RESEARCH PROPOSAL

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Perspectives in Random Matrix Theory: statistical methods for correlated random variables with applications to complex and quantum systems.

1. Introduction

Ensembles of matrices with random entries have been successfully employed in the most diverse areas of physics and mathematics for almost a century. Many years before the well-known pioneering applications of Gaussian random matrices to nuclear spectra by Wigner and Dyson [1], John Wishart had already introduced some thirty years earlier random covariance matrices in his studies of multivariate populations [2]. The *Wishart ensemble* and its progeny have enjoyed a broad range of unexpected applications in more recent years, from wireless communications [3] and financial risk [4] to effective theories of strong interactions [5] and random fluctuating interfaces [6]. The *Jacobi ensemble* is yet another set of random matrices *beyond* the classical Gaussian paradigm, which has found recent applications in the theory of quantum transport in mesoscopic devices [7].

The two examples of classical ensembles stated above fall into the broad class of *invariant* matrix models, whose probability distribution of entries remains unchanged after a global rotation of basis. Such powerful symmetry is highly welcomed by random matrix theorists as it allows to write down explicitly the joint probability density of the N eigenvalues, a crucial piece of information which is otherwise unavailable. The lack of such symmetry, only partially compensated by the requirement of *independent entries*, characterizes the second main class of matrix models usually considered in the literature: examples of the latter include the adjacency matrix of random graphs [8] and Lévy matrices [9].

The technical tools available to deal with random matrices depend crucially on whether the joint density of eigenvalues is known (invariant ensemble) or not. In the former case, a wealth of analytical methods (such as orthogonal polynomials, the Coulomb gas technique and symmetric function expansions) are available, while in the latter the analytical results are usually limited to the average level density, obtained by replicas and Green's function techniques.

The joint density of real eigenvalues $\mathcal{P}(\lambda_1, \ldots, \lambda_N)$, when available, is a remarkable object. As a consequence of integrating out the eigenvector components, a Vandermonde all-to-all coupling term $\prod_{j<k} |\lambda_j - \lambda_k|^\beta$ appears, which is responsible for the peculiar strong correlations (long-range) among the eigenvalues. The Dyson index β is classically quantized and can take only the values $\beta = 1, 2, 4$ according to the number of variables needed to specify a single entry (real, complex or quaternion numbers). The same index in turn identifies the symmetry group of the ensemble (Orthogonal, Unitary and Symplectic respectively).

The presence of this Vandermonde term leads to a natural and far-reaching interpretation of the set of N real eigenvalues for $N \gg 1$ as particles of a 2-dimensional fluid confined to a line and subject to a logarithmic (Coulomb) repulsion, a confining external potential plus (if needed) a series of other constraints. This *Coulomb gas analogy*, originally due to Dyson, and in particular its constrained version have recently seen a fruitful revival [10]. It indeed appears to be the most convenient way to tackle very difficult problems such as full probability distributions of linear statistics on the eigenvalues of random matrices [11] and a wealth of recent results appeared recently in the literature.

Along with remarkable progresses in establishing connections between different areas and tools of statistical physics, recent times have witnessed a remarkable boost towards formal developments of the theory. The beautiful tools of symmetric functions and group theory [12] have been invaluable in tackling very hard problems at the boundary between physics and mathematics. Questions related to numerical efficiency of Random Matrix Theory (RMT) algorithms have attracted much attention [13], not to mention the very recent 'topological recursion' approach to free energy expansions of matrix models [14]. At present, RMT provides one of the most promising roads towards a proof of the celebrated Riemann's hypothesis, a marvellous example of cross-fertilization between physics and pure mathematics [15] with far-reaching and possibly spectacular outcomes.

