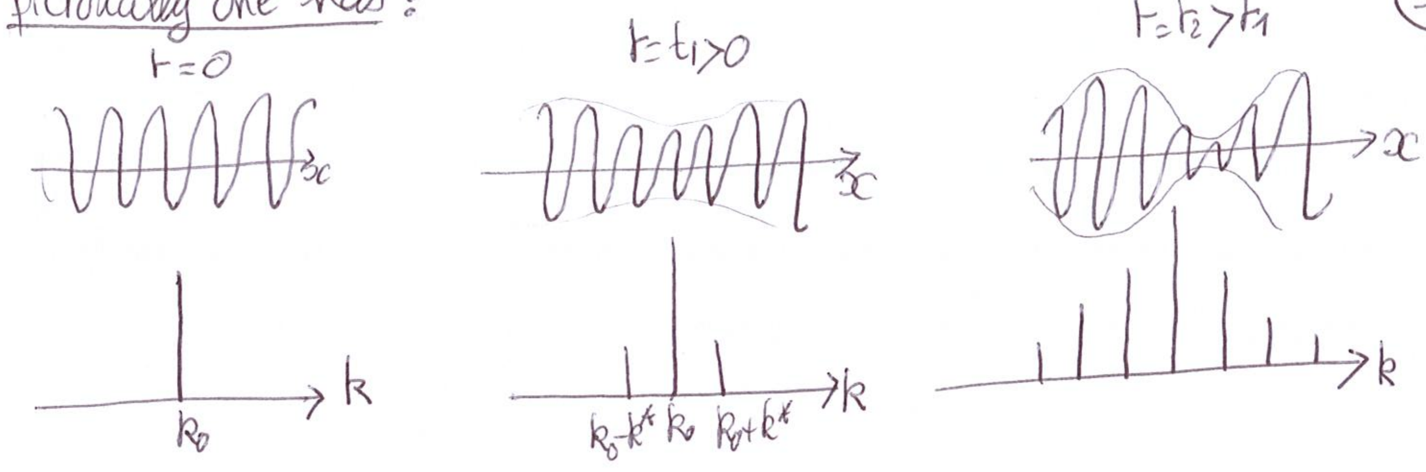
pictorially one has :



TUTO 2

- Soliton solutions for ~~de~~focusing : (ie unstable plane wave)

a change of variable in (x,τ,ϵ) leads to an eq. of the form :

$$i\partial_t \varphi = -\tfrac{1}{2}\varphi_{xx} - |\varphi|^2\varphi$$

assume that $\varphi = \varphi(\xi = x - Vt)$, then one has to remember that one might have a general time-dependent phase : one has to be more general

$$\varphi = A(x-Vt)\, e^{iS(x-Vt)+if(t)}$$

$$\begin{cases} \varphi_t = [i(f' - V S')A - VA']\, e^{iS+if} \\ \varphi_x = (A' + iS'A)\, e^{iS+if} \\ \varphi_{xx} = (A'' - AS'^2 + 2iS'A' + iS''A)\, e^{iS+if} \end{cases}$$

inserting back into NLS :

$$-iVA' - (f' - VS')A = -\tfrac{1}{2}(A'' - AS'^2) - \tfrac{i}{2}(2S'A' + AS'') - A^3$$

→ imaginary part : $-VA' + \underbrace{S'A' + \frac{A}{2}S''}_{\frac{1}{2A}(S'A^2)'} = 0$ → $\boxed{A^2(S'-V) = C^{ste}}$

for the soliton $A \xrightarrow[\xi\to\pm\infty]{} 0$ hence $C^{st} = 0$

hence $S' = V$

→ real part :

$$f' = \frac{1}{A}\left[V^2 A + \frac{A''}{2} - A\frac{V^2}{2} + A^3\right]$$

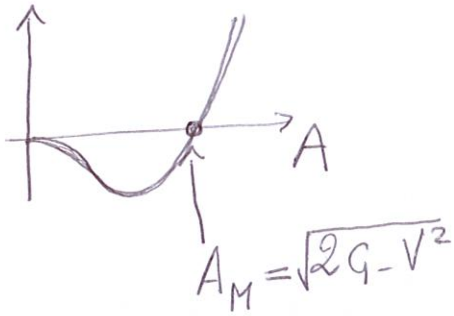
fct(t) fct(ξ)

hence both are constant : let's denote by G the common ~~value~~ constant value.

