Does wave ~~beat~~ breaking really always lead to such a simple shock structure? In general the answer is no, but one can argue that in the presence of viscosity this is indeed the case.

For the traffic equations let's slighty modify the model in which instead $Q(\rho) \longrightarrow Q(\rho) - \nu\rho_x$ (with $\nu>0$) this means that if a driver sees an increasing density ahead he will slow down. In this case the conservation equation becomes:

$$\rho_t + [Q(\rho) - \nu\rho_x]_x = 0$$

leading to $\boxed{\rho_t + c(\rho)\rho_x = \nu\rho_{xx}}$ where $Q'(\rho) = c(\rho)$ as usual.

this is called an advection-diffusion equation: $c(\rho)\rho_x$ is the advective part and $\nu\rho_{xx}$ the diffusive term.

nonlinear

indeed without the advective term one has a simple heat equation: $\rho_t = \nu\rho_{xx}$ which is THE diffusion equation

note that the term $\nu\rho_{xx}$ is indeed a viscous term: (1) it is physical intuitive that it will tend to damp sharp fluctuations but (2) it can also be seen by studying the propagation of small perturbations:

$$\rho(x,t) = \rho_0 + \rho_1(x,t) \qquad |\rho_1| \ll \rho_0 \qquad c(\rho_0) \equiv c_0$$

one has $\rho_{1t} + c_0\rho_{1x} = \nu\rho_{1xx}$. If one look for plane wave solutions of the form:

$$\rho_1 = A e^{i(kx-\omega t)}$$

one gets

$$\omega = c_0 k - i\nu k^2 \Rightarrow \text{hence}$$

$$\rho_1(x,t) = A e^{ik(x-c_0 t)} \, e^{-\nu k^2 t}$$

damping term. Note that ν has to be positive otherwise: instability (ρ_1 diverges). But this was clear from the beginning. (Q should ↓ when $\rho_x > 0$)

one aims at describing a shock structure propagating at constant velocity U for the new equation (which would be the $\nu \neq 0$ version of the stupid shock of pages 13 & 14 if everything goes ok).

one looks for a steady profile $\rho = \rho(X)$ with $X = x - Ut$. then the eq. reads:

$$[-U+c(\rho)]\rho_X = \nu \rho_{XX} \xrightarrow[\text{integral}]{\text{first}} -U\rho + \Phi(\rho) + A = \nu \rho_X$$

(A = constant of integration)

writing $\int dX = \int \frac{d\rho}{\rho_X}$ one can compute, at least formally, the shape of $\rho(X)$. But first a remark: One wants an asymptotic profile such that

$$\lim_{X \to \pm\infty} \rho(X) = \rho_{2/1}$$

one thus has: $-U\rho_1 + \Phi(\rho_1) + A = 0 = -U\rho_2 + \Phi(\rho_2) + A$

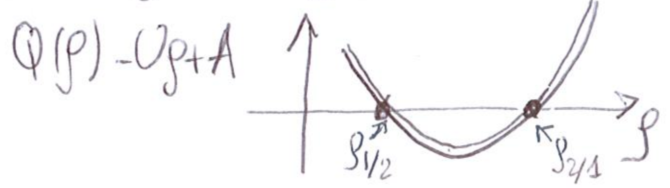
hence: $\boxed{U = \frac{\Phi(\rho_2) - \Phi(\rho_1)}{\rho_2 - \rho_1}}$ this is the shock velocity determined page (13)!

shape $\rho(X)$: one has $\frac{X}{\nu} = \int \frac{d\rho}{\Phi(\rho) - U\rho + A}$

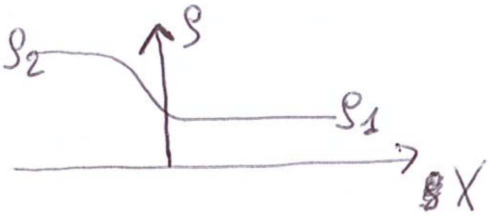
ρ_1 and ρ_2 are zero of $\Phi(\rho) - U\rho + A$: hence as $\rho \to \rho_{1/2}$, X diverges and tends to $\pm\infty$ as required. Let's ~~consider~~ discuss the signs in the case where $c'(\rho) > 0$ (which is the case considered in pages 13-14 (actually starting from the end of page 9) and does not corres-pond to the situation in traffic flows.

$\nu > 0$, then $\text{sign}(\rho_X) = \begin{bmatrix} \text{sign}(\Phi(\rho) - U\rho + A) \\ \text{between } \rho_1 \text{ and } \rho_2 \end{bmatrix}$

~~if $\rho_1 > \rho_2$~~ then since $c'(\rho) = \Phi''(\rho) > 0$, Φ is convex and one has:



$\Phi(\rho) - U\rho + A < 0$ between the roots thus $\rho_X < 0$ and one must have $\rho_1 > \rho_2$ for describing the shock which looks like:



and indeed, this is the physically expected configuration for a shock when $c(\rho) > 0$.

Finally it is clear that if ρ_1 and ρ_2 are kept fixed (so that U and A are fixed) a change in ν can be absorbed by a change in the X scale. One has:

$$\rho = F(X/\nu)$$

where F depends on ρ_1 and ρ_2 but not on ν: as $\nu \to 0$ the profile is compressed in the X direction and tends to a step function having all the characteristics computed previously for $\nu = 0$. The shock structure is only a special solution of $\rho_t + c(\rho)\rho_x = \nu\rho_{xx}$, but from it we might expect in general that when $\nu \to 0$ the solution of this eq. tends to the one of $\rho_t + c(\rho)\rho_x = 0$.