In the realm of quantum phenomena, entanglement has played a major role as a fundamental breaking point with respect to the classical world. The properties of *typical* entangled states (pure or mixed) have been recently addressed assuming that the coefficients of such states in a given basis are taken at random from a certain distribution [16]. The properties of the reduced density matrices ρ of such states have been conveniently framed in terms of Wishart eigenvalues with a fixed-trace constraint [17]. Notwithstanding remarkable progresses in the analysis of such eigenvalue problems, fundamental questions are still open, such as the distribution of extreme eigenvalues in the case of arbitrary dimensions of the corresponding Hilbert spaces [18] and distributions of entanglement quantifiers (entropy, purity, concurrence...) beyond the large-N paradigm [19].

Taking a broader perspective, random matrices are also rather useful in many interdisciplinary contexts. Both biology and financial engineering, for instance, are disciplines beyond the traditional borders which highly exploited RMT-related tools and techniques to tackle problems such as protein folding [20], molecular dynamics [21], and risk assessment [4]. Such interdisciplinary applications are expected to grow in number over the years and furtherly establish RMT as a unifying and powerful language for a variety of scientific purposes.

In full generality, the eigenvalues of invariant matrix models are also of purely theoretical interest as one of the rare examples of *strongly correlated* random variables for which a wealth of analytical tools are available. Extreme value statistics for correlated variables is a young and fertile field which is currently under intense scrutiny. The ultimate goal would be to establish universality classes for the statistics of extremes mimicking the well-known Gumbel-Frechet-Weibull paradigm holding for uncorrelated variables. At present, the Tracy-Widom distribution appears to be very robust and fairly ubiquitous, even though hybridizations with other statistics have been reported in the literature [22].

Recent developments in the theory clearly highlight a few critical directions of research to be undertaken in the future, as detailed in the following section.

2. Future directions of research

Statistical physics of disordered systems provides the technical tools to understand and classify the behaviour of systems composed by a large number of interacting constituents. The introduction of random ingredients (the *disorder*), at the microscopic (interactions) or macroscopic (environment) level, poses formidable technical challenges and has unveiled the existence of a rich zoology of new phenomena, such as freezing, aging, hybrid transitions and multifractality, whose experimental testing in a variety of systems constitutes a remarkable success of the theory.

In recent years, the statistical mechanics community has also witnessed a growing interest in the properties of the largest (or smallest) element in a sequence of *correlated* random variables. Going beyond the traditional Gumbel-Frechet-Weibull paradigm holding for independent sequences, the presence of strong correlations poses indeed quite serious challenges, and few analytical methods are presently available.

My research activity so far has been mostly devoted to the study of rare events (large deviations) in the behaviour of eigenvalues of random matrices as a prominent example of strongly correlated random variables and the thermodynamics of Coulomb fluids with constraints. A coherent description of such systems in a unifying framework is still far from being completed, as many fundamental questions remain unanswered.

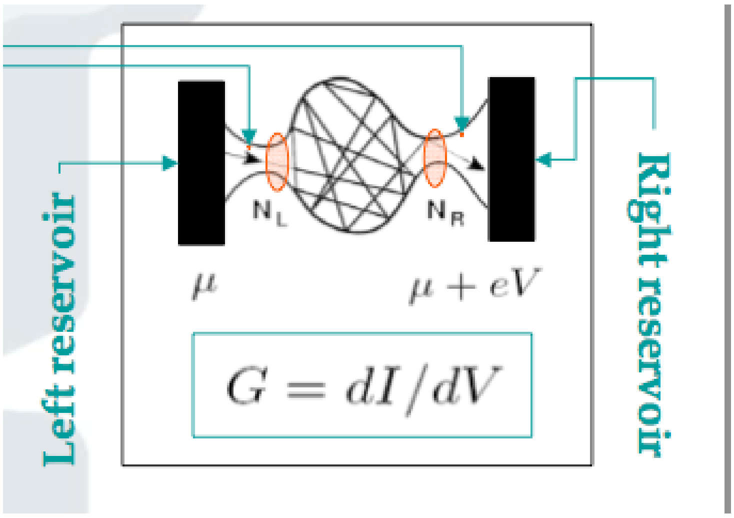
Looking at recent developments in the statistical mechanics community through the prism of random matrix theory, a few outstanding problems emerge. The main research lines I shall be pursuing are then as follows:

- **Theory of quantum transport in non-ideal chaotic cavities.**
  Consider a cavity of submicron dimensions etched in a semiconductor, as sketched in next figure. The cavity is connected by two leads to two electron reservoirs. When an external voltage is applied, electrons flow inside the cavity, get scattered on the boundary and may leave the cavity from either of the two leads. The statistical properties of electronic transport inside the cavity may be described by an appropriate random matrix model of the scattering operator $\mathcal{S}$. The random scattering matrix framework is well established in the (idealised) case of perfectly transparent leads. In real experiments, though, the leads are not perfectly transparent to the incoming electrons: the presence of so-called *tunnel barriers* in the leads crucially affect the chaotic motion of electrons inside the cavity and the probability of experimental outcomes. One of my research goals is to develop a statistical theory of electronic motion inside non-ideal cavities, based on the following technical steps:

  1. computation of the joint probability density of transmission eigenvalues for a scattering matrix $\mathcal{S}$ distributed according to the so-called Poisson's kernel [23]. This framework is suitable for treating the case of partially transparent junctions between the leads and the cavity.
  2. full probability distribution of conductance and shot noise computed via a Coulomb gas technique in Laplace space [11].

  It is expected that this research line should begin by dealing with the simplest case of average scattering matrix proportional to the identity $\langle \mathcal{S} \rangle = z\mathbb{I}$, for which some results are already available.

- **Tracy-Widom distribution for continuous β from Pandey-Mehta model.**
  The Pandey-Mehta matrix model was introduced in [24] as an analytical tool to study transitions between different symmetry classes, namely the orthogonal ($\beta = 1$) and unitary ($\beta = 2$), with possible applications to the study of intermediate spectral statistics in quantum chaos. The model, depending on a continuous parameter α, has the merit of being exactly solvable, i.e. n-point correlation functions of energy levels can be written down explicitly in terms of pfaffians of a non-standard α-dependent kernel. Since this ensemble has been around for long time, it has somehow been overlooked when, many years after its introduction, Tracy and Widom discovered the distribution $F_\beta(x)$ that bears their names. The idea is now to dig this model out again as a possible source of analytical transitions between Tracy-Widom distributions with a *continuous* β. More precisely, the distribution of the largest eigenvalue of the Pandey-Mehta model, when



suitably scaled, should converge to $F_2(x)$ or $F_1(x)$ for special values of $\alpha = 0, 1$, and to a new interpolating distribution for general α. It is then of great interest to study how the Painlevé integrable structure (governing the distributions for $\beta = 1, 2$) gets deformed when a continuous transition between the two classes is allowed.

- **Statistics of number of filtered-out directions in Principal Component Analysis.** Principal Component Analysis (PCA) is a commonly used tool in detecting relevant patterns in multivariate statistical data. The main idea is to collect eigenvalues and eigenvectors of the covariance matrix of empirical data and then 'filter out' the data which lie along the 'weakest' directions (corresponding to the smallest eigenvalues). This procedure eventually produces a simplified backbone of relevant correlations among data, washing out the noise-dressed components that do not carry important informations. The question is then: how many directions (eigenvalues) should we neglect? There are no obvious and unique criteria telling us when to stop! However, some 'rules of thumb' do actually exist: the simplest and most widely used (although highly questionable) is the so-called *Kaiser-Guttman* criterion [25], which prescribes to keep all the directions corresponding to eigenvalues *larger than the average* ($\lambda_j > \langle\lambda\rangle$) while discarding the others. Suppose now that the data were completely random (e.g. drawn independently from a Gaussian distribution). Still, according to this criterion, we would keep $\mathcal{N}_+$ directions for each instance and dub them 'significant', even though they clearly are not! In real-world experiments, we thus have to offset the effect of spurious (or false positive) correlations which would arise even if the data were completely uncorrelated. In order to do so, we first need to understand the statistics of the random variable $\mathcal{N}_+$. This can be done in the framework of Wishart-Laguerre ensemble of random covariance matrices, extending the recent similar treatment of Gaussian matrices [26].