one thus has: $\cancel{\xi} A'' + (V^2 - 2G)A + 2A^3 = 0$

$\times A'$ and integrate $\Rightarrow$ $\boxed{\dfrac{A'^2}{2} + \underbrace{\left(\dfrac{V^2}{2} - G\right)A^2 + \dfrac{1}{2}A^4}_{W(A)} = C_2}$

one whishes here to study a solition with $A \xrightarrow[|\xi|\to\infty]{} 0$ and $A' \xrightarrow[|\xi|\to\infty]{} 0$, hence $C_2 = 0$. It is easy to check that to get an acceptable solution one needs $\frac{V^2}{2} - G < 0$ and then:



one can write: $W(A) = \dfrac{A^2}{2}(A^2 - A_M^2)$

one has $\dfrac{dA}{d\xi} = \pm\sqrt{A^2(A_M^2 - A^2)}$

forget about the sign: $d\xi = \dfrac{dA}{A\sqrt{A_M^2 - A^2}}$

change of variable: $\left(A = \dfrac{A_M}{\mathrm{ch}\,\theta} \quad dA = -A_M \dfrac{\mathrm{sh}\,\theta\, d\theta}{\mathrm{ch}^2\theta} \right)$

hence $d\xi = \dfrac{1}{A_M} \dfrac{-\mathrm{sh}\,\theta\, d\theta}{\mathrm{ch}^2\theta \dfrac{1}{\mathrm{ch}\,\theta}\sqrt{1 - \dfrac{1}{\mathrm{ch}^2\theta}}} = -\dfrac{d\theta}{A_M}$

(je m'en fous)

thus $\boxed{A(\xi) = \dfrac{A_M}{\mathrm{ch}(A_M \xi)}}$

and

$$\varphi(x,t) = \frac{A_M e^{iV(x-Vt)} e^{i(A_M^2+V^2)\frac{t}{2}}}{\mathrm{ch}[A_M(x-Vt)]}$$

other possible way to get the result: on has $A'^2 + A^2(A^2 - A_M^2) = 0$: on looks for solutions of the form $A = \dfrac{\alpha}{\mathrm{ch}(\beta\xi)} \to A' = \dfrac{-\alpha\beta\,\mathrm{sh}\,\theta}{\mathrm{ch}^2\theta}$

and one finds:

$\alpha^2\beta^2 \dfrac{\mathrm{sh}^2\theta}{\mathrm{ch}^2\theta} + \alpha^2\left(\dfrac{\alpha^2}{\mathrm{ch}^2\theta} - A_M^2\right) = 0$

using $\mathrm{sh}^2\theta = 1 + \mathrm{ch}^2\theta$ this gives:

$\mathrm{ch}^2\theta\,[\alpha^2\beta^2 - \alpha^2 A_M^2] + [\alpha^4 - \alpha^2\beta^2] = 0$

Thus $\alpha = \beta = A_M$

one whishes to compute its energy and the # of "particles" it contains ie

$N = \int dx\, |\varphi|^2$

(N makes sense only for BEC, no for the envelope soliton. But at least it is in all the cases a conserved quantity) [remember one has typically an envelope soliton

]

[change of variable: $u = \mathrm{th}\,\theta = \frac{\mathrm{sh}\,\theta}{\mathrm{ch}\,\theta}$

$du = \left(1 - \frac{\mathrm{sh}^2\theta}{\mathrm{ch}^2\theta}\right) d\theta$

$= \frac{1}{\mathrm{ch}^2\theta} d\theta$

hence $\int_{\mathbb{R}} \frac{d\theta}{\mathrm{ch}^2\theta} = \int_{-1}^{1} du = 2$]

$$N = \int dx\, |\varphi|^2 = A_n^2 \int_{\mathbb{R}} \frac{d\xi}{\mathrm{ch}^2(A_n \xi)} = A_n \int_{\mathbb{R}} \frac{d\theta}{\mathrm{ch}^2\theta}$$

thus $\boxed{N = 2A_M}$

▽ a lagrangian density for NLS

$$\mathcal{L} = \frac{i}{2}(\varphi^* \varphi_t - \varphi \varphi_t^*) - \frac{1}{2}|\varphi_x|^2 + \frac{\sigma}{2}|\varphi|^4$$

