar^2}{2\mu} \Delta_x \Psi + U_0(x) \Psi + g|\Psi|^2\Psi \]

(Non-linear Schrödinger (or Gross-Pitaevskii) equation)

- MFG equations, specifying to \( V[m](x) \equiv U_0(x) + gm(x, t) \)

\[
\begin{align*}
\mu \sigma^2 \partial_t \Gamma &= \frac{\mu \sigma^4}{2} \Delta_x \Gamma + U_0(x) \Gamma + g m \Gamma \\
-\mu \sigma^2 \partial_t \Phi &= \frac{\mu \sigma^4}{2} \Delta_x \Phi + U_0(x) \Phi + g m \Phi
\end{align*}
\]
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-\mu \sigma^2 \partial_t \Phi &= \frac{\mu \sigma^4}{2} \Delta_x \Phi + U_0(x) \Phi + g m \Phi
\end{align*}
\]

Formal change \( (\Psi, \Psi^*, \hbar) \rightarrow (\Phi, \Gamma, i\mu \sigma^2) \) maps NLS to MFG !!!
Why the excitement?

- Man Field Games exist since 2005-2006, the Non-Linear Schrödinger equation since at least the work of Landau and Ginzburg on superconductivity in 1950.

- NSL applies to many field of physics: superconductivity, non-linear optic, gravity waves in inviscid fluids, Bose-Einstein condensates, etc..

→ huge literature on the subject

- We feel we have a good qualitative understanding of the “physics” of NLS, together with a large variety of technical tools to study its solutions.

[NB : Change of variable giving NLS known by Guéant, (2011)]
A case study: a quadratic mean field game in the strong positive coordination regime

To illustrate how this ‘transfer of knowledge’ works, consider a simple (but non-trivial) quadratic mean field game:

- \( d = 1 \)
- Local interaction \( V[m](x) = U_0(x) + g m \)
- Strong positive coordination (large positive \( g \))

(If it helps, think of it as a population dynamics model for a aquatic specie living in a river:

- \( U_0(x) \equiv \) intrinsic quality of the location (e.g. for food gathering)
- \( g \) measure the protection from predator by other members of the group.
- \( T = \) daylight duration, \( m_0(x) = \) initial distribution in the morning, \( c_T(x) = \) quality of shelter for the night)
Schrödinger vs Heisenberg representation and Ehrenfest relations

Quantum mechanics

- State of the system $\equiv$ wave function $\Psi(x, t)$
- Observables $\equiv$ operators: $\hat{O} = f(\hat{p}, \hat{x})$
- Average $\langle \hat{O} \rangle \equiv \int dx \Psi^*(x) \hat{O} \Psi(x)$
- Hamiltonian $\equiv \hat{H} = \frac{\hat{p}^2}{2\mu} + V(x) = -\frac{\hbar^2}{2\mu} \Delta_x + V(x)$

\[
\begin{align*}
\hat{x} & \equiv x \times \\
\hat{p} & \equiv i\hbar \partial_x
\end{align*}
\]

\[
i\hbar \partial_t \Psi = \hat{H} \Psi \quad \Rightarrow \quad i\hbar \frac{d}{dt} \langle \hat{O} \rangle = \langle [\hat{H}, \hat{O}] \rangle
\]

\[
\begin{cases}
\frac{d}{dt} \langle \hat{x} \rangle = \frac{1}{\mu} \langle \hat{p} \rangle \\
\frac{d}{dt} \langle \hat{p} \rangle = -\langle \nabla_x V(\hat{x}) \rangle
\end{cases}
\]

(Ehrenfest)
Quadratic Mean Field Games

- Operators: \( \hat{X} \equiv x \times \hat{\Pi} \equiv \mu \sigma^2 \partial_x \hat{O} = f(\hat{\Pi}, \hat{X}) \)

- Average: \( \langle \hat{O} \rangle(t) \equiv \int dx \Gamma(x, t) \hat{O} \Phi(x, t) \)
  
  \[ m = \Gamma \Phi \]

  \[ \Rightarrow \quad \text{if } \hat{O} = O(\hat{X}, \hat{\Pi}) \quad \langle \hat{O} \rangle \equiv \int dx \, m(x) O(x) \]

  \[ \left( \langle \hat{1} \rangle \equiv \int dx \, m(x) = 1 \quad \langle \hat{X} \rangle \equiv \int dx \, xm(x) \right) \]