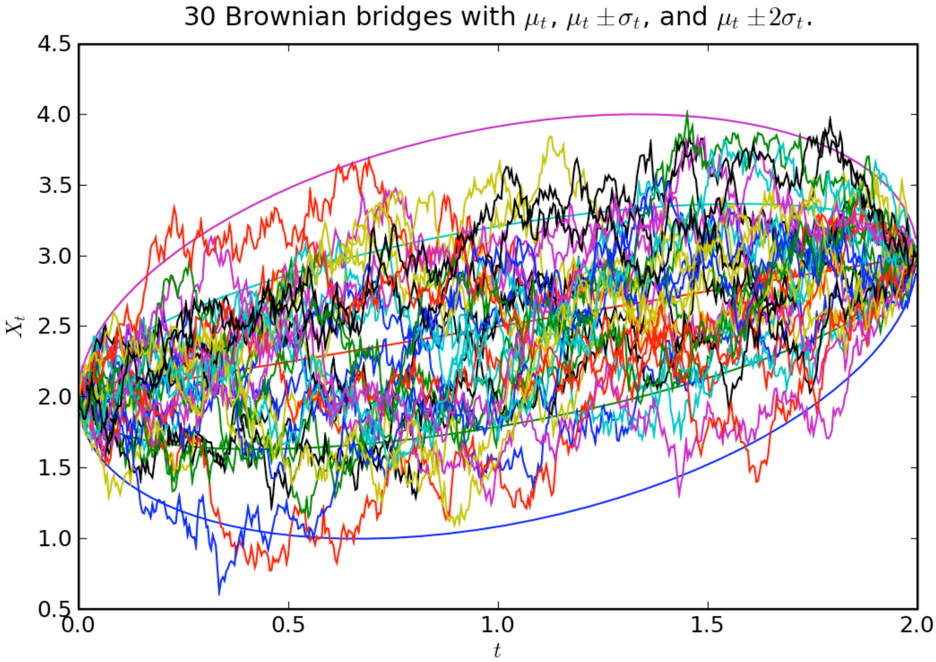
- **Continuous β models for complex matrices.** The discovery of β-ensembles of random matrices by Dumitriu and Edelman in 2002 [27] has been a major breakthrough in the field. It is a beautiful mathematical construction leading to a formidable increase in numerical efficiency, and it goes beyond the traditional quantization of Dyson's index $\beta = 1, 2, 4$ present in all traditional ensembles. Natural questions that I intend to address are: is it possible to produce β-ensembles for matrices whose eigenvalues are scattered in the complex plane (β-Ginibre ensembles)? What are the properties of the extreme eigenvalues, e.g. does the largest eigenvalue follow the Tracy-Widom distribution as the standard 'quantized' ensembles? The tridiagonal mapping used in [27] appears non-trivial to generalize to the complex plane, and yet there are hints that such a construction must exist. Equally interesting for applications (in particular for numerical simulations) is a tridiagonal construction for Dyson's circular ensembles [12] which I plan to pursue along the same lines.

