ADVANCED NONLINEAR PHYSICS

Duration: 3 hours. Problems A and B correspond to the first part of the course, problems C and D to the second part. They are all independent of each other. Please use different sheets for writing up the solution of each problem.

Dictionaries, handwritten notes on the courses and tutorials are allowed. All the material distributed during the first part of the course is allowed. Printed documents related to the second part of the course are forbidden. Books as well as computers, telephones and other electronic devices are forbidden.

A Method of characteristics

One considers the partial differential equation

$$u_t + x\, u_x = -u \;, \tag{A1}$$

subject to the initial condition $u(x,0) = g(x)$, where $g(x)$ is a regular, bounded function which verifies $\lim_{x\to\pm\infty} g(x) = 0$ and $-\infty < \int_{\mathbb{R}} g(x)\mathrm{d}x < +\infty$.

(a) Give the expression of $u(x,t)$ for $t > 0$.

(b) Show that, for any fixed $x \in \mathbb{R}$, $\lim_{t\to\infty} u(x,t)$ reaches a x-independent value.

(c) Compute the integral $\int_{\mathbb{R}} \mathrm{d}x\, u(x,t)$.

(d) Discuss the compatibility of the results of questions (b) and (c).

B Long Josephson junction

0/ Useful results: Integrating over an interval comprised in $[0, 2\pi]$ one has

$$\int^{\varphi} \frac{\mathrm{d}\varphi'}{\sqrt{1-\cos\varphi'}} = \sqrt{2}\ln\left(\tan\frac{\varphi}{4}\right) \;, \qquad \int^{\varphi} \mathrm{d}\varphi' \sqrt{1-\cos\varphi'} = -2\sqrt{2}\cos\frac{\varphi}{2} \;. \tag{B0}$$

A Josephson junction is a sandwich formed by two superconductors separated by a thin insulating layer as represented in Fig. 1.

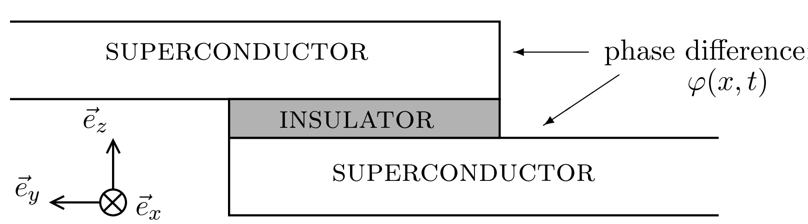

Figure 1: *Schematic representation of a vertical cut of a Josephson junction extended in the x-direction.*

Neglecting the thickness of the insulating layer in the y and z directions, the phase difference between the wave function of the superconducting layers of a long Josephson junction is a function $\varphi(x,t)$ with x and t in $\mathbb{R}$. One can show that φ obeys the nonlinear equation[1]

$$\varphi_{tt} - \varphi_{xx} + \sin\varphi = 0 \ . \tag{B1}$$

1/ One defines the energy as

$$E = \int_{\mathbb{R}} \mathcal{H}(x,t)\,\mathrm{d}x \ , \quad \text{where} \quad \mathcal{H}(x,t) = \tfrac{1}{2}\varphi_t^2 + \tfrac{1}{2}\varphi_x^2 + (1-\cos\varphi) \tag{B2}$$

is the energy density. Show that E is a conserved finite quantity[2] when $\varphi(x,t)$ tends to constant, time independent values at spatial boundaries (i.e., when $x \to +\infty$ and $x \to -\infty$).

2/ One studies traveling waves. These are solutions of (B1) for which φ depends on a single variable $\xi = x - V\,t$, where V is the constant velocity of the wave. One will henceforth only consider[3] the case $V \in [0,1]$.

(a) One notes $\varphi' = \mathrm{d}\varphi/\mathrm{d}\xi$. Show that upon integration one may write (B1) under the form

$$\tfrac{1}{2}(1-V^2)(\varphi')^2 + \cos\varphi = E_{\text{cl}} \ . \tag{B3}$$

where E_{cl} is an integration constant.

(b) One looks for "kink solitons" $\Phi(\xi)$ which are solutions of (B3) satisfying the boundary conditions $\Phi(\xi \to -\infty) = 0$, $\Phi(\xi \to +\infty) = 2\pi$, and $\Phi'(\xi \to \pm\infty) = 0$. Give the explicit expression of $\Phi(\xi)$ by furthermore imposing that $\Phi(\xi = 0) = \pi$. Show that[4] the energy of a kink is

$$E = 8\,(1-V^2)^{-1/2} \ . \tag{B4}$$

3/ In a more realistic treatment of the physics of the Josephson junction, Eq. (B1) is modified to

$$\varphi_{tt} - \varphi_{xx} + \sin\varphi = R[\varphi] \ , \quad \text{where} \quad R[\varphi] = -\gamma - \alpha\,\varphi_t \ , \tag{B5}$$

α and γ being positive constants.

(a) Show that in this case the energy (B2) is typically not a conserved quantity because

$$\frac{\mathrm{d}E}{\mathrm{d}t} = \int_{\mathbb{R}} \mathrm{d}x\, R[\varphi]\,\varphi_t \ . \tag{B6}$$

(b) If the perturbation $R[\varphi]$ is small, one may assume that the kink soliton still exists, but is now of the form $\Phi(x - X(t))$ where Φ has been determined above and X is a yet unknown function of time. The velocity of the kink is now $V(t) = \mathrm{d}X/\mathrm{d}t$. Show that under this assumption one gets

$$\frac{\mathrm{d}V}{\mathrm{d}t} = \frac{\pi\,\gamma}{4}(1-V^2)^{3/2} - \alpha V(1-V^2) \ . \tag{B7}$$

(c) Show that there exists two particular velocities V for which the kink soliton is not affected by the perturbation. If time and motivation permits show that one of these velocities is an attractive fixed point.

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[1] The space and time coordinates have been re-scaled to reach a dimensionless form. Eq. (B1) is known as the "sine-Gordon equation".

[2] Hint: show that $\mathrm{d}E/\mathrm{d}t = \int_{\mathbb{R}} \mathrm{d}x\,(\varphi_x\varphi_t)_x$.

[3] One can show that the solutions with $V > 1$ are unstable.

[4] Hint: show that $E = \int_{\mathbb{R}} \mathrm{d}\xi\,(\Phi')^2$ and make a simple change of variable. If you cannot prove the result (B4), admit it and continue the exercise.