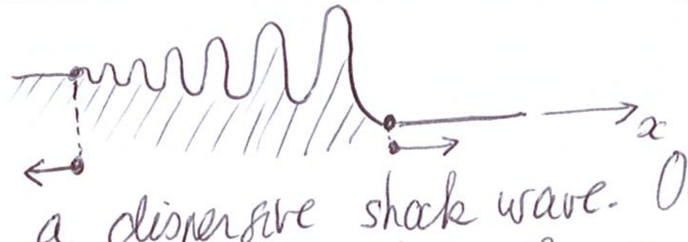
However in some cases the shock structure is not resolved by viscosity, but by dispersion effets instead. This is the case for the ~~KdV~~ eq for instance where:

$$\rho_t + c(\rho)\,\rho_x + \nu\rho_{xxx} = 0$$

(here the sign of ν is not of great importance: $\omega = c(\rho_0)k - \nu k^3$)

this has been studied alot, and for KdV, when $c(\rho) = \rho$ an initial condition:

leads to:



this is called a dispersive shock wave. Occurs in tidal bore for instance. Let's now study the combined effects of non linearity and dispersion.

one start with $\rho_t + c(\rho)\rho_x + \nu\rho_{xxx} = 0$

weak non linearity $c(\rho) \simeq c_0 + c_1\rho$

weak dispersive effect. At least, the 1st one can think of.

$$\begin{cases}\xi = x - c_0 t\\ \tau = t\end{cases} \rightarrow \begin{cases}\partial_x = \partial_\xi\\ \partial_t = -c_0\partial_\xi + \partial_\tau\end{cases} \rightarrow -c_0\rho_\xi + \rho_\tau + (c_0 + c_1\rho)\rho_\xi + \nu\rho_{\xi\xi\xi} = 0$$

$$\rightarrow \rho_\tau + c_1\rho\rho_\xi + \nu\rho_{\xi\xi\xi} = 0$$

note: one can change the sign of ν by doing $\tau \to -\tau$, $\xi \to -\xi$, $\rho \to -\rho$. So let's assume that $\nu > 0$

then one defines: $\begin{cases} t = \alpha\tau\\ x = \beta\xi\\ \rho = \gamma u\end{cases}$ this leads to:

$$\alpha\gamma u_t + c_1\beta\gamma^2 u u_x + \nu\beta^3\gamma u_{xxx} = 0$$

take $\alpha = 1$, $c_1\beta\gamma = 1$ and $\nu\beta^2 = 1$ $\rightarrow \boxed{u_t + uu_x + u_{xxx} = 0}$

KdV equation first derived in the study of shallow water waves in 1895. here the eq. is written in dimensionless form.

Let's do the same as we did for ~~the~~ Burger's equation: look for solutions depending on $x - Vt$ only: $u(x,t) = u(x - Vt)$. Then one gets

$$-Vu' + uu' + u''' = 0$$

first integral: $-Vu + \frac{u^2}{2} + u'' = C_1$

multiply by u' and integrate again:

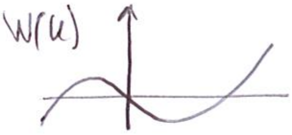
$$-V\frac{u^2}{2} + \frac{u^3}{6} + \frac{1}{2}u'^2 = C_1 u + C_2$$

this is of the form

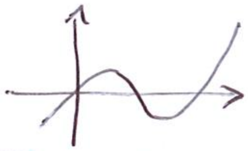
$$\boxed{\frac{u'^2}{2} + \underbrace{W(u)}_{\frac{u^3}{6} - V\frac{u^2}{2} - C_1 u} = C_2}$$

(I assume hence forth that $V > 0$. change the sign of V by changing the sign of t)

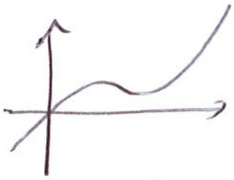
if $q>0$ one has:



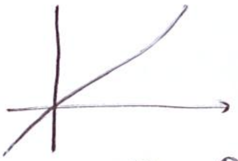
if $q<0$ one has



or



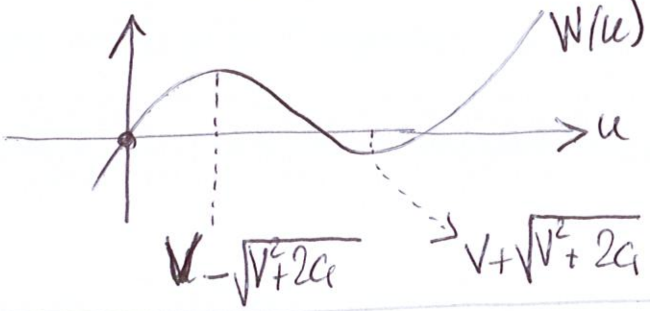
or



for having a bounded solution one needs that W(u) has a local minimum = $W'(u) = \frac{u^2}{2} - Vu - q = 0$ one has real roots iff $V^2 + 2q > 0$ ← this will be assumed as verified henceforth