- **Theory of multipartite entanglement of random states.**
Consider a Hilbert space $\mathcal{H}$ wich is bipartite as the tensor product of two subspaces $\mathcal{H}_A$ and $\mathcal{H}_B$. Quantum states in $\mathcal{H}$ can then be decomposed (*factorized states*) or not (*entangled states*) as products of states, one belonging to $\mathcal{H}_A$ and the other to $\mathcal{H}_B$. Entangled states whose coefficient in a given basis are random Gaussian variables are of paramount interest, as they provide information about *typical* (unbiased) properties we should expect for such states, given the dimensions M and N of the two subspaces. For random *pure* states, there is a natural unitarily invariant measure on the eigenvalues of the reduced density matrix ϱ_A (so-called *Schmidt eigenvalues*) which is nothing but a Wishart distribution with an extra trace-fixing constraint ($\sum_i \lambda_i = 1$). An outstanding problems which I plan to address is the distribution of Von Neumann and Rényi entropies for *finite* N, M as quantitative measures of entanglement. For such project, which involve the computation of non-trivial Selberg-like integrals with several constraint, the collaboration with J.-G. Luque in Rouen (an expert in the theory of symmetric functions) and with O. Giraud at LPTMS (who has already produced interesting works on the same problems) will prove crucial.

  The case of random *mixed* states is even more interesting, as several measures on the eigenvalues of the reduced density matrix can be used, and the available analytical results are surprisingly scarce. For the Bures measure (enjoying many important mathematical properties), the full distribution of entanglement quantifiers such as purity and von Neumann entropy is still unknown [28]. In the Paris area, I will enjoy interactions with Satya N. Majumdar, Gregory Scher and Massimo Vergassola, in an attempt to tailor the constrained Coulomb gas technique [10] to the entanglement problem. The idea is to try to formulate a suitable singular integral equation (or scalar Riemann-Hilbert problem) for the eigenvalue distribution of the reduced density matrix. The solution will likely provide the rate (or large deviation) function for entanglement quantifiers in the limits of large dimensions M, N. The question of *finite* N, M corrections is clearly much more challenging and could perhaps be addressed by an appropriate use of the machinery of symmetric functions. At present, such idea is merely speculative but I plan to pursue it and attempt to tackle this very difficult problem, whose solution is very much called for.

- **Statistical properties of configurations of many brownian walkers.**
Sets of one-dimensional random walkers with different geometrical constraints are known to have deep connections with random matrix theory (see next picture for so-called brownian bridges). For example, self-avoiding walks are characterized by the property that the paths of different walkers never intersect as time progresses. When several walkers starting at the origin have prescribed ending points, an interesting connection with random matrices emerges, since the joint probability density of the positions of the walkers at a fixed time τ generically displays a Vandermonde-like repulsion. This sort of interaction arises from the use of a quantum-mechanic formulation of the problem in terms of fermionic propagators [29]. This technique is very powerful and allows to tackle many statistical questions related, for example, to the properties of maximal excursions [30]. An interesting transition in the density of walkers on the real line at a fixed time τ is expected if additional constraints are provided. For instance, one may impose that a certain

fraction f of walkers ends up at the same location x_1, while $1-f$ walkers end up at x_2. What happens to the statistics of observables due to the presence of this hard extra-constraint? The question is relevant as new universality classes for distribution of observables may arise from the interplay between strong repulsion and hard constraints. In seeking signatures of new universal distributions, I shall highly benefit from discussions and interactions with Satya N. Majumdar, Alain Comtet and Alberto Rosso at LPTMS and Gregory Scher at LPT as leading experts in this field.



- **Adjacency Matrices of Random Graphs beyond the replica trick.**
  The adjacency matrix of random graphs is an ensemble of sparse matrices whose entry (i, j) is equal to 1 if the nodes i and j of the corresponding graph are connected and 0 otherwise. The eigenvalues of the adjacency matrices affect the static and dynamical properties of the underlying graph in a non-trivial way [31]. It is therefore highly desirable to compute analytically spectral properties of a graph given its topology. At present, the situation is however rather unsatisfactory, as our knowledge is basically limited to the average spectral density for a few topologies [32], and even when available, the resulting expressions are not quite explicit - typically, the density is computed via replica calculations and comes out as the implicit solution of integral equations, which can then be solved only numerically. Very little is known about level spacing distributions or properties of extreme eigenvalues (with the exception of some algebraic bounds [33]), which are however very effective in shaping dynamical processes occurring on networks [31]. Our ignorance about the joint distribution of the eigenvalues is to blame, and it is a major theoretical challenge with countless applications to overcome such limitations. As a large-scale project, I would try to focus on such fundamental issues and attempt to make some progress towards our understanding of the statistics of the eigenvalues of random graphs in a more systematic way.

