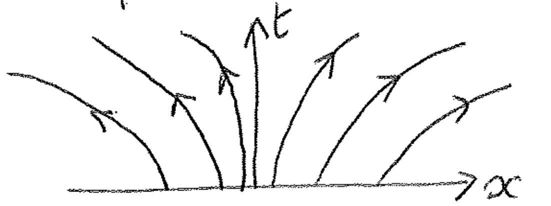
method of characteristics

define $\varphi(t) = u(x_c(t), t)$ where $x_c(t)$ is yet unknown.

$$\frac{d\varphi}{dt} = \frac{dx_c}{dt} u_x + u_t$$ if one chooses to impose $\frac{dx_c}{dt} = x_c$ one gets $\frac{d\varphi}{dt} = -\varphi$ ie $\varphi(t) = \varphi(0) e^{-t}$.

the characteristics have equation $x_c(t) = x_c(0)\, e^{t}$. In the (x,t) plane they look like:

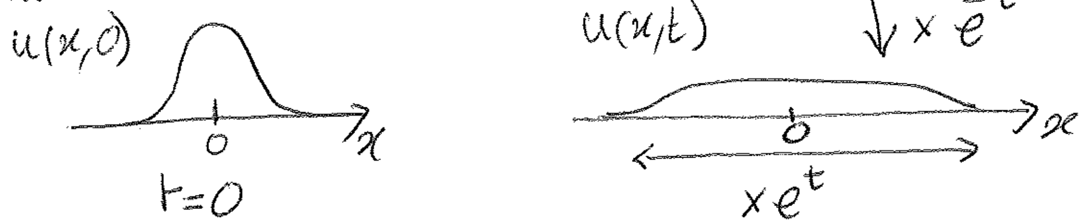


one has:

$$u(x,t) = e^{-t}\, u(x_c(0), 0) = e^{-t} g(x_c(0))$$ where $x_c(t) = x$.
this fixes $x_c(0) = x e^{-t}$

and thus $\boxed{u(x,t) = e^{-t} g(x e^{-t})}$

- one has clearly $\lim_{t \to \infty} u(x,t) = 0$
- $$\int_{\mathbb{R}} u(x,t)\,dx = \int_{\mathbb{R}} dx\, e^{-t} g(x e^{-t}) = \int_{\mathbb{R}} dy\, g(y) = C^{st}$$ (time independent)
  $y = x e^{-t}$
- It might seem strange at first sight that $\int u(x,t)\,dx$ remains constant while $\lim_{t\to\infty} u(x,t) = 0$. The reason is that the exponential decrease of u is exactly compensated by an exponential increase of its spatial extend, as illustrated by the typical behavior:



Long Josephson junction

(LJJ1)

1/ $E = \int_{\mathbb{R}} \mathcal{H}\, dx \qquad \frac{dE}{dt} = \int_{\mathbb{R}} \partial_t \mathcal{H}\, dx = \int_{\mathbb{R}} (\varphi_t \varphi_{tt} + \varphi_x \varphi_{xt} + \varphi_t \sin\varphi)\, dx$

using (B1) one gets $\frac{dE}{dt} = \int_{\mathbb{R}} (\varphi_t \varphi_{xx} + \varphi_x \varphi_{xt})\, dx = [\varphi_t \varphi_x]_{-\infty}^{+\infty}$

this is zero under the assumption that $\lim_{x\to\pm\infty} \varphi$ is time-independent

2/ if $\varphi(x,t) = \varphi(\xi = x - Vt)$, (B1) reads:

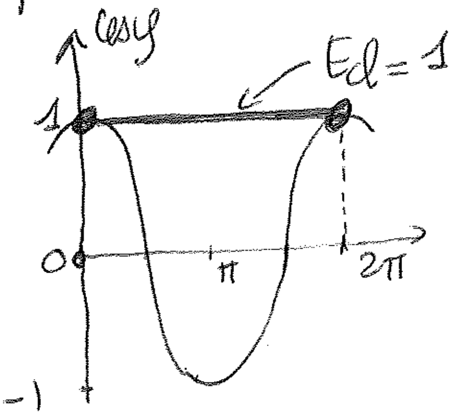
$$(V^2-1)\varphi'' + \sin\varphi = 0$$

× plying by φ' yields a first-integrated form: $\frac{1-V^2}{2}(\varphi')^2 + \cos\varphi = E_d$

integration constant

consider solutions $\Phi(\xi)$ of this eq. with $\lim_{\xi\to-\infty} \Phi = 0$ and $\lim_{\xi\to+\infty} \Phi = 2\pi$.