(where $\sigma = \pm 1$ - $\sigma = +1$ for attractive NLS, $\sigma = -1$ for repulsive)

one treats φ and φ^* as independent fields.
the eq. of motion is: $\partial_t\left(\frac{\partial \mathcal{L}}{\partial \varphi_t^*}\right) + \partial_x\left(\frac{\partial \mathcal{L}}{\partial \varphi_x^*}\right) = \frac{\partial \mathcal{L}}{\partial \varphi^*}$

which reads
$$\partial_t\left(-\frac{i}{2}\varphi\right) + \partial_x\left(-\frac{1}{2}\varphi_x\right) = \frac{i}{2}\varphi_t + \sigma \varphi^2 \varphi^*$$

hence $-\frac{1}{2}\varphi_{xx} - \sigma|\varphi|^2\varphi = \frac{i}{2}\varphi_t$: indeed this is NLS.

the energy is:
$$E = \int dx \left[\varphi_t \frac{\partial \mathcal{L}}{\partial \varphi_t} + \varphi_t^* \frac{\partial \mathcal{L}}{\partial \varphi_t^*} - \mathcal{L}\right] = \int dx \left[\frac{1}{2}|\varphi_x|^2 - \frac{\sigma}{2}|\varphi|^4\right]$$

let's compute this quantity for the "bright soliton" (ie $\sigma = +1$)
one can do a brute force computation. But there is a better way:

$E = \int d\xi \left[\frac{A'^2}{2} - \frac{1}{2}A^4\right]$ and $\frac{A'^2}{2} + \frac{A^4}{2} - \frac{A^2}{2}A_M^2 = 0$ hence one can eliminate A^4 and write...

$$E = \int_{\mathbb{R}} d\xi \left[A'^2 - \frac{A^2}{2}A_M^2\right] = 2\int_{-\infty}^{\otimes} \underbrace{d\xi\, A'}_{dA} \times \underbrace{A'}_{= A\sqrt{A_n^2 - A^2}} - \frac{A_n^2}{2} \times N$$

thus
$$E = 2\underbrace{\int_0^{A_n} dA\, A\sqrt{A_n^2 - A^2}}_{A_n^3 \int_0^1 dx\, x\sqrt{1-x^2}} - \frac{A_n^2}{2} \times \underbrace{N}_{2A_M} = \frac{2}{3}A_n^3 - A_n^3 = -\frac{1}{3}A_M^3$$

$$A_n^3 \int_0^1 dx\, x\sqrt{1-x^2} = A_M^3 \left[-\frac{1}{3}(1-x^2)^{3/2}\right]_0^1 = A_n \times \frac{1}{3}$$

(note pour moi = calcul brute force:

$$A'^2 = A_n^4 \frac{sh^2(A_n\xi)}{ch^4(A_n\xi)} \quad \text{et donc} \quad E = \frac{A_n^4}{2}\int d\xi \frac{1}{ch^4(A_n\xi)}\underbrace{\left[sh^2(A_n\xi) - 1\right]}_{ch^2(A_n\xi)-2}$$

en posant $\theta = A_n\xi$ cela donne:

$$E = \frac{A_n^3}{2}\int d\theta\left(\frac{1}{ch^2\theta} - \frac{2}{ch^4\theta}\right)$$

or, on a vu que $\int \frac{d\theta}{ch^2\theta} = 2$ et tjs en posant $u = th\,\theta$ il est facile de voir que

$$\int_{-\infty}^{\infty}\frac{d\theta}{ch^4\theta} = \int_{-1}^{1} du\,(1-u^2) = 2\left[u - \frac{u^3}{3}\right]_0^1 = 4/3$$

donc $E = \frac{A_n^3}{2}\left(2 - \frac{8}{3}\right) = -\frac{1}{3} A_n^3$

soliton solutions for defocusing. (ie stable plane wave)

here also $\psi = A(x-Vt)\,e^{iS(x-Vt)+if(t)}$

one plugs this back into NLS. One gets: $A^2(S'-V) = C^{ste} = J$

here J is not necessarily zero (A does not $\to 0$ at $\pm\infty$). The real part of NLS is similar to the last eq. of p31 with $A^3 \to -A^3 \Rightarrow$ (using here $S' = \frac{J}{A^2} + V$, thus $VS'A - \frac{1}{2}AS'^2 = -\frac{J^2}{2A^3} + V^2\frac{A}{2}$):