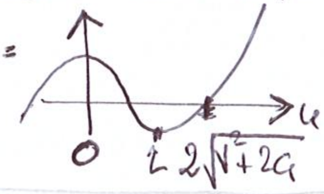
then W' cancels for $u = V \pm \sqrt{V^2+2q}$

one thus has:



(plot for $q<0$)

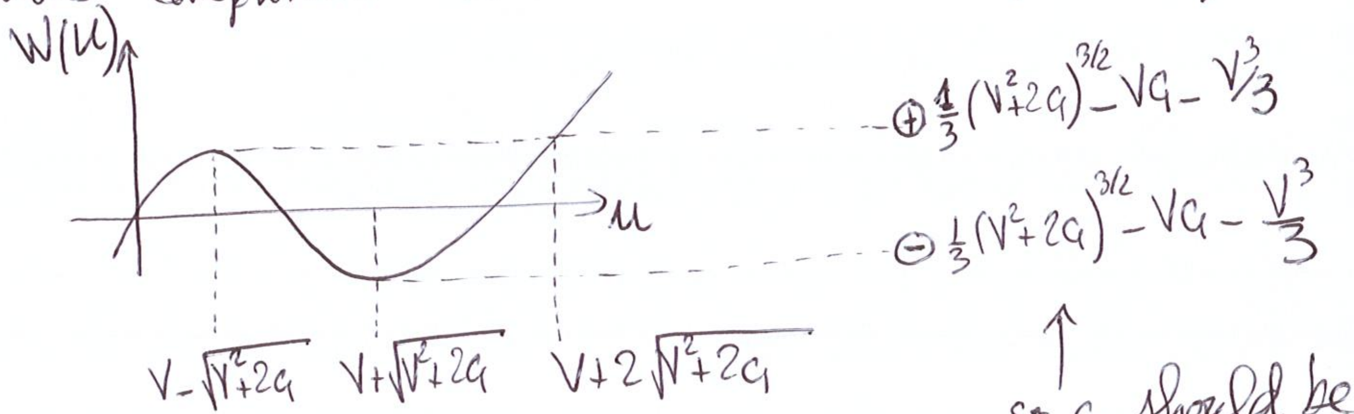
by a change of variable $u = v(x-Vt) + V - \sqrt{V^2+2q}$ one can work in a configuration with:



but this is not necessary.

INUTILE

a teadious computation shown that one has (cf page 20)



↑

so, C_2 should be in this range for having a bounded solution.

side computation: let's denote $u_0 = V - \sqrt{V^2+2a}$ the local maximum

$$W(u_0) = \frac{1}{6}u_0^3 - \frac{V}{2}u_0^2 - a u_0^* = \frac{u_0}{3}\Big(\underbrace{\frac{u_0^2}{2}}_{\boxed{Vu_0+a}} - \frac{3}{2}Vu_0^* - 3a\Big) = \frac{u_0}{3}\Big(-\frac{V}{2}u_0 - 2a\Big) =$$

$$= -\frac{V}{3}\underbrace{\frac{u_0^2}{2}}_{Vu_0+a} - \frac{2}{3}a u_0 = -\frac{u_0}{3}(V^2+2a) - Va/3 = -\frac{1}{3}(V-\sqrt{V^2+2a})(V^2+2a) - Va/3$$

$$= +\frac{1}{3}(V^2+2a)^{3/2} - Va - V^3/3$$

look for the solution of $W(u) = W(u_0)$. u_0 is a double solution, so this equation, which reads $\boxed{u^3 - 3Vu^2 - 6au = 2(V^2+2a)^{3/2} - 6Va - 2V^3}$ can be writen under the form:

$$(u-u_0)^2(u-u_M) = 0$$

$$\Leftrightarrow (u^2 - 2u_0u + u_0^2)(u-u_M) = 0$$

$$\Leftrightarrow u^3 + u^2(-u_M - 2u_0) + u(\underbrace{u_0^2}_{2Vu_0+2a} + 2u_0u_M) = u_M\underbrace{u_0^2}_{2Vu_0+2a}$$

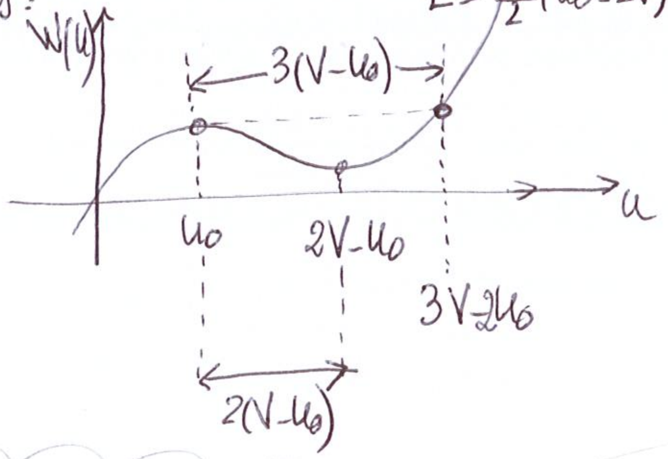
$$\Leftrightarrow \boxed{u^3 - u^2(u_M + 2u_0) + u(2(u_M+V)u_0 + 2a) = 2(Vu_0 + a)u_M}$$

comparing the 2 framed eqs one gets $u_M = V + 2\sqrt{V^2+2a}$

this value makes it possible to match the coeffs of ~~the~~ all the powers of u in the framed eqs.