This eq. can be interpreted as the conservation of the mechanical energy of an effective particle of mass $1-V^2$, position φ, oscillating in an external potential $\cos\varphi$ (ξ playing the role of an effective time).



for the boundary condition considered $E_d = 1$ and thus

$$\Phi' = \sqrt{\frac{2}{1-V^2}}\sqrt{1-\cos\Phi}$$

this reads $\frac{d\Phi}{\sqrt{1-\cos\Phi}} = \sqrt{\frac{2}{1-V^2}}\, d\xi$

using the first of eqs. (B0) this yields: --/--

LJJ2

$\Phi = 4 \arctan\left[\exp\left(\frac{\xi - \xi_0}{\sqrt{1-V^2}}\right)\right]$ where ξ_0 is an integration constant.
$(\Phi(\xi_0) = \pi/2)$.

this expression indeed fulfills $\Phi(+\infty) = 2\pi$ and $\Phi(-\infty) = 0$.

the corresponding energy is from (B3):

$$E = \int_{\mathbb{R}} dx \left(\frac{V^2+1}{2} (\Phi')^2 + \underbrace{1 - \cos\Phi}_{\frac{1}{2}(1-V^2)(\Phi')^2 \text{ from (B3) with } E_{cl}=1} \right) = \int_{\mathbb{R}} d\xi (\Phi')^2$$

changing variable to Φ ($d\xi\, \Phi' = d\Phi$) one gets:

$$E = \int_0^{2\pi} d\Phi\, \Phi' = \sqrt{\frac{2}{1-V^2}} \int_0^{2\pi} d\Phi \sqrt{1-\cos\Phi} \underset{\text{using the second of eqs. (B0)}}{=} -\frac{4}{\sqrt{1-V^2}} \left[\cos\frac{\Phi}{2}\right]_0^{2\pi} = \frac{8}{\sqrt{1-V^2}}$$

3/ when φ fulfills (B5) one gets:

$$\frac{dE}{dt} = \int dx \left(R[\varphi] \varphi_t + \underbrace{\varphi_t \varphi_{xx} + \varphi_x \varphi_{xt}}_{\text{as previously: this term cancels upon integration.}} \right)$$

It remains $\frac{dE}{dt} = \int dx\, R[\varphi] \varphi_t$

- For an expression $\Phi(x - X(t))$ one still gets as previously:

$E = \frac{8}{\sqrt{1-V^2}}$ where here $V = \frac{dX}{dt}$

proof: indeed, writing $\varphi(x,t) = \Phi(x - X(t))$ one gets:

$\varphi_t^2 = \left(\frac{dX}{dt}\right)^2 (\Phi')^2 \qquad \varphi_x^2 = (\Phi')^2$ and $1 - \cos\Phi \underset{\text{property of function } \Phi}{=} \frac{1-V^2}{2} (\Phi')^2$

But, if one makes the identification $V=\frac{dX}{dt}$ one has

$\Phi(x-X(t)) = f\left(\frac{x-X}{\sqrt{1-\dot{X}^2}}\right)$ where $f(y)=4\,\mathrm{arctg}(\exp(y))$

thus $\frac{d}{dt}\Phi$ has an extra contribution with respect to the one given at the end of the previous page. This contribution is small if $|\ddot{X}|\ll 1$ which we assume henceforth.

Now (B6) reads
$$\frac{dE}{dt} = \int_{\mathbb{R}} dx\left(-\gamma\varphi_t - \alpha\varphi_t^2\right) = \int_{\mathbb{R}} d\xi\left[\gamma V\Phi' - \alpha V^2(\Phi')^2\right]$$

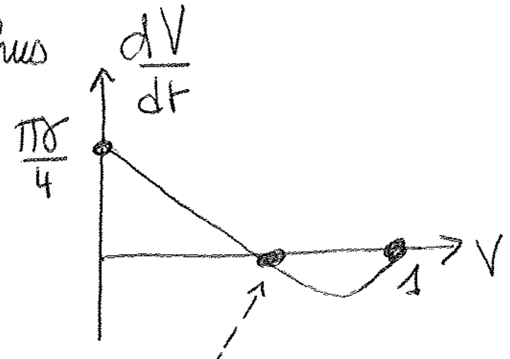
one has here $\int_{\mathbb{R}} d\xi\,\Phi' = [\Phi]_{-\infty}^{+\infty} = 2\pi$ and $\int_{\mathbb{R}} d\xi\,(\Phi')^2 = E$ and thus:

$$\frac{dE}{dt} = 2\pi\gamma V - \alpha V^2 E = 2V\left(\pi\gamma - \frac{4\alpha V}{\sqrt{1-V^2}}\right)$$

and $\frac{dE}{dt} = \frac{d}{dt}\left(\frac{8}{\sqrt{1-V^2}}\right) = \frac{8V}{(1-V^2)^{3/2}}\frac{dV}{dt}$ which yields:

$$\frac{dV}{dt} = \frac{\pi\gamma}{4}(1-V^2)^{3/2} - \alpha V(1-V^2) = \frac{\pi\gamma}{4}(1-V^2)\underbrace{\left[\sqrt{1-V^2} - \frac{4\alpha}{\pi\gamma}V\right]}_{\text{zero when } V^2 = \frac{1}{1+\left(\frac{4\alpha}{\pi\gamma}\right)^2}}$$

thus



$$V = V^* = \frac{1}{\sqrt{1+\left(\frac{4\alpha}{\pi\gamma}\right)^2}}$$

this plot shows that V^* is an attractive fixed point:
$$\begin{cases} \text{if } V > V^* & \frac{dV}{dt} < 0 \\ \text{if } V < V^* & \frac{dV}{dt} > 0 \end{cases}$$