- Hamiltonian \( \hat{H} = \frac{\hat{\Pi}^2}{2\mu} + V[m](x) = \frac{\mu \sigma^4}{2} \Delta_x + V[m](x) \)

\[
\begin{cases} 
+ \mu \sigma^2 \partial_t \Gamma = \hat{H} \Gamma \\
- \mu \sigma^2 \partial_t \Phi = \hat{H} \Phi 
\end{cases} \Rightarrow \quad \mu \sigma^2 \frac{d}{dt} \langle \hat{O} \rangle = \langle [\hat{H}, \hat{O}] \rangle \]
Exact relations

Force operator: \( \hat{F}[m_t] \equiv -\nabla_x V[m_t](\hat{X}) \)

\[
(V[m_t] = U_0 + gm_t \rightarrow \hat{F}[m_t] \equiv \begin{cases} \hat{F}_0 & -g \nabla_x m_t \\ -\nabla_x U_0 \end{cases})
\]

\( \Sigma^2 \equiv \langle (\hat{X}^2) - \langle \hat{X} \rangle^2 \rangle \quad \Lambda \equiv \langle \hat{X} \hat{\Pi} + \hat{\Pi} \hat{X} \rangle - 2\langle \hat{\Pi} \rangle \langle \hat{X} \rangle \)

\[
\begin{aligned}
\frac{d}{dt} \langle \hat{X} \rangle &= \frac{1}{\mu} \langle \hat{\Pi} \rangle \\
\frac{d}{dt} \langle \hat{\Pi} \rangle &= \langle \hat{F}[m_t] \rangle
\end{aligned}
\]

\[
\begin{aligned}
\frac{d}{dt} \Sigma^2 &= \frac{1}{\mu} \left( \langle \hat{X} \hat{\Pi} + \hat{\Pi} \hat{X} \rangle - 2\langle \hat{\Pi} \rangle \langle \hat{X} \rangle \right) \\
\frac{d}{dt} \Lambda &= -2\langle \hat{X} \hat{F}[m_t] \rangle + 2\langle \hat{\Pi}^2 \rangle
\end{aligned}
\]

\( \mathcal{E}_{\text{tot}}(t) \equiv \frac{1}{2\mu} \langle \hat{\Pi}^2 \rangle + \langle U_0(\hat{X}) \rangle + \langle \hat{H}_{\text{int}} \rangle \quad \equiv \quad \text{conserved quantity} \)

\( \langle \hat{H}_{\text{int}} \rangle \equiv \frac{g}{2} \int dx \: m_t(x)^2 \)
Ergodic solution

Stationary non-linear Schrödinger

Let $\Psi_e(x)$ the solution of the stationary NLS

$$\lambda \Psi_e = \frac{\mu \sigma^4}{2} \Delta_x \Psi_e + U_0(x) \Psi_e + g |\Psi_e|^2 \Psi_e$$

Define

$$\begin{align*}
\Gamma_e(x, t) & \equiv \exp \left( + \frac{\lambda}{\mu \sigma^2} t \right) \Psi_e(x) \\
\Phi_e(x, t) & \equiv \exp \left( - \frac{\lambda}{\mu \sigma^2} t \right) \Psi_e(x)
\end{align*}$$

$$\Rightarrow \text{ solution of } \begin{cases} 
\mu \sigma^2 \partial_t \Gamma = \frac{\mu \sigma^4}{2} \Delta_x \Gamma + U_0(x) \Gamma + g m \Gamma \\
-\mu \sigma^2 \partial_t \Phi = \frac{\mu \sigma^4}{2} \Delta_x \Phi + U_0(x) \Phi + g m \Phi
\end{cases}$$

with $m_e(x) \equiv \Gamma_e(x, t) \Phi_e(x, t) = |\Psi_e(x)|^2 = \text{const.}$
Limiting case $U_0(x) \equiv 0$  (NB: $g > 0$)

In that case solution of stationary NLS known (bright soliton)

$$\Psi_e(x) = \frac{\sqrt{\eta}}{2} \frac{1}{\cosh \left( \frac{x}{2\eta} \right)}$$

$\eta \equiv 2\mu\sigma^4/g$

characteristic length scale

“Strong coordination” regime

- meaning: variations of $U_0(x)$ on the scale $\eta$ are small

- ergodic state

$$m_e(x) \simeq \frac{\eta}{4} \frac{1}{\cosh^2 \left( \frac{x - x_{\text{max}}}{2\eta} \right)}$$

$x_{\text{max}} = \arg\max[U_0]$
Ehrenfest relations & ergodic solution → most of the story (for strong positive coordination)

Generic scenario

1. Herd formation: extension = \eta, position \( x_0 = \langle x \rangle_{m_0} \) (very short time process)
2. Propagation of the group:
   - as a classical particle of mass \( \mu \) in pot \( U_0(x) \)
   - initial position \( x(0) = x_0 \)
   - final velocity \( \dot{x}(T) = -\partial_x c_T(x(T)) \)
3. Herd dislocation near \( t = T \) (again very short time process).

