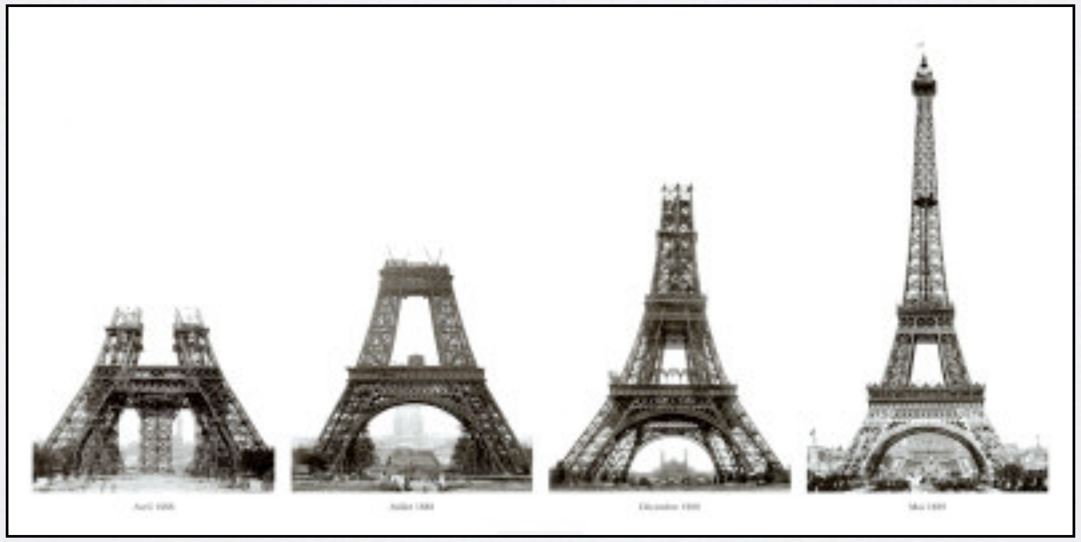
TOP EIGENVALUE OF CAUCHY RANDOM MATRICES

with Satya N. Majumdar, Gregory Schehr and Dario Villamaina

Pierpaolo Vivo
(LPTMS - CNRS - Paris XI)

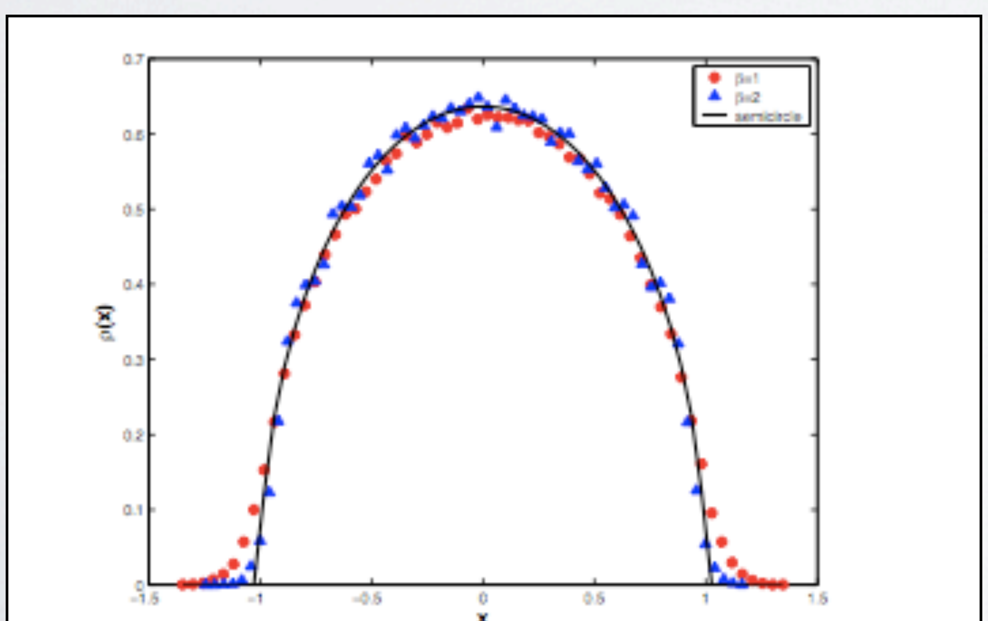


Gaussian Ensembles

N = 5

$$\begin{pmatrix} 0.5377 & 0.2631 & -1.8044 & 0.3286 & 0.4951 \\ 0.2631 & -0.4336 & 1.6888 & 1.7271 & 0.7810 \\ -1.8044 & 1.6888 & 0.7254 & 0.7133 & 0.7160 \\ 0.3286 & 1.7271 & 0.7133 & 1.4090 & 1.5237 \\ 0.4951 & 0.7810 & 0.7160 & 1.5237 & 0.4889 \end{pmatrix}$$

$$\vec{\lambda} = \left[-2.4341 \;\; -0.8386 \;\; -0.5203 \;\; 2.2594 \;\; 4.2610 \right]$$