$u_0 = V - \sqrt{V^2+2a}$

other parametrization: It is more appropriate to ~~define~~ use u_0 as a parameter. Then $a = \begin{cases} \frac{1}{2}(V-u_0)^2 - \frac{V^2}{2} \\ = \frac{u_0}{2}(u_0 - 2V) \end{cases}$ and $\sqrt{V^2+2a} = V - u_0$ and one has:



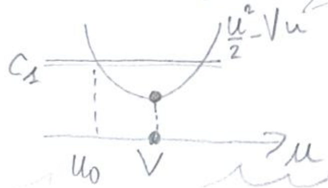
and the condition $V^2+2a > 0$ is always fulfilled.

one has also always $V > u_0$

(but u_0 can be >0 or <0)

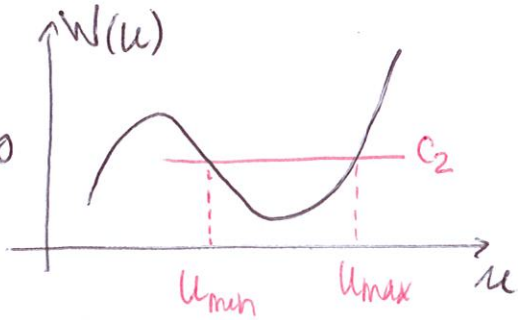
of course: $W'(u) = 0$ reads $\frac{u^2}{2} - Vu = c_1$

so, one has:



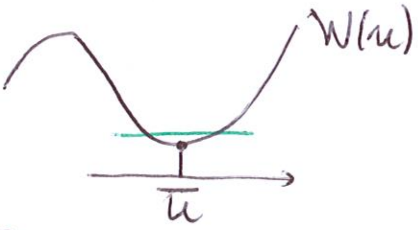
= indeed $u_0 < V$

so for given c_2 one has:
u has non linear oscillations between u_{min} and u_{max}



* when c_2 is close to the minimum

$$\left[\begin{array}{l} W(u)=\frac{u^3}{6}-V\frac{u^2}{2}-c_1u \rightarrow \\ W'(u)=\frac{u^2}{2}-Vu-c_1=\underset{\text{at }\bar{u}}{\text{zero}} \\ W''(u)=u-V \end{array}\right.$$



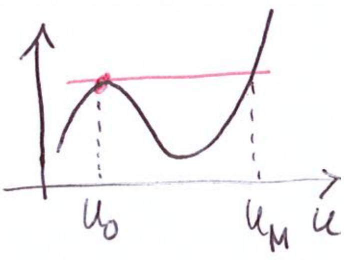
close to $\bar{u}$ one writes $W(u) \simeq W(\bar{u})+\frac{1}{2}(u-\bar{u})^2 W''(\bar{u})$
then the eq. $\frac{u'^2}{2}+W(u)=c_2$ leads (after derivation) to =

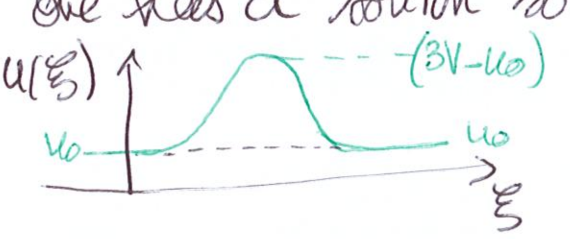
$$u''+(u-\bar{u})W''(\bar{u})=0$$

sol $u(x-Vt=\xi)=\bar{u}+u^{(1)}(\xi)$ where $u^{(1)}(\xi) \propto e^{\pm i\sqrt{W''(\bar{u})}\,\xi}$

this is of the form $\pm i(kx-\omega t)$ with $k=\sqrt{W''(\bar{u})} \rightarrow k^2=\bar{u}-V$
and $\omega=V\sqrt{W''(\bar{u})}=Vk$
since $V=\bar{u}-k^2$ the dispersion relation reads $\omega=(\bar{u}-k^2)k$ as expected.

* more interesting = when c_2 is at the maximum:
then one has a soliton solution =





one has $\frac{u'^2}{2}+W(u)=W(u_0) \rightarrow \frac{u'^2}{2}=W(u_0)-W(u)=\frac{1}{6}(u-u_0)^2(u_M-u)$

writing $u=u_0+v(\xi) \rightarrow v'^2=\frac{1}{3}v^2(v_M-v)$ where $v_M=u_M-u_0$
$=3(V-u_0)$

hence $\cancel{v'^2}$ $\frac{dv}{d\xi}=\pm\frac{v}{\sqrt{3}}(v_M-v)^{1/2}$

let's consider the case where $\frac{dv}{d\xi}>0$ = $\frac{d\xi}{\sqrt{3}}=\frac{(\pm)\,dv}{v(v_M-v)^{1/2}}$

one writes $v = \frac{v_M}{w^2} \rightarrow dv = -\frac{2 v_M \, dw}{w^3}$

and $\frac{d\xi}{\sqrt{3}} = (\pm) \frac{-2 v_M \, dw / w^3}{\frac{v_M^{3/2}}{w^2}\left(1 - \frac{1}{w^2}\right)^{1/2}} = -(\pm) \frac{2\, dw}{\sqrt{v_M}\sqrt{w^2-1}}$

then $w = ch\,\theta \quad dw = sh\,\theta\, d\theta = (\mp)\sqrt{1-w^2}\, d\theta$ with $\left[\begin{array}{l} - \text{ when } \theta < \\ + \text{ when } \theta > \end{array}\right.$

hence one gets $\frac{d\xi}{\sqrt{3}} = \frac{2\, d\theta}{\sqrt{v_M}} \rightarrow \theta = \frac{\sqrt{v_M/3}}{2}\xi = \frac{\sqrt{V-u_0}}{2} \equiv \xi$