NB: boundary pb, rather than initial value pb → possibly more than one solution
Propagation phase in the large $T$ regime (Cardialaguet)

Assuming $U_0(x)$ bounded and with a single maximum (at $x_{\text{max}}$), the only way not to be sent to $\infty$ as $T \to \infty$ is to spend most of the time near $x_{\text{max}}$, which is an unstable fix point.

Thus, propagation phase decomposes into:

a. Start from $x_0$ with a total energy $U_0(x_{\text{max}})$

b. Approach $x_{\text{max}}$ following its stable manifold

c. Stay close to $x_{\text{max}}$ as long as necessary

d. Move away from $x_{\text{max}}$ following its unstable manifold

e. Arrive at $T$ with final velocity $\dot{x}(T) = -\partial_x c_T(x(T))$

If more than one maximum, possible phase transition (discontinuous variation of the solution) as $T$ increases, as the system switches from one maximum to another.
Herd formation

First stage of dynamic = herd formation.

- It takes place on a short time scale.
- Can we be more precise?

- Assume initial distribution $m_0(x)$ "featureless", i.e. well characterized by its mean $x_0$ and variance $\Sigma^2$

- Neglect $U_0$ during the herd formation phase

\[\Gamma(x, t) = e^{-\gamma(t)/\sigma^2} \frac{1}{\left(2\pi \Sigma^2 \sum_{x_i}^2\right)^{1/4}} e^{-\frac{(x-x_0)^2}{4\Sigma^2}(1-\frac{\Lambda(t)}{\sigma^2})}\]

\[\Phi(x, t) = e^{+\gamma(t)/\sigma^2} \frac{1}{\left(2\pi \Sigma^2\right)^{1/4}} e^{-\frac{(x-x_0)^2}{4\Sigma^2}(1+\frac{\Lambda(t)}{\sigma^2})}\]
Action:

\[
S[\Gamma(x, t), \Phi(x, t)] \equiv \int dt \, dx \, \left[ \frac{\sigma^2}{2} (\partial_t \Phi \Gamma - \Phi \partial_t \Gamma) - \frac{\sigma^4}{2\mu} \nabla \Phi \cdot \nabla \Gamma + U_0(x) \Phi \Gamma + \frac{g}{2} \Phi^2 \Gamma^2 \right]
\]

\[
\begin{aligned}
\Lambda & = -\frac{\sigma^4}{2\mu} (1 - \frac{\Lambda^2}{4}) \frac{1}{\Sigma^2} + \frac{g}{2\sqrt{\pi} \Sigma} \\
\Sigma^2 & = \frac{\Lambda}{\mu}
\end{aligned}
\]

hyperbolic fixed point: \( \Lambda^* = 0 \) \( \Sigma^* = \sqrt{\pi} \frac{\mu \sigma^4}{g} \)

\(~\text{soliton scale} \, \eta\)
Flow near the fix point

Large $T$: need to stay on stable and unstable manifold of the fixed point.

\[
\frac{d^2}{dt^2} \sum^2 = \frac{g}{2\mu \sqrt{\pi}} \left( \frac{1}{\Sigma_*} - \frac{1}{\Sigma(t)} \right)
\]

\[-(z_t - z_i) - \log \left( \frac{1 - z_t}{1 - z_i} \right) = \frac{t}{\tau^*} \]

\[z_t \equiv \frac{\Sigma}{\Sigma_*} \]

\[\tau^* \equiv 2\pi \sqrt{\frac{\mu \eta^3}{g}} \]
Comparison with numerical simulation
Mean field games = new tool to study a variety of socio-economic problems

Formal, but deep, relation between a class of mean field games and the Non-Linear Schrödinger equation dear to the heart of physicists

Classical tools developed in that context (Ehrenfest relations, solitons, variational methods, etc..) can be used to analyze mean field games

Here: application to a simple population dynamics model
→ rather thorough understanding of this model

It seems rather clear that the connection with NLS will eventually provide a good level of understanding for all quadratic mean field games
Two open (longer term) questions

- Quadratic mean field games represent a kind of paradigm of mean field game. How much is this true?
  - Can we find realistic (application oriented) mean field games in that class?
  - Is the qualitative behavior of quadratic mean field games generic?

- Can fishes solve the MFG equations (even in their NLS form)?