This may even require to change the way we represent graphs, in the spirit of *portraits* put forward in a beautiful but not widely appreciated paper [34]. A related fundamental question is how to implement rotational invariance (i.e. basis independence) in random graphs. Clearly, the sparsity prescription is the major obstacle towards this goal. Is it possible to waive such requirement and yet provide a reliable and faithful description of complex networks?

Perhaps a first step towards a better understanding of the complete eigenvalue statistics of complex networks would be the study of *two-population* matrices: the entries of such matrices are drawn partly from a Gaussian centred in zero, and partly from a Gaussian centred in one. In the limit of very small variances, one expects to recover a sparsity pattern typical of adjacency matrices, but for finite variances one may fully exploit the benefit of dealing with Gaussian variables, a notorious technical advantage. Such two-population matrices have received very little attention so far, even though a careful search reveals that some examples do actually exist [35]. It is my intention to start a deep investigation of such ensembles, using standard diagrammatic and Green's function techniques.

Another route towards this goal would be to exploit ideas from superstatistics [36], which has already shaped a few existing matrix models [37, 38, 39]. Taking the *average* of matrix elements as a fluctuating quantity (instead of the commonly used *variance*), one may preserve rotational invariance and yet mimic the sought sparsity pattern through a judicious choice of the superstatistical distribution (for example, using a bimodal function with peaks in zero and one). Such ideas may constitute the core of a long-term research activity I would like to pursue throughout the next few years.

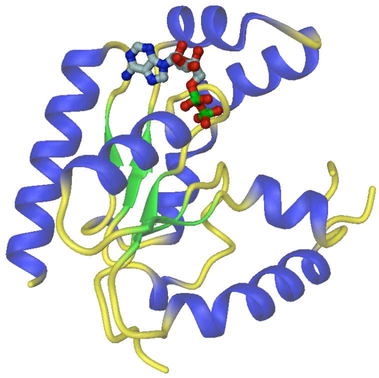
- **Nonlinear Selberg-like integrals.**
  The theory of invariant random matrices heavily relies on evaluations of multiple integrals involving the Vandermonde determinant. The theory of Selberg-like integrals is still under developments and many exciting results, highlighting the connection with the theory of symmetric functions, have recently appeared in the mathematical literature [40]. One of the most interesting case for applications has been nevertheless poorly considered so far, and it is related to certain nonlinear averages involving products of different eigenvalues. Such integrals are notoriously harder to compute than their 'linear' counterparts. Surprisingly, a very efficient algorithmic solution was found recently for the Jacobi case [41], whose main merit being that its complexity does not grow at all with N, the number of integration variables. Such remarkable feature allows to perform a $N \to \infty$ asymptotic analysis, revealing a fascinating combinatorial structure lurking behind. This research field is very young and many interesting mathematical developments can be foreseen, such as nonlinear extensions of q-Selberg integrals [40] and the generalization of integral formulas to the Gaussian and Wishart case. Given the remarkable connection between such nonlinear Selberg integrals and the theory of Jack polynomials, I look forward to collaborating with Raoul Santachiara at LPTMS who has recently completed some exciting works on this topic [42].

- **Universality for products of random matrices.**
  In many physical applications, one single random matrix is not enough: consider for example the case of correlation matrices $\mathcal{C}$ among different data tables $\mathcal{X}$ and $\mathcal{Y}$, $\mathcal{C} = \mathcal{X}\mathcal{Y}^T$. Spectral properties of products of random matrices have received adequate consideration only very recently [43]. A preliminary result which is very appealing is the existence of a *universal* distribution law for the eigenvalues of such products in the complex plane. Such universal law lies somehow on the same footing as the celebrated Girko's and Wigner's laws, as the above mentioned distribution

is independent on fine details of the matrix distributions. A new and very promising direction of research I shall pursue is the extension of such works to the case of rectangular matrices and heavy-tailed distributions of matrix entries, for which the diagrammatic technique used in [43] is not directly applicable. A more insightful way to look at this problem is to try to compute the joint distribution of the entries of the product matrix. A polar decomposition shall then bring into play the complex eigenvalues whose universal density (albeit with the limitations stated above) has been computed in [43]. Possible concrete applications are related to the study of covariance matrices of pairs of independent random datasets.

- **Theory of quantum quenches for random Hamiltonians.**
The condensed matter physics community has recently witnessed a growing interest in physical systems which are brought out-of-equilibrium by a sudden change in an external control parameter. For example, one can consider a slab of magnetic material at room temperature which is suddenly cooled well below the critical temperature separating the paramagnetic and ferromagnetic phases. One can ask: what happens to the inner structure of magnetic domains due to such an abrupt change (a *quench*) in the external conditions? This question has been investigated in detail for the case of *specific* models of microscopic interactions, on a case-by-case basis. My goal is to formulate a random matrix theory of quantum quenches. In this approach, specific interaction models are replaced by random energy matrices, whose entries depend continuously on some parameter λ. Typically, the statistics of energy levels will oscillate between two extreme cases (the so-called Poisson and Wigner-Dyson distributions). In particular, it is of paramount interest to describe *universal* (i.e. model-independent) features of the work needed to change the symmetry class of the ensemble. This general goal can be in principle achieved at a technical level by introducing a parameter-dependent Rosenzweig-Porter matrix model [44] and then computing the so-called Loschmidt echo [45], i.e. the squared overlap between the ground state eigenvectors corresponding to two arbitrary values λ and λ' of the parameter. I do expect this computation to be technically involved but likely feasible with current (or slightly improved) analytical tools.

- **Protein displacements and Molecular Dynamics.**
The biological functionality of a protein often relies on its capability to undergo large-scale conformational changes. In order to study such motions, one may formulate elastic network models, where the protein structure is approximated by a network of anisotropic springs connecting aminoacids within a certain cutoff distance. The *covariance matrix* of aminoacid displacements contains essential information about the vibrational modes of the protein around its reference (equilibrium) structure. Together with my collaborators [21], I recently suggested that comparisons with randomly generated covariance matrices should help answering in a solid, quantitative way the following question: how many vibrational modes should we take into account to give a sufficiently reliable description of a protein's overall motion? Our recent proposal opens up a very promising direction of research, where the marriage between biophysical issues and random matrix techniques is expected to play a fruitful role. Such interplay is in fact not completely new. One of the most prominent result in this respect is the formulation of the RNA folding problem (i.e. the prediction of the spatial arrangements of pairings between canonical base pairs) in terms of a matrix field theory [20]. This proposal is rather fascinating and in the Paris area I will have the opportunity to interact also with H. Orland at Saclay as one of the leading expert in the interplay between field theories, random matrices and biological problems.



3. Summary and Outlook

At present, there is an ongoing rapid expansion of the range of applicability of RMT and I have witnessed the beginning of a cross-fertilization between different areas of statistical mechanics and beyond. I have identified and detailed above a few fundamental and timely research issues involving random matrix theory and correlated random variables that I plan to undertake at CNRS. While it is admittedly difficult to provide a reliable forecast about the chances of success of the aforementioned projects, I have the feeling that it is not unreasonable to expect tha the majority of the issues outlined above will lead sooner or later to a positive outcome. For a few of my research topics, I tried to identify eminent scientists in the Paris area with whom I expect to collaborate closely towards such goals, and the Laboratoire I chose (LPTMS - Orsay) clearly provides the most fertile environment where my research plans can be developed with the highest probability of success.

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