$\hookrightarrow x - Vt$

hence = $\boxed{u(\xi) = u_0 + \frac{3(V-u_0)}{ch^2\left[\frac{\sqrt{V-u_0}}{2}\xi\right]}}$

Better way = one writes $u = u_0 + v$

then $W(u) = \frac{u^3}{6} - \frac{Vu^2}{2} - \frac{u_0}{2}(u_0 - 2V)u$ $\left[\text{using } c_1 = \frac{u_0}{2}(u_0 - 2V)\right]$

and simple manipulations show that

$$W(u) - W(u_0) = \frac{v^3}{6} + \frac{v^2}{2}(u_0 - V)$$

then the eq. $\frac{u'^2}{2} + W(u) = W(u_0)$ reads $\boxed{v'^2 + \frac{v^3}{3} + (u_0 - V)v^2 = 0}$

and there one look for a solution of the type $v = \frac{a}{ch^2(B\xi)} \rightarrow v' = -2aB\frac{sh(B\xi)}{ch^3(B\xi)}$

and plugging this back in the framed equation yields

$$4a^2B^2\frac{sh^2}{ch^6} + \frac{a^3}{3\,ch^6} + \frac{a^2(u_0 - V)}{ch^4} = 0$$

$$4a^2B^2 \underset{\substack{\swarrow \\ (ch^2-1)}}{sh^2} + \frac{a^3}{3} + a^2(u_0 - V)ch^2 \longrightarrow \left\{\begin{array}{l} B^2 = \frac{V-u_0}{4} \\ a = 12B^2 = 3(V-u_0) \end{array}\right.$$

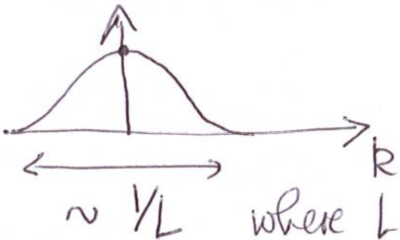
on retrouve bien-sûr la formule en cadrée !

~~note = $V \propto u_M - u_0$~~

hand-waving argument

(23)

the dispersion relation $\omega = u_0 k - k^3$ gives the phase velocity $V_\varphi = u_0 - k^2$ for our soliton we have a Fourier transform (of $u(x) - u_0$) =



$\sim 1/L$ where L is the (yet unknown) width of the soliton.

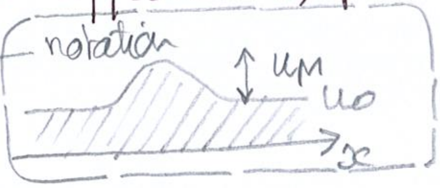
↳ note = exact result

$$\int_{\mathbb{R}} dx \frac{e^{ikx}}{ch^2(\beta x)} = \frac{\pi k/\beta}{sh\left(\frac{\pi k}{2\beta}\right)}$$

so

$$\int dx\, e^{ikx}(u - u_0) = \frac{3\sqrt{V-u_0}\,\pi k/2}{sh\left(\frac{\pi k}{\sqrt{V-u_0}}\right)}$$

* usually ≠ k's have ≠ V_φ ⇒ spreading. this spreading is compensated by non linear effects. So, for a given k $V_\varphi \approx u_0 + u_M)_{typ} - k^2)_{typ}$

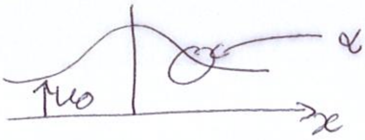
notation


these 2 effects should compensate ⇒ (ad hoc description of NL effect)

$$\boxed{u_M \sim k^2)_{typ} \sim 1/L^2}$$

relies on the argument that soliton moves without changing its shape

note that $u_M > 0$ = where the sign of dispersion different ($\omega = u_0 k + k^3$) then $u_M < 0$

* real space  $\propto \exp[-\tilde k(x - Vt)]$ since in this region the soliton is a small perturbation of the background, one can perform a linear expansion similar to what is done for $e^{i(kx - \omega t)}$

hence $\tilde k \leftrightarrow -ik$ $\quad -i\omega \leftrightarrow V\tilde k$ and $V = \frac{-i\omega(k = i\tilde k)}{\tilde k}$

since $V = \frac{\omega(k)}{k}$

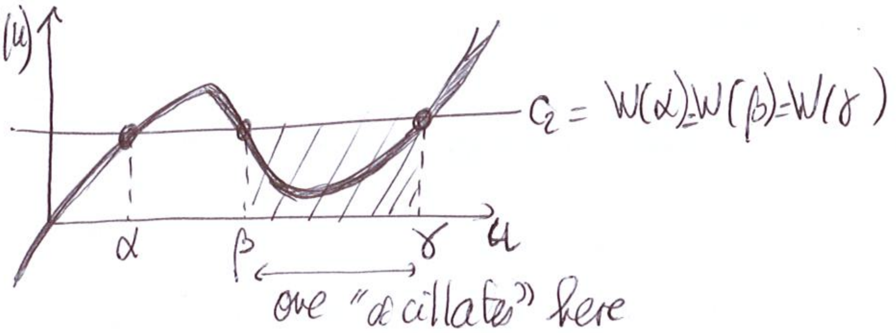
here $\omega(k) = u_0 k - k^3$ this leads to $\left\{ \begin{array}{l} V = \frac{-i(u_0 i\tilde k + i\tilde k^3)}{\tilde k} \\ \quad = u_0 + \tilde k^2 \end{array} \right.$

↳ of course $\tilde k = 1/L$ and one gets $\boxed{V \sim u_0 + 1/L^2}$

* gathering the two evaluations one gets = $\left\{ \begin{array}{l} L \sim \frac{1}{\sqrt{V-u_0}} \\ u_M \sim V - u_0 \end{array} \right.$

(in accordance with the exact result)

→ there are also NL waves (KdV 1895): CNOIDAL WAVES



one has $\frac{u'^2}{2} = \underbrace{c_2 - W(u)}_{>0 \text{ in the allowed region}}$ = it's a 3rd order polynomial = can be writen as $= \frac{1}{6}(u-\alpha)(u-\beta)(\gamma-u)$

then $\int \frac{d\xi}{\sqrt{3}} = \underbrace{\int_\beta^u \frac{du}{\sqrt{(u-\alpha)(u-\beta)(\gamma-u)}}}_{\text{elliptic integral}}$ → $u(\xi)$ is an elliptic ~~integral~~ function. these are denoted as cn, sn or dn hence one speaks of "cnoidal waves".

when c_2 changes one goes from sinusoidal, to cnoidal an then to a soliton.

---

ubiquity of KdV =

it is a very natural PDE for who aims at describing both non linearity and dispersion - Start from: $\omega = ck$, and then try to add these two effects =

$$c \rightarrow c_0 + \underset{\substack{\uparrow \\ c_1 = 1 \text{ for correctly chosen units}}}{c_1} \times \left(\begin{matrix}\text{wave} \\ \text{amplitude} = u\end{matrix}\right) + \overset{\pm}{\underset{\substack{\uparrow \\ \text{there is a possible prefactor, I put it} = 1}}{\nu}} k^2$$

of course k^2 = only the wavelength matters, no its "sign"!

this leads to =

$\omega = c_0 k + u k + \nu k^3$

and then $\begin{bmatrix} \omega \leftrightarrow -i\partial_t \\ k \leftrightarrow +i\partial_x \end{bmatrix}$ leads to = $-iu_t = ic_0 u_x + iuu_x - \nu i u_{xxx}$ the c_0 term can be re-absorbed by a re-definition of x (work in the moving frame, cf p18) leading to =

$$\boxed{u_t + u u_x - \nu u_{xxx} = 0}$$

KdV = Conserved quantities and lagrangian density =

for a general $u(x,t)$ which goes to a constant when $x \to \pm\infty$. then, by a change of function ($u \to u - u_0$) one can assume that this constant is zero [$v_t + (u_0+v)v_x + v_{xxx} = 0$ and the terms u_0 disappears if $x \to x - u_0 t$].

⇨ the quantity $\boxed{\int_{\mathbb{R}} u\,dx}$ is conserved since KdV reads

$$u_t + \left(\frac{u^2}{2} + u_{xx}\right)_x = 0$$

⇨ one also has $\boxed{\left(\frac{u^2}{2}\right)_t + \left[\frac{u^3}{3} + uu_{xx} - \frac{1}{2}(u_x)^2\right]_x = 0}$

(momentum density see below)

indeed this reads $uu_t + u^2u_x + uu_{xxx} - \cancel{u_x}u_{xx} - u_x\cancel{u_{xx}} =$

which agrees with KdV

hence $\boxed{\int_{\mathbb{R}} u^2\,dx}$ is also conserved.

⇨ one has also $\underbrace{(u^3 - 3u_x^2)_t}_{3u^2u_t - 6u_xu_{xt}} + \left[\frac{3}{4}u^4 + 3u^2u_{xx} - 6uu_x^2 + 3u_{xx}^2 - 6u_xu_{xxx}\right]_x = 0$

energy density (see below)

$$3u^2u_t - 6u_xu_{xt} = -3u^2u_x - 3u^2u_{xxx} \overset{\text{(using KdV)}}{+} 6uu_xu_{xx} + 6u_xu_{(4x)}$$

$u_t = -uu_x - u_{xxx}$

$u_{xt} = -u_x^2 - uu_{xx} - u_{(4x)}$

the divergence term reads:

$$\text{div} = 3u^2u_x + 6uu_xu_{xx} + 3u^2u_{xxx} - 6u_x^3 - 12uu_xu_{xx}$$
$$+6\cancel{u_{xx}u_{xxx}} - 6\cancel{u_{xx}u_{xxx}} - 6u_xu_{(4x)} =$$

this seems up to zero when added to the time-derivative terme

▨ these 2 last invariants are actually natural: they come from the lagrangian character of KdV =

$$\boxed{S = \int dx\,dt\,\mathcal{L} \quad \text{where } \mathcal{L} = \frac{1}{2}\varphi_x\varphi_t + \frac{1}{6}\varphi_x^3 - \frac{1}{2}(\varphi_{xx})^2}$$

here it's unusual because $\mathcal{L}$ depends on φ_{xx} ! so, the Euler-Lagrange eqs read:

$$\frac{\partial\mathcal{L}}{\partial\varphi} = \partial_t\left(\frac{\partial\mathcal{L}}{\partial\varphi_t}\right) + \partial_x\left(\frac{\partial\mathcal{L}}{\partial\varphi_x}\right) - \partial_x^2\left(\frac{\partial\mathcal{L}}{\partial\varphi_{xx}}\right.$$

(unusual term)

- in our system this gives: $0 = \frac{1}{2}\varphi_{xt} + \left(\frac{1}{2}\varphi_{\cancel{x}t} + \frac{1}{2}\varphi_x^2\right)_x + \varphi_{xxxx} = 0$

$\Leftrightarrow 0 = \varphi_{xt} + \varphi_x \varphi_{xx} + \varphi_{xxxx} = 0$ = this is KdV for $u = \varphi_x$.

- the usual way of getting energy conservation is here slightly modified by the φ_{xx} contribution to $\mathcal{L}$: on writes

$$\partial_t \mathcal{L} = \underbrace{\frac{\partial \mathcal{L}}{\partial \varphi}}\varphi_t + \frac{\partial \mathcal{L}}{\partial \varphi_{\cancel{t}}}\varphi_{tt} + \frac{\partial \mathcal{L}}{\partial \varphi_x}\varphi_{xt} + \frac{\partial \mathcal{L}}{\partial \varphi_{xx}}\varphi_{xxt}$$

$\hookrightarrow$ Euler-Lagrange $= \partial_t\left(\frac{\partial \mathcal{L}}{\partial \varphi_t}\right) + \partial_x\left(\frac{\partial \mathcal{L}}{\partial \varphi_x}\right) - \partial_x^2\left(\frac{\partial \mathcal{L}}{\partial \varphi_{xx}}\right)$

this writes (regrouping the terms):

$$\partial_t \mathcal{L} = \partial_t\left[\frac{\partial \mathcal{L}}{\partial \varphi_t}\varphi_t\right] + \partial_x\left[\frac{\partial \mathcal{L}}{\partial \varphi_x}\varphi_t\right] + \underbrace{\partial_x\left[-\partial_x\left(\frac{\partial \mathcal{L}}{\partial \varphi_{xx}}\right)\varphi_t + \frac{\partial \mathcal{L}}{\partial \varphi_{xx}}\varphi_{xt}\right]}_{\text{unusual contribution}}$$

hence one has:

$$0 = \partial_t\left(\underbrace{\frac{\partial \mathcal{L}}{\partial \varphi_t}\varphi_t - \mathcal{L}}\right) + \partial_x\left[\frac{\partial \mathcal{L}}{\partial \varphi_x}\varphi_t - \partial_x\left(\frac{\partial \mathcal{L}}{\partial \varphi_{xx}}\right)\varphi_t + \frac{\partial \mathcal{L}}{\partial \varphi_{xx}}\varphi_{xt}\right]$$

usual for of the energy density: $\frac{1}{2}\varphi_x\varphi_t - \left(\frac{1}{2}\varphi_{x\cancel{t}}\varphi_t + \frac{1}{6}\varphi_x^3 - \frac{1}{2}(\varphi_{xx})^2\right) = \boxed{-\frac{1}{6}u^3 + \frac{1}{2}u_x^2 = \text{energy density} =: \mathcal{E}}$

and here the Poynting vector is:

$$S = \frac{\partial \mathcal{L}}{\partial \varphi_x}\varphi_t - \partial_x\left(\frac{\partial \mathcal{L}}{\partial \varphi_{xx}}\right)\varphi_t + \frac{\partial \mathcal{L}}{\partial \varphi_{xx}}\varphi_{xt} = \left(\frac{1}{2}\varphi_t + \frac{1}{2}\varphi_x^2\right)\varphi_t - \partial_x(-\varphi_{xx})\varphi_t - \varphi_{xx}\varphi_{xt}$$

$$= \left(\frac{1}{2}\varphi_t + \frac{1}{2}\varphi_x^2 + \varphi_{xxx}\right)\varphi_t - \varphi_{xx}\underbrace{\varphi_{xt}}_{-uu_x - u_{xxx}}$$

φ_t is no nice, but one notices that, from the eqs of motion one has $(\varphi_{\cancel{x}t})_x = -\partial_x\left(\frac{1}{2}\varphi_x^2 + \varphi_{xxx}\right) = -\left(\frac{u^2}{2} + u_{xx}\right)_x$ since this appears in a x derivative, I <u>allow myself</u> to replace φ_t by $-\frac{u^2}{2} - u_{xx}$. then one gets:

$$S = -\left(\frac{1}{4}u^2 + \frac{1}{2}u_{xx}\right)\left(\frac{1}{2}u^2 + u_{xx}\right) + u_x(u u_x + u_{xxx})$$

$$= -\frac{1}{8}u^4 - \frac{1}{2}u^2 u_{xx} - \frac{1}{2}(u_{xx})^2 + u u_x^2 + u_x u_{xxx}$$

and $\boxed{\mathcal{E}_t + S_x = 0}$ = same as the last eq. up to a factor (-6)
conservation → green frame