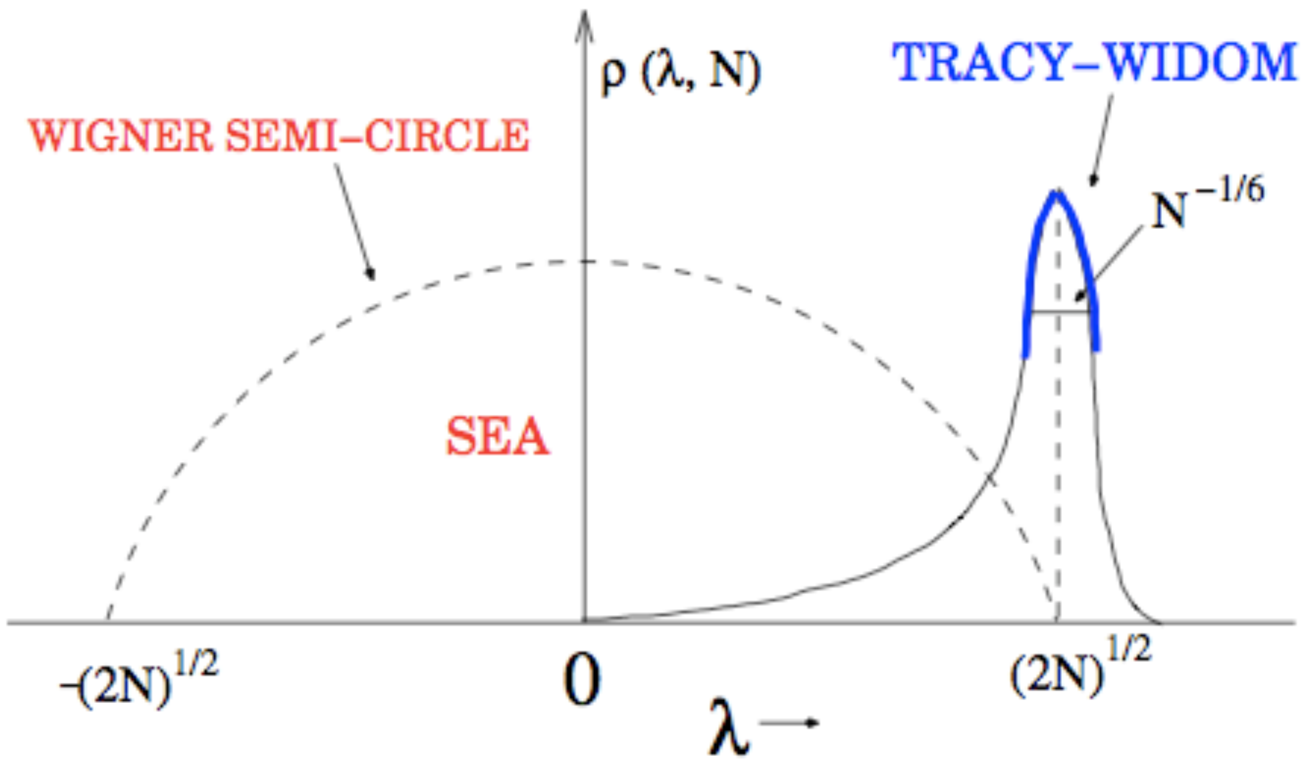
Semicircle Law

$$\rho(\lambda) = \frac{1}{2\sqrt{N}} f\left(\frac{\lambda}{2\sqrt{N}}\right)$$

$$f(x) = \frac{2}{\pi}\sqrt{1 - x^2}$$

LARGEST EIGENVALUE

Tracy-Widom distribution for λ_{max}



- $\langle \lambda_{\max} \rangle = \sqrt{2N}$; typical fluctuation: $|\lambda_{\max} - \sqrt{2N}| \sim N^{-1/6}$ (small)
- typical fluctuations are distributed via Tracy-Widom (1994):
- cumulative distribution:

$$\mathrm{Prob}[\lambda_{\max} \leq t, N] \to F_\beta\left(\sqrt{2}N^{1/6}(t - \sqrt{2N})\right)$$

- Prob. density (pdf): $f_\beta(z) = dF_\beta(z)/dz$
- $F_\beta(z) \to$ obtained from solution of Painlevé-II equation

$$\lambda_{\max} \approx \sqrt{2N} + a_\beta N^{-1/6} \underline{\chi}$$

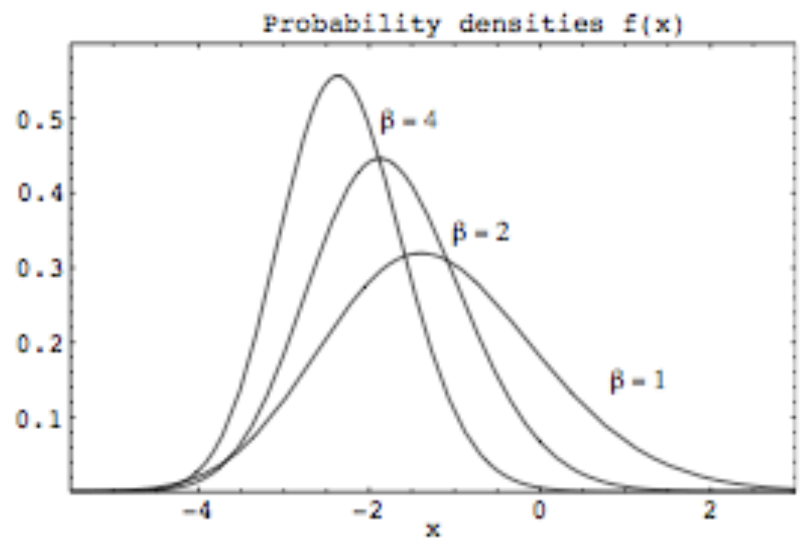
$$\mathcal{P}(\underline{\chi} \leq x) = F_\beta(x)$$

$$F_2(x) = \exp\left[-\int_x^\infty (z-x) q^2(z) dz\right]$$

$$q'' = 2q^3 + zq$$

Painlevé II

Tracy-Widom distribution for λ_{max}



- Tracy-Widom density $f_\beta(x)$ depends explicitly on β.
- Asymptotics: $f_\beta(x) \sim \exp\left[-\frac{\beta}{24}|x|^3\right]$ as $x \to -\infty$

$$\sim \exp\left[-\frac{2\beta}{3}\, x^{3/2}\right] \quad \text{as} \quad x \to \infty$$

Applications: Growth models, Directed polymer, Sequence Matching .....
(Baik, Deift, Johansson, Prahofer, Spohn, Johnstone,....)

A recent 'simpler' derivation of Tracy-Widom for $\beta = 2 \to$ [Nadal and Majumdar 2011]

[Takeuchi and Sano, PRL 2010]