$$AC_1 = \frac{1}{2}A'' - \frac{J^2}{2A^3} + \frac{V^2}{2}A - A^3 \xRightarrow[\text{integrate}]{(\times 2A')\text{ and}} A^2C_1 = \frac{1}{2}A'^2 + \frac{J^2}{2A^2} + \frac{V^2}{2}A^2 - \frac{A^4}{2}$$

hence

$$\boxed{\begin{array}{l}\frac{1}{2}A'^2 + W(A) = C_2 \\ \text{where } W(A) = \frac{J^2}{2A^2} + \left(\frac{V^2}{2} - C_1\right)A^2 - \frac{A^4}{2}\end{array}}$$

if $J = 0$ one needs $V^2 > 2C_1$ so that

if $J \neq 0$ one needs

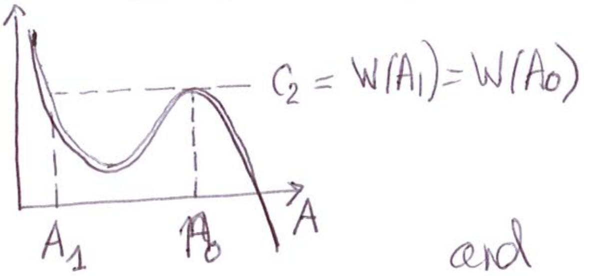
ie $W'(A) = -\frac{J^2}{A^3} + 2\left(\frac{V^2}{2} - C_1\right)A - 2A^3$ must have enough zeros. Ie one needs 2 solutions at:

$$\frac{J^2}{A^4} + 2A^2 = V^2 - 2C_1$$

minimum reached for $A^6 = J^2$. the value of the minimum is ~~$3/J^{2/3}$~~ $3J^{2/3}$

so, one ~~much~~ must have $\boxed{V^2 - 2C_1 > 3J^{2/3}}$

and then the solitonic solution corresponds to =



the eq. reads =

$$A'^2 = 2\left[W(A_0) - W(A)\right]$$

and $2W(A) = \frac{1}{A^2} \times$ (polynom of 3^{rd} degree in A^2, with higher term: $-A^6$)

so $$\boxed{2\left[W(A_0) - W(A)\right] = \frac{1}{A^2}\left(A_0^2 - A^2\right)^2\left(A^2 - A_1^2\right)}$$

by the way, if $n = A^2$
then $n' = 2AA'$

and one has = $\frac{1}{4} n'^2 = (n_0 - n)^2 (n - n_1)$ where $n_1 < n(\xi) < n_0$

this reads = $\frac{1}{2}\frac{dn}{d\xi} = \pm (n_0 - n)\sqrt{n - n_1}$ define $\rho(\xi) = n_0 - n(\xi)$
$0 < \rho < \rho_M = n_0 - n_1$

then one has $$\boxed{\frac{1}{2}\frac{d\rho}{d\xi} = \pm \rho\sqrt{\rho_M - \rho}}$$

this integration has already been performed for the KdV case (page 22)

one has $2\,d\xi = \frac{d\rho}{\rho\sqrt{\rho_M - \rho}}$ --- instead, look for the solution under the form

$$\rho(\xi) = \frac{\rho_M}{ch^2[\beta\xi]}$$ then =

$$\frac{1}{2}\frac{d\rho}{d\xi} = \frac{1}{2}(-2)\beta\rho_M\frac{sh(\beta\xi)}{ch^3(\beta\xi)}$$

$$\pm\rho\sqrt{\rho_M - \rho} = \pm\frac{\rho_M^{3/2}}{ch^2(\beta\xi)}\sqrt{1 - \frac{1}{ch^2(\beta\xi)}} = \pm\frac{\rho_M^{3/2}}{ch^3(\beta\xi)}\,sh(\beta\xi)$$

these 2 should be equal, hence $\beta = \sqrt{\rho_M} \rightarrow \rho(\xi) = \frac{\rho_M}{ch^2[\sqrt{\rho_M}\,\xi]}$

and $$\boxed{n(x,t) = n_0 - \frac{(n_0 - n_1)}{ch^2\left[\sqrt{n_0 - n_1}\,(x - Vt)\right]}}$$

determination of the integration constants:

one imposes $S' \xrightarrow[\infty]{} 0$ (ie steady flow at ∞) and since $n \xrightarrow[\infty]{} n_0$ the current conservation law $J = A^2(S'-V)$ yields $\boxed{J = -Vn_0}$

also remember that $C_1 = f' = c^{st}$ hence the wave-function φ at ∞ reads:

$$\varphi(x,t) \xrightarrow[\pm\infty]{} \sqrt{n_0}\, e^{iS(\pm\infty)}\, e^{iC_1 t}$$

plugging this back into NLS gives at once $\boxed{C_1 = -n_0}$ and $W(A)$ reads:

$$W(A) = \frac{V^2 n_0^2}{2n} + \left(\frac{V^2}{2} + n_0\right) n - \frac{n^2}{2}$$

$$W'(A) = \frac{dW}{dn}\frac{dn}{dA} = \left(-\frac{V^2 n_0^2}{2n^2} + \frac{V^2}{2} + n_0 - n\right)\cdot 2\sqrt{n}$$

cancels for $n = n_0$ as it should.

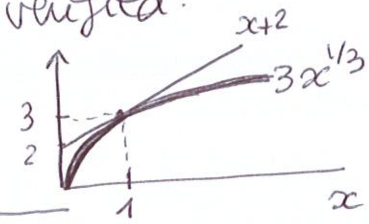
$$W''(A) = \underbrace{\frac{d^2W}{dn^2}}_{\frac{V^2 n_0^2}{n^3} - 1 \,=\, \frac{V^2}{n_0} - 1 \text{ for } n=n_0} \cdot 2\sqrt{n} + \underbrace{\frac{dW}{dn}}_{=0 \text{ for } n=n_0} \cdot \frac{1}{\sqrt{n}}$$

one wants $W''(A_0) < 0$, hence $\boxed{V < \sqrt{n_0}}$

by the way, in the case considered, the condition $V^2 - 2C_1 > 3J^{2/3}$ reads $V^2 + 2n_0 > 3(V^2 n_0^2)^{1/3}$

or $\frac{V^2}{n_0} + 2 > 3\left(\frac{V^2}{n_0}\right)^{1/3}$

which is always verified:



one has $W(A_0) = V^2 n_0 + \frac{n_0^2}{2}$ and $W(A) - W(A_0) = \frac{1}{2n}\left[V^2 n_0^2 + (V^2 + 2n_0)n^2 - n^3 - (2V^2 n_0 + n_0^2)n\right]$

thus $\boxed{W(A) - W(A_0) = \frac{1}{2n}(n - n_0)^2 (V^2 - n)}$ (easy to check).

comparing with the framed formula page 35 this shows that $\boxed{n_1 = V^2}$ and the above condition $V < \sqrt{n_0}$ amounts to say that $n_1 < n_0$ as it should.

then, the phase: $S' = \frac{J}{A^2} + V = V\left(1 - \frac{n_0}{n}\right)$

I define $\boxed{\frac{n_1}{n_0} = V^2/n_0 = \sin^2\theta}$ then $\boxed{\frac{n}{n_0} = 1 - \frac{\cos^2\theta}{\mathrm{ch}^2[\sqrt{n_0}\cos\theta\,\xi]}}$

and

$$\frac{1}{V}\frac{dS}{d\xi} = 1 - \frac{1}{1 - \frac{\cos^2\theta}{\mathrm{ch}^2[\cdots]}} = \frac{-\cos^2\theta}{\mathrm{ch}^2[\cdots] - \cos^2\theta}$$

thus

$$\boxed{\frac{1}{\sin\theta}\frac{dS}{\sqrt{n_0}\,d\xi} = \frac{-\cos^2\theta}{\mathrm{ch}^2[\cos\theta\sqrt{n_0}\,\xi] - \cos^2\theta}}$$

show that the solution is of the form $\mathrm{tg}(\alpha S) = \beta\,\mathrm{th}(\cos\theta\sqrt{n_0}\,\xi)$
where α and β are yet unknown.
Differentiating yields: $\left(\begin{array}{l}\text{since } \mathrm{tg}' = 1 + \mathrm{tg}^2 \\ \text{and } \mathrm{th}' = 1 - \mathrm{th}^2 = \frac{1}{\mathrm{ch}^2}\end{array}\right)$