⇨ momentum conservation :

$$\partial_x \mathcal{L} = \underbrace{\frac{\partial \mathcal{L}}{\partial \varphi}\varphi_x}_{} + \frac{\partial \mathcal{L}}{\partial \varphi_t}\varphi_{xt} + \frac{\partial \mathcal{L}}{\partial \varphi_x}\varphi_{xx} + \frac{\partial \mathcal{L}}{\partial \varphi_{xx}}\varphi_{xxx}$$

$$\hookrightarrow \partial_t\left(\frac{\partial \mathcal{L}}{\partial \varphi_t}\right)\varphi_x + \partial_x\left(\frac{\partial \mathcal{L}}{\partial \varphi_x}\right)\varphi_x - \partial_x^2\left(\frac{\partial \mathcal{L}}{\partial \varphi_{xx}}\right)\varphi_x$$

hence

$$\partial_x \mathcal{L} = \partial_t\left(\frac{\partial \mathcal{L}}{\partial \varphi_t}\varphi_x\right) + \partial_x\left(\frac{\partial \mathcal{L}}{\partial \varphi_x}\varphi_x\right) + \partial_x\left[-\partial_x\left(\frac{\partial \mathcal{L}}{\partial \varphi_{xx}}\right)\varphi_x + \frac{\partial \mathcal{L}}{\partial \varphi_{xx}}\varphi_{xx}\right]$$

thus

$$\partial_t\left(\frac{\partial \mathcal{L}}{\partial \varphi_t}\varphi_x\right) + \partial_x\left(\frac{\partial \mathcal{L}}{\partial \varphi_x}\varphi_x - \mathcal{L} - \partial_x\left(\frac{\partial \mathcal{L}}{\partial \varphi_{xx}}\right)\varphi_x + \frac{\partial \mathcal{L}}{\partial \varphi_{xx}}\varphi_{xx}\right) = 0$$

momentum density $= \frac{1}{2}(\varphi_x)^2 = \frac{u^2}{2}$

$$\cancel{\frac{1}{2}\varphi_x\varphi_t} + \frac{1}{2}\varphi_x^3 - \cancel{\frac{1}{2}\varphi_x\varphi_t} - 1/6\,\varphi_x^3 + \frac{1}{2}(\varphi_{xx})^2$$
$$+ \varphi_{xxx}\varphi_x - (\varphi_{xx})^2 = \frac{u^3}{3} - \frac{1}{2}(u_x)^2 + u u_{xx}$$

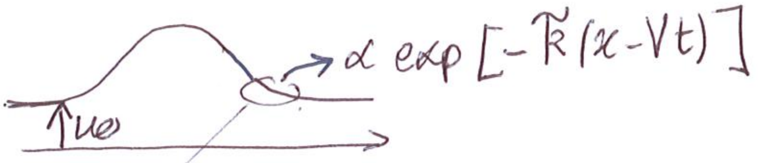
on retrouve $\left(\frac{u^2}{2}\right)_t + \left(\frac{u^3}{3} + u u_{xx} - \frac{1}{2}u_x^2\right)_x = 0$ comme page 26.

⇨ note culturelle. KdV a ceci de particulier qu'en + de ces 3 quantités conservées, elle en a une infinité d'autres, ~~voici~~ en voici quelques unes :

$$\begin{cases} \frac{u^4}{4} - 3 u u_x^2 + \frac{2}{5} u_{xx}^2 \\ \frac{u^5}{5} - 6 u^2 u_x^2 + \frac{36}{5} u u_{xx}^2 - \frac{108}{35} u_{xxx}^2 \\ \text{--- etc --} \end{cases}$$

⟵ note on replac^t of φ_t by $-\left(\frac{u^2}{2} + u_{xx}\right)$: when one computes S_x, there remain
this replac^t is allowed because
a term φ_t, but it is in factor of $\varphi_{xt} + \frac{1}{2}(\varphi_x^2)_x + \varphi_{4x}$ which is zero.
Then, all what remains is either φ_{xt} $(= -\varphi_x\varphi_{xx} - \varphi_{4x} = -uu_x - u_{xxx})$
or spatial derivatives of φ

* Real space :



since in this region the soliton is a small perturba° of the background, on can perform a linear expansion similar to what is done for $\exp[ikx - i\omega t]$

with $\begin{cases} \tilde{k} = -ik \\ \tilde{k}V = -i\omega \end{cases}$

since the dispersion relation is $\omega = fct(k)$ one will have :

$\frac{\tilde{k}V}{-i} = fct\left(\frac{\tilde{k}}{-i} = i\tilde{k}\right)$ where $fct(k) = u_0 k - k^3$

this yields $\tilde{k}V = -i\left(u_0(i\tilde{k}) - (i\tilde{k})^3\right) = u_0\tilde{k} + \tilde{k}^3$

hence $V = u_0 + \tilde{k}^2$ since, of course $\tilde{k} \sim 1/L$ (of the above plot)

one gets $\boxed{V \sim u_0 + 1/L^2}$

↖ better derivation than in page 23