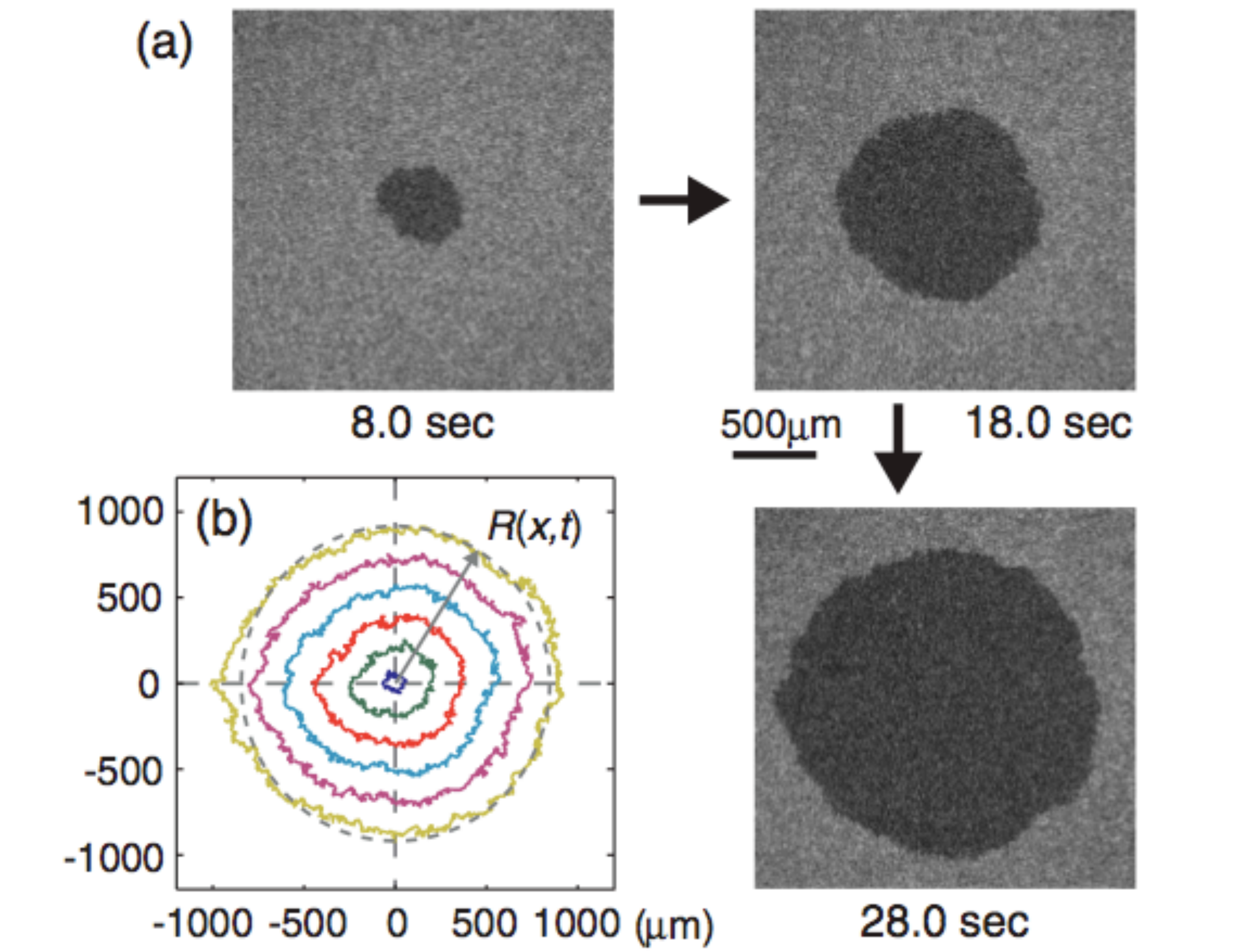
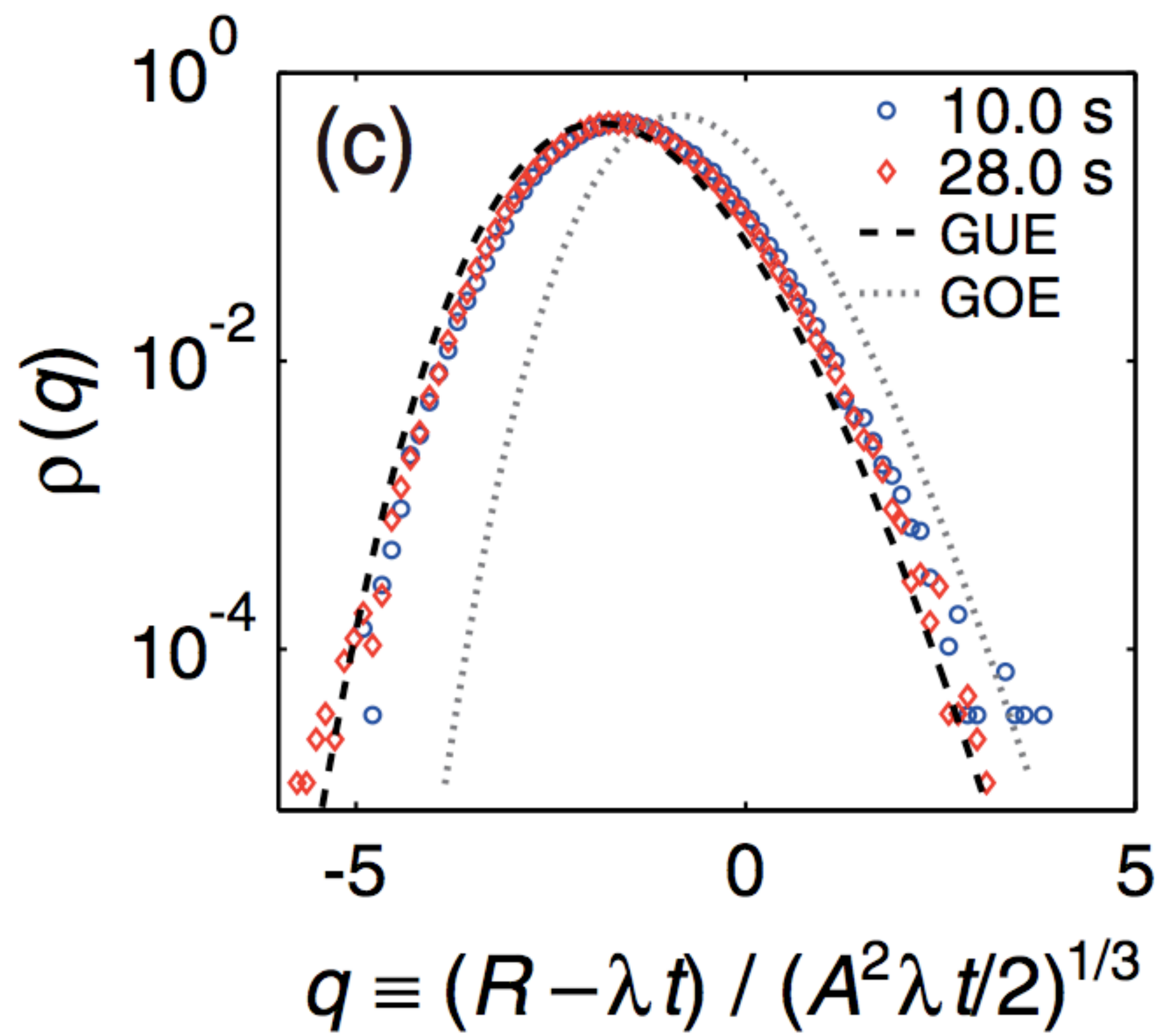
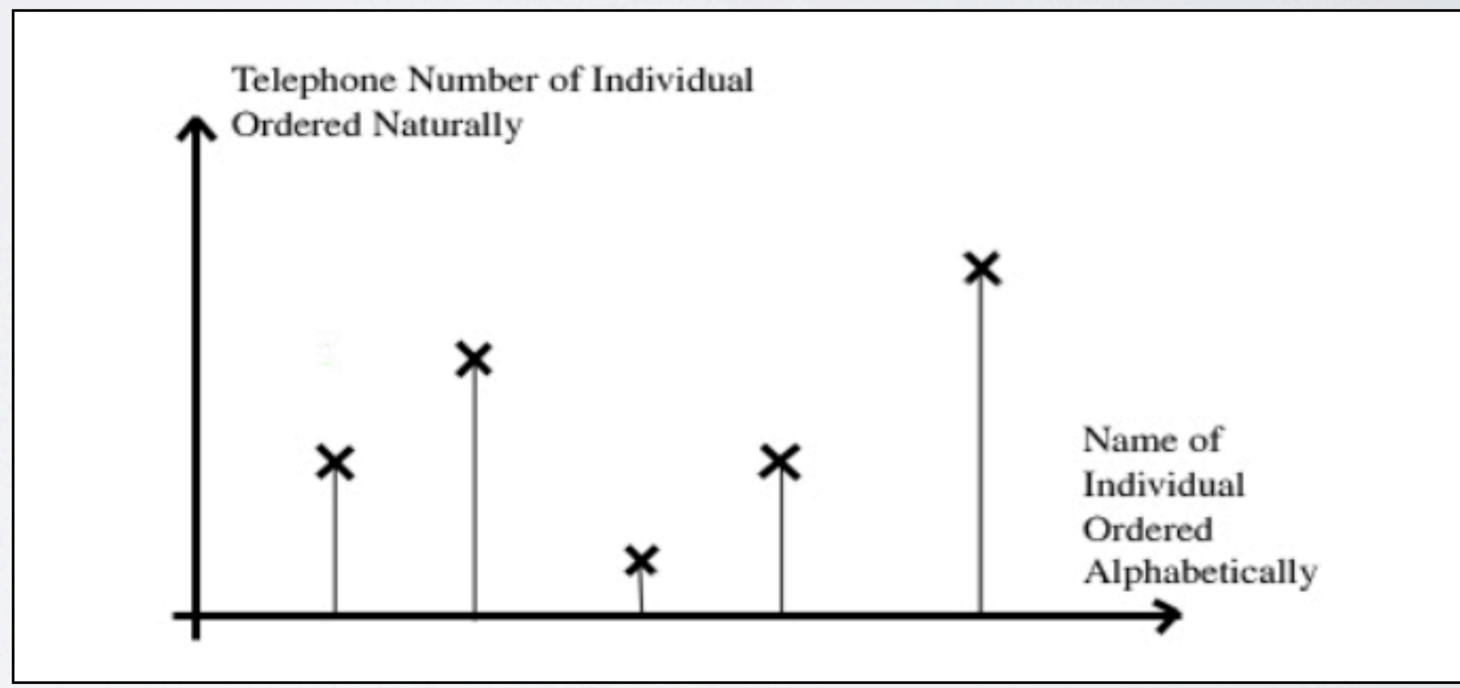


FIG. 1 (color online). Growing DSM2 cluster. (a) Images. Indicated below is the elapsed time after the emission of laser pulses. (b) Snapshots of the interfaces taken every 5 s in the range 2 s $\leq t \leq$ 27 s. The gray dashed circle shows the mean radius of all the droplets at $t = 27$ s. The coordinate x at this time is defined along this circle.



“Are Tracy and Widom in Your Local Telephone Directory?”

Ryan Witko
Advisor: Percy Deift



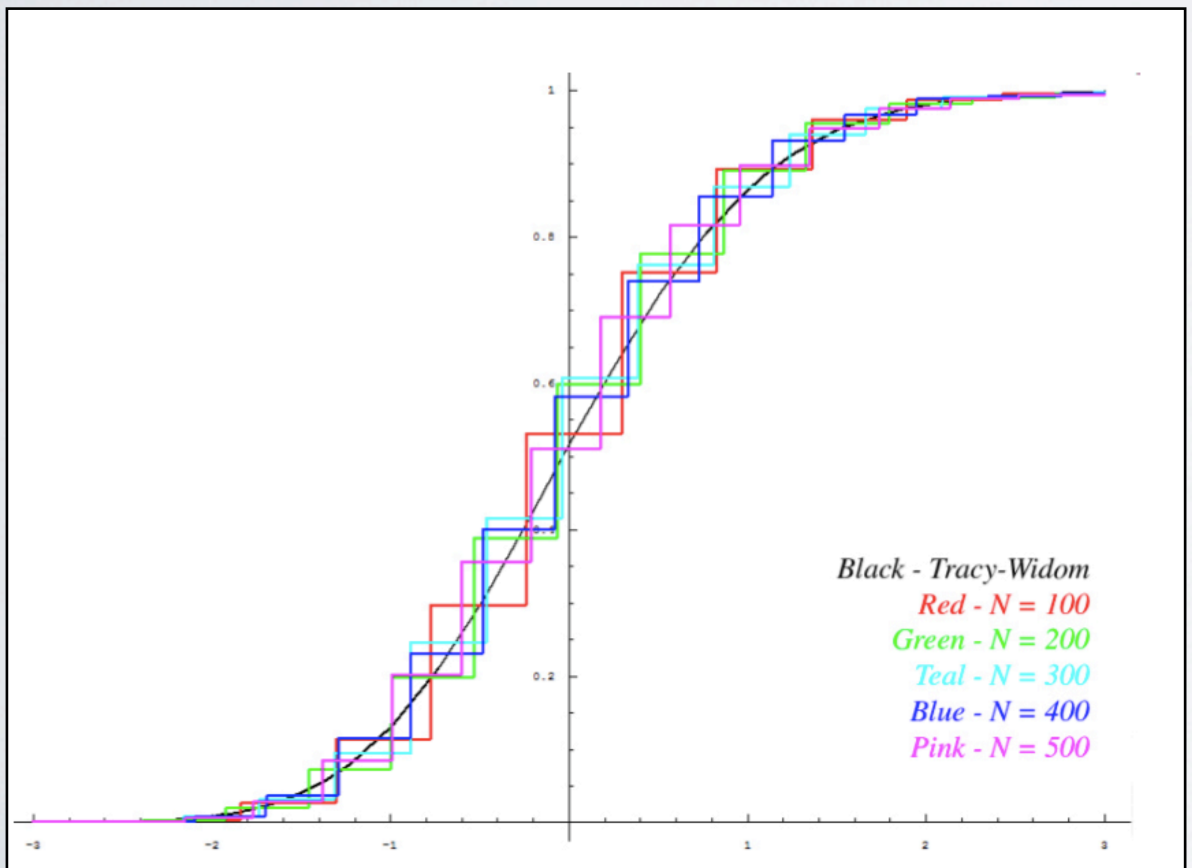
Definition:

The longest increasing (contiguous) subsequence of a given sequence is the subsequence of increasing terms containing the largest number of elements. For example, the longest increasing subsequence of the permutation {6, 3, 4, 8, 10, 5, 7, 1, 9, 2} is {3, 4, 8, 10}.

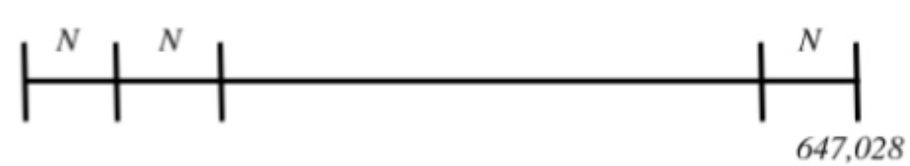
It can be coded in *Mathematica* as follows.

```
<<Combintorica`
LongestContinguousIncreasingSubsequence[p_] :=
  Last[
  Split[Sort[Runs[p]], Length[#1] >= Length[#2]&]
  ]
```

More information »



We broke the 647,028 entries into successive samples each containing *N* entries.

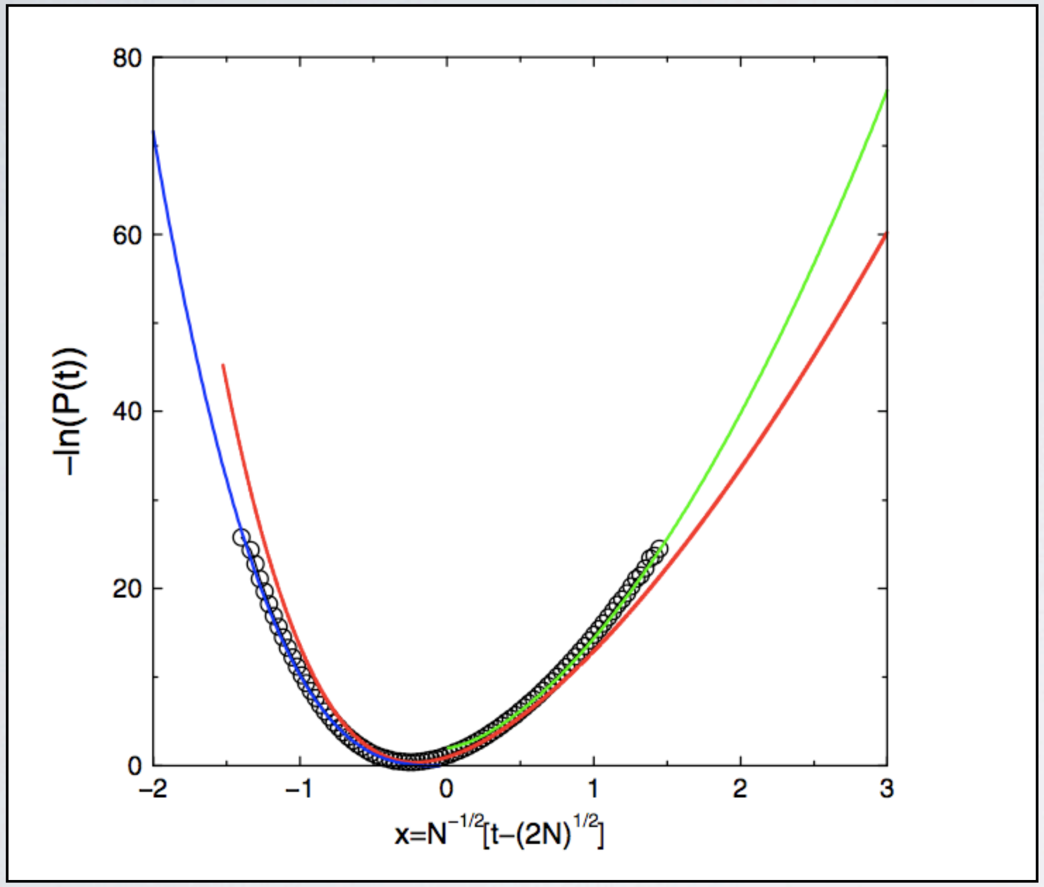


Jinho Baik, Kurt Johansson and Percy Deift showed that as $N \to \infty$

(3) $$\text{Prob}\left(\frac{\ell_N - 2\sqrt{N}}{N^{1/6}} \le t\right) \to F(t)$$

The function *F(t)* was shown by Craig Tracy and Harold Widom to be the distribution of the largest eigenvalue of a random matrix in the Gaussian Unitary Ensemble (GUE). It

TYPICAL VS. ATYPICAL



PRL 102, 060601 (2009) PHYSICAL REVIEW LETTERS week ending 13 FEBRUARY 2009

Large Deviations of the Maximum Eigenvalue for Wishart and Gaussian Random Matrices

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Probab. Theory Relat. Fields 120, 1–67 (2001)
Digital Object Identifier (DOI) 10.1007/s004400000115

G. Ben Arous · A. Dembo · A. Guionnet

Aging of spherical spin glasses

$$\mathcal{P}(\lambda_{\max} = t) \approx \begin{cases} \exp\left(-\beta N^2 \psi_-\left(\frac{t}{\sqrt{N}}\right) + \ldots\right) & \text{for } t < \sqrt{2N} \text{ and } |t-\sqrt{2N}| \approx \mathcal{O}(N) \\ \frac{1}{a_\beta N^{-1/6}} F'_\beta\left(\frac{t-\sqrt{2N}}{a_\beta N^{-1/6}}\right) & \text{for } |t-\sqrt{2N}| \approx \mathcal{O}(N^{-1/6}) \\ \exp\left(-\beta N \psi_+\left(\frac{t}{\sqrt{N}}\right) + \ldots\right) & \text{for } t > \sqrt{2N} \text{ and } |t-\sqrt{2N}| \approx \mathcal{O}(N) \end{cases}$$

$$\lim_{N\to\infty} \frac{1}{\beta N^2} \ln \mathcal{P}(\lambda_{\max} = z\sqrt{N}) = -\psi_-(z) \qquad \text{for } z < \sqrt{2}$$

$$\lim_{N\to\infty} \frac{1}{\beta N} \ln \mathcal{P}(\lambda_{\max} = z\sqrt{N}) = -\psi_+(z) \qquad \text{for } z > \sqrt{2}$$

A simple example of large deviation tails

- Let $M \to$ no. of heads in N tosses of an unbiased coin
- Clearly $P(M,N) = \binom{N}{M} 2^{-N}$ $(M = 0, 1, \ldots, N) \to$ binomial distribution

  with mean$= \langle M \rangle = \frac{N}{2}$ and variance$=\sigma^2 = \langle \left(M - \frac{N}{2}\right)^2 \rangle = \frac{N}{4}$
- typical fluctuations $M - \frac{N}{2} \sim O(\sqrt{N})$ are well described by the Gaussian form: $P(M,N) \sim \exp\left[-\frac{2}{N}\left(M - \frac{N}{2}\right)^2\right]$
- Atypical large fluctuations $M - \frac{N}{2} \sim O(N)$ are not described by Gaussian form
- Setting $M/N = x$ and using Stirling's formula $N! \sim N^{N+1/2} e^{-N}$ gives

$$P(M = Nx, N) \sim \exp\left[-N\Phi(x)\right] \qquad \text{where}$$

$$\boxed{\Phi(x) = x \log(x) + (1-x)\log(1-x) + \log 2} \to \text{large deviation function}$$

- $\Phi(x) \to$ symmetric with a minimum at $x = 1/2$ and for small arguments $|x - 1/2| << 1$, $\Phi(x) \approx 2(x - 1/2)^2$

  $\to$ recovers the Gaussian form near the peak

RAPID COMMUNICATIONS

PHYSICAL REVIEW E **85**, 020101(R) (2012)

Measuring maximal eigenvalue distribution of Wishart random matrices with coupled lasers

Moti Fridman, Rami Pugatch, Micha Nixon, Asher A. Friesem, and Nir Davidson*

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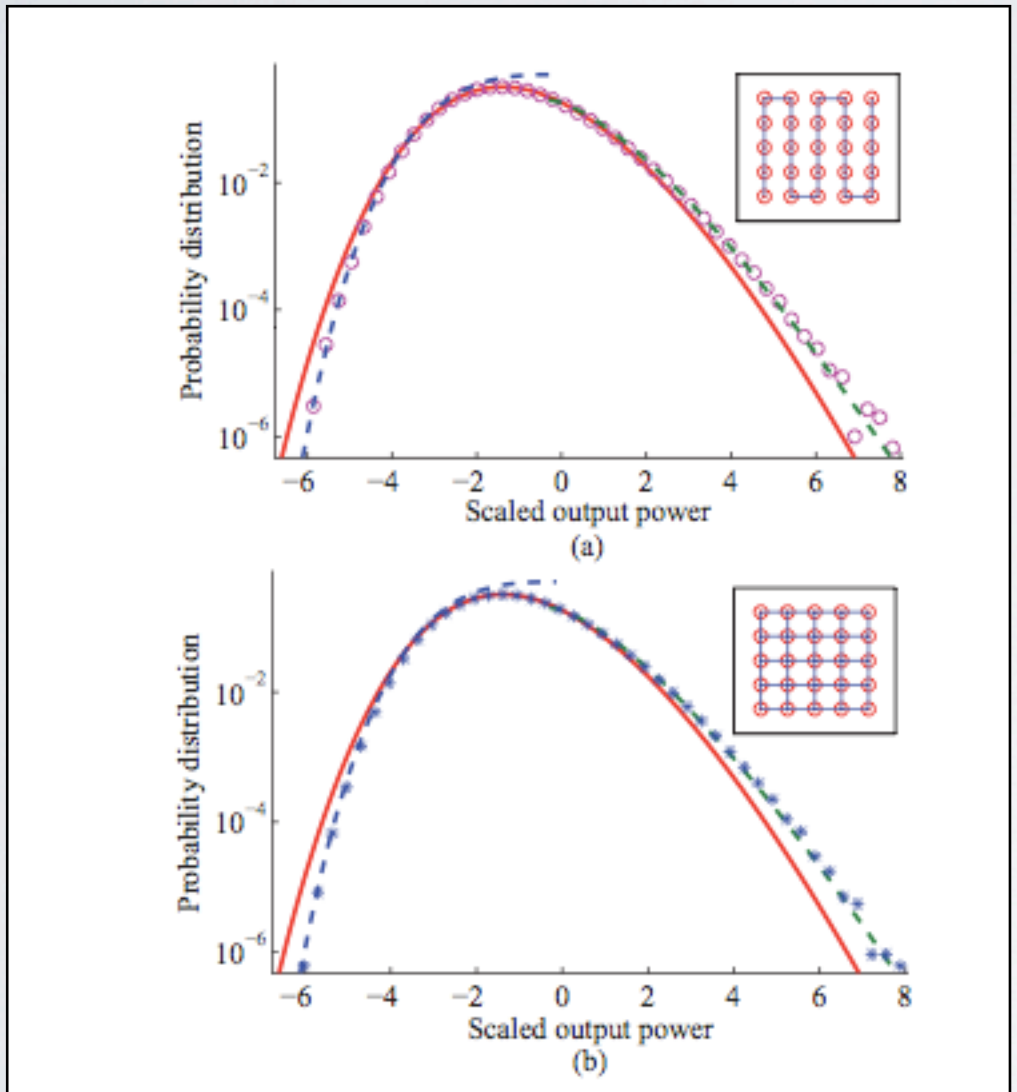

Recently, Majumdar and Vergassola (MV) calculated the probability of large deviations of the maximal eigenvalue [12–14] above the mean and Pierpaolo, Majumdar, and Bohigas (PMB) calculated below the mean. The MV and the PMB distributions were numerically confirmed, but so far eluded experimental demonstration.

BEYOND GAUSS...

- i.i.d. entries
  - All moments are finite [Soshnikov (2004)]: TW
  - Power-law decay $\sim |M_{ij}|^{-1-\mu}$

- rotationally invariant
  - Classical Wishart and Jacobi : TW
  - Critical Ensembles [Claeys, Its and Krasovski (2009)] : gen. TW
  - Disordered Ensembles [Bohigas et al. (2009)] : transitions
  - Lévy-Smirnov ensembles [Wieczorek (2002)]

THE CAUCHY ENSEMBLE

$$P(\mathbf{H}) \propto \left[\det(\mathbf{1}_N + \mathbf{H}^2)\right]^{-\beta N/2}$$

$$P(\lambda_1, \ldots, \lambda_N) \propto \prod_{i=1}^{N} \frac{1}{(1+\lambda_i^2)^{\beta N/2}} \prod_{j<k} |\lambda_j - \lambda_k|^\beta .$$

3 interesting properties...

- If $\mathbf{H}$ is distributed according to (1), then I) $\mathbf{H}^{-1}$ is also distributed according to (1), and II) every $n \times n$ submatrix of $\mathbf{H}$ obtained by erasing $N - n$ rows and columns is distributed according to (1) with N replaced by n. These properties have been crucial in establishing the Poisson kernel law in the context of mesoscopic transport in non-ideal quantum dots [4].

- The orthogonal polynomials with respect to the Cauchy weight are Jacobi polynomials analytically continued to complex arguments. In contrast to the classical cases, only a *finite* number of orthogonal polynomials do exist for this ensemble.

- The average density of eigenvalues is given by:

$$\rho(x) = \frac{1}{\pi} \frac{1}{1 + x^2}, \qquad x \in (-\infty, \infty) \tag{3}$$

exactly *for any* N (and not just asymptotically for large N).

Free Random Lévy Matrices [Burda *et al.* (2000)]

cond-mat/9501025

Generalized circular ensemble of scattering matrices for a chaotic cavity with non-ideal leads

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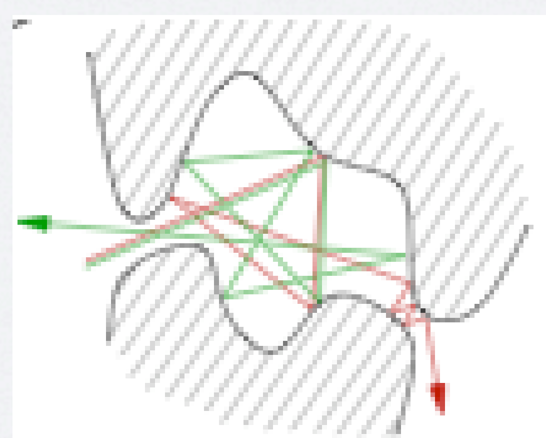
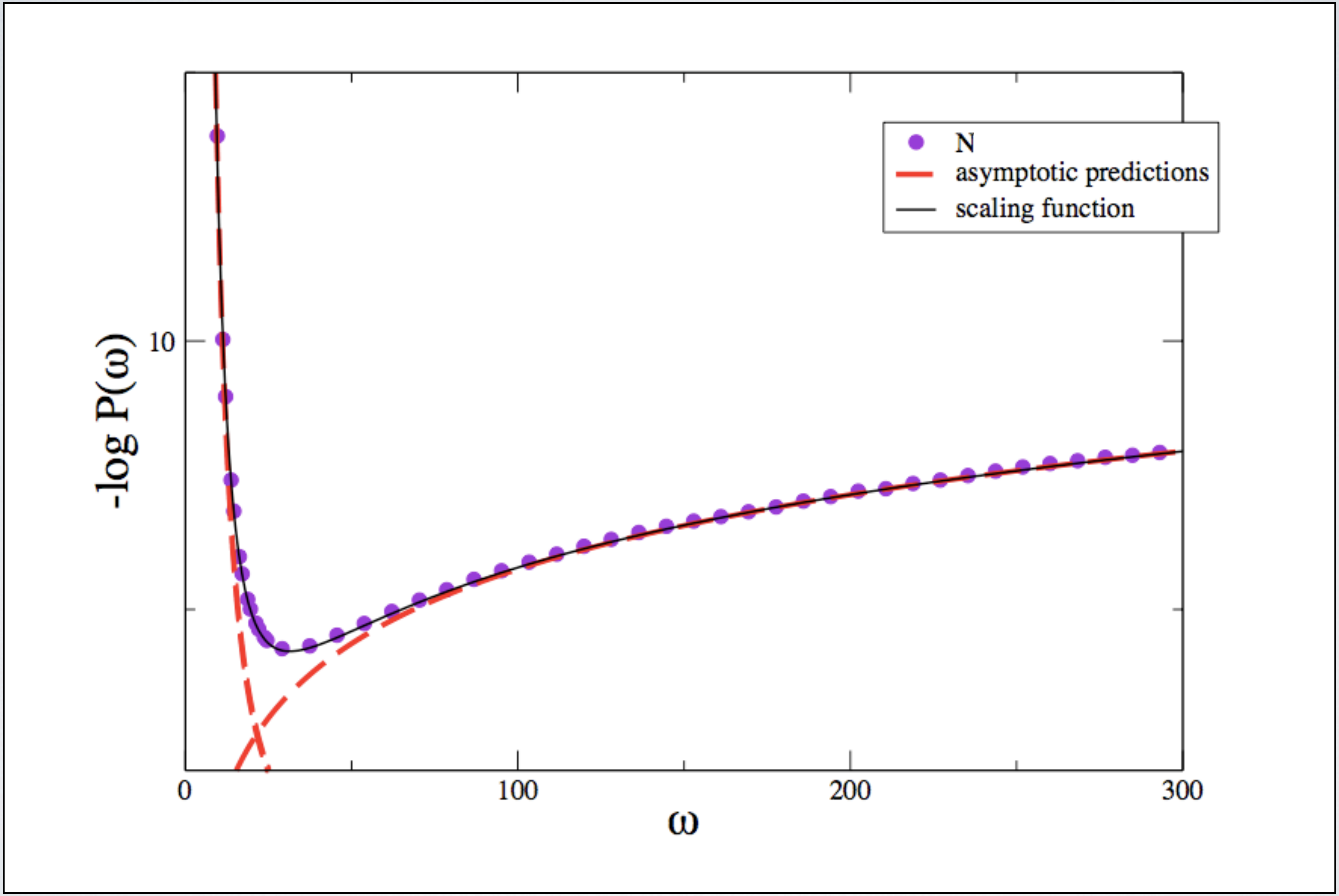
Abstract

We consider the problem of the statistics of the scattering matrix S of a chaotic cavity (quantum dot), which is coupled to the outside world by non-ideal leads containing N scattering channels. The Hamiltonian H of the quantum dot is assumed to be an $M \times M$ hermitian matrix with probability distribution $P(H) \propto \det[\lambda^2 + (H - \varepsilon)^2]^{-(\beta M + 2 - \beta)/2}$, where λ and ε are arbitrary coefficients and $\beta = 1, 2, 4$ depending on the presence or




N
asymptotic predictions
scaling function
10
-log P(ω)
0
100
200
300
ω

$$P(w,N) \approx \begin{cases} \exp[-\beta N^2 \psi(w)] & w \ll N \\ N^{-1} f(w/N) & w \sim N \\ N\phi(w) & w \gg N \end{cases}$$

How to determine the typical scale with N?

$$\int_{\lambda_{\max}}^{\infty} \rho(x)dx \approx 1/N$$

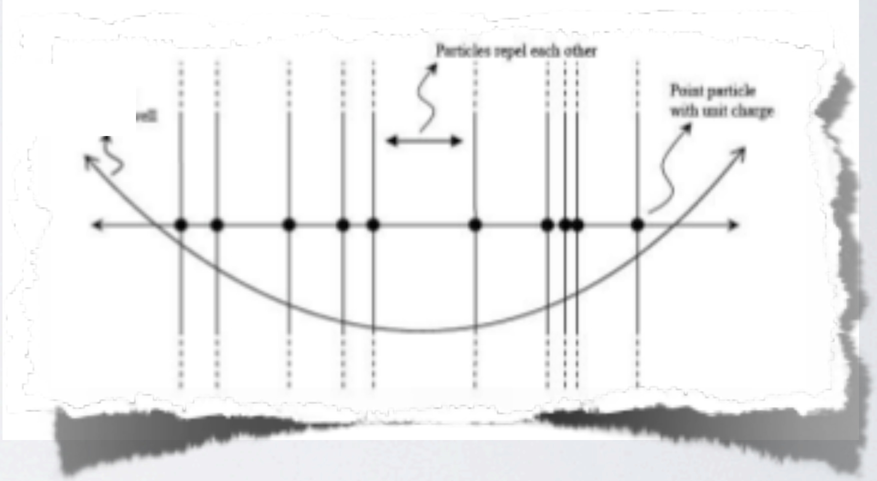
$$\arctan(\lambda_{\max}) \approx \pi/2 - \pi/N$$

$$\lambda_{\max} \sim \mathcal{O}(N)$$

$$\mathcal{P}[\lambda_{\max} < w] = \frac{\int_{(-\infty,w)^N} \prod_{i=1}^N d\lambda_i \; P(\lambda_1,\ldots,\lambda_N)}{\int_{(-\infty,\infty)^N} \prod_{i=1}^N d\lambda_i \; P(\lambda_1,\ldots,\lambda_N)}$$

$$P(\lambda_1,\ldots,\lambda_N) \;\propto\; \exp(-(\beta/2)E[\{\boldsymbol{\lambda}\}])$$

$$E[\{\boldsymbol{\lambda}\}] = N\sum_{i=1}^N \ln(1+\lambda_i^2) - \sum_{i\neq j} \ln|\lambda_i - \lambda_j|$$



$$\varrho_w(x) = (1/N)\sum_i \delta(x-\lambda_i)$$

$$\sum_{i=1}^N f(\lambda_i) = N\int d\lambda \varrho_w(\lambda) f(\lambda)$$

$$\mathcal{P}[\lambda_{\max} < w] \propto \int \mathcal{D}[\varrho_w] \exp\left(-\frac{\beta N^2}{2} \mathcal{S}_w[\varrho_w] \right)$$

$$\mathcal{S}_w[\varrho_w] = \int_{-\infty}^{w} \ln(1+x^2)\varrho_w(x)dx - \int_{-\infty}^{w}\int_{-\infty}^{w} dx\ dx'\, \varrho_w(x)\varrho_w(x') \ln|x-x'| + C\left(\int_{-\infty}^{w} dx \varrho_w(x) - 1 \right)$$

$$\ln(1+x^2) + C = 2\int_{-\infty}^{w} dx'\, \varrho_w^{\star}(x') \ln|x-x'|$$

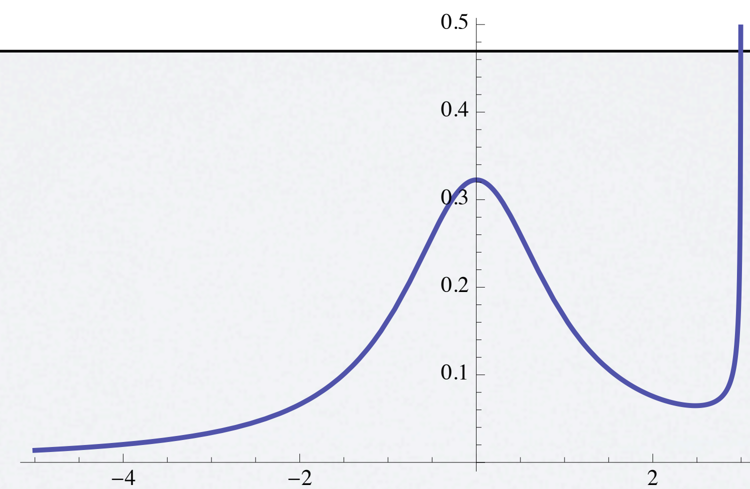
$$\Pr \int_0^{\infty} d\tau \frac{\hat{\varrho}_w(\tau)}{\tau - z} = \frac{w-z}{1+(w-z)^2}$$

Half-Hilbert transform

$$\hat{\varrho}_w(\tau) = -\frac{1}{\pi^2\sqrt{\tau}}\left\{\underbrace{\Pr\int_0^\infty ds \frac{1}{s-\tau}\frac{\sqrt{s}(w-s)}{1+(w-s)^2}}_{I_w(\tau)} + B\right\}