$$\alpha\,[1 + \mathrm{tg}^2(\alpha S)]\frac{dS}{d\xi} = \beta\cos\theta\sqrt{n_0}\,\frac{1}{\mathrm{ch}^2(\cdots)}$$

$$1 + \beta^2\,\mathrm{th}^2(\cdots) = 1 + \beta^2\left(1 - \frac{1}{\mathrm{ch}^2(\cdots)}\right)$$

hence

$$\alpha\frac{dS}{d\xi} = \beta\cos\theta\sqrt{n_0}\,\frac{1}{\mathrm{ch}^2(\cdots)} \times \frac{1}{1+\beta^2 - \frac{\beta^2}{\mathrm{ch}^2(\cdots)}} = \frac{\beta\cos\theta\sqrt{n_0}}{1+\beta^2}\,\frac{1}{\mathrm{ch}^2(\cdots) - \frac{\beta^2}{1+\beta^2}}$$

thus

$$\frac{\alpha(1+\beta)}{\beta\cos\theta\sqrt{n_0}}\,\frac{dS}{d\xi} = \frac{1}{\mathrm{ch}^2(\cdots) - \frac{\beta^2}{1+\beta^2}}$$

comparing with the framed formula above: $\cos^2\theta = \frac{\beta^2}{1+\beta} \longrightarrow \beta^2 = \frac{\cos^2\theta}{\sin^2\theta}$

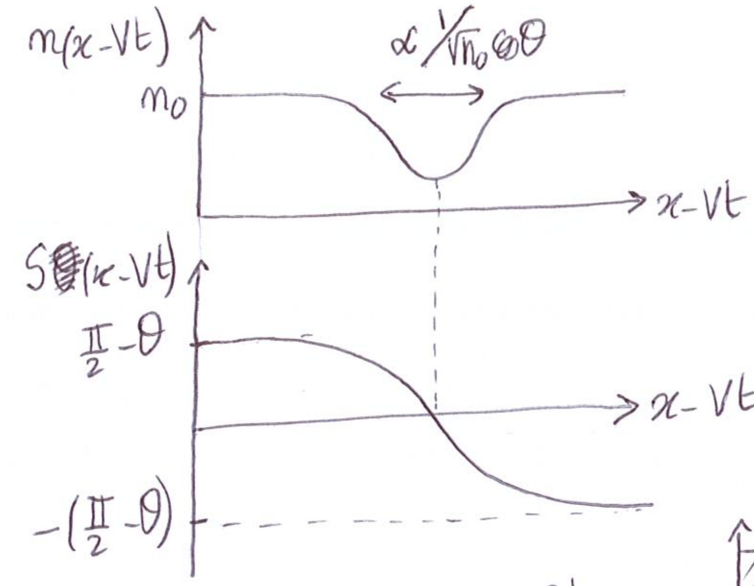
and

$$-\frac{\alpha(1+\beta^2)\cos\theta}{\beta\sqrt{n_0}}\,\frac{dS}{d\xi} = \frac{-\cos^2\theta}{\mathrm{ch}^2(\cdots) - \cos^2\theta}$$

comparing again with the above framed expression one finds $\alpha = -1$

and thus: $\boxed{\mathrm{tg}\,S = -\frac{1}{\mathrm{tg}\,\theta}\,\mathrm{th}(\cos\theta\sqrt{n_0}\,\xi)} \longrightarrow \boxed{\text{note: } \frac{1}{\mathrm{tg}\,\theta} = \mathrm{tg}\left(\frac{\pi}{2} - \theta\right)}$

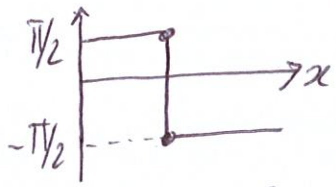
so, one has



dark soliton

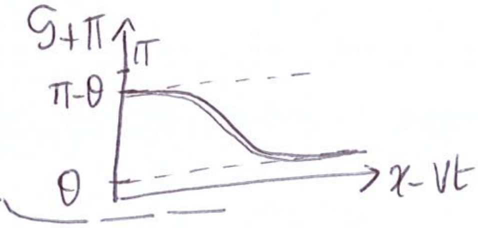
when $\theta=0$, $V=0$ and one has a black soliton =

with a phase =



Note that there is an upper threshold for the velocity: $V < \sqrt{n_0} = c$ = "speed of sound" in the system as can be checked at the end of page 29.

remark: the use is to add to S a phase $\pi/2$ (global phase: does not affect the result) then



$\varphi = \sqrt{n}\, e^{iS - in_0 t}$

$$\sqrt{n}\cos S = \sqrt{n_0}\sqrt{1-\frac{\cos^2\theta}{ch^2(\ldots)}} \times \frac{1}{\sqrt{1+\frac{1}{tg^2\theta}th^2(\ldots)}} \overset{\text{easy computation}}{=} \sqrt{n_0}\sin\theta$$

$\cos S = \frac{1}{\sqrt{1+tg^2 S}}$ $(\cos S > 0)$

and $\sqrt{n}\sin S = \underbrace{\sqrt{n}\cos S}_{\sqrt{n_0}\sin\theta}\, tg\, S = -\sqrt{n_0}\cos\theta\, th(\cos\theta\sqrt{n_0}\,\xi)$

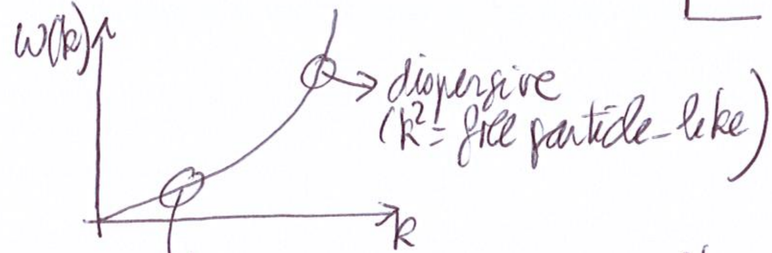
thus $$\boxed{\varphi(x,t) = \sqrt{n_0}\, e^{-in_0 t}\left\{\sin\theta - i\cos\theta\, th\left(\cos\theta\sqrt{n_0}(x-Vt)\right)\right\}}$$

up to a global phase...

by the way much more easy: look for a solution of the initial NLS of the form $\rightarrow \sqrt{n_0}\, e^{-in_0 t}[\alpha\, th(\beta\xi) + i\gamma]$

hand waving argument for NLS dark solitons.

→ the dispersion relation is $\boxed{\omega^2 = \underset{A_0^2}{n_0} k^2 + k^4/4}$ (cf page 29 with $\alpha = \Omega'' \equiv 1$ and $\epsilon_0^2 = n_0$)

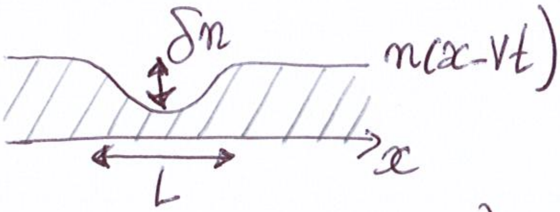


non dispersive = sound-like with $\boxed{c = \sqrt{n_0}}$

→ phase velocity V_φ : $V_\varphi^2 = \left(\frac{\omega}{k}\right)^2 = n_0 + \frac{k^2}{2}$. For a soliton to keep its shape, the NL effects should compensate the dispersive effect. One writes : $(V_\varphi)^2 = n_0 + \delta n + k^2/2$ ad hoc prescription to include NL effects

one should thus have $\delta n + k_{typ}^2 = 0$

hence $\delta n < 0$ and one has :



$\boxed{\delta n \sim -1/L^2}$

→ the outskirt of the soliton : $\boxed{\begin{array}{l} n(x,t) \simeq n_0 + e^{-\tilde{k}(x-Vt)} \\ \text{where } \tilde{k} \sim 1/L \end{array}}$

the math are the same as for small plane wave perturbations with $\left\{\begin{array}{l} -\tilde{k} \leftrightarrow ik \\ \tilde{k}V \leftrightarrow -i\omega \end{array}\right.$

since one has $\omega = fct(k)$
↳ this is the dispersion relation.

one has also : $i\tilde{k}V = fct(i\tilde{k}) \Rightarrow -\tilde{k}^2 V^2 = -n_0 \tilde{k}^2 + \tilde{k}^4/4$

this yields (since $\tilde{k}^2 \sim 1/L^2$) $-V^2 \sim -c^2 + \tilde{k}^2$ ($c^2 = n_0$)

Thus $\boxed{c^2 - V^2 \sim 1/L^2}$

in agreement with the red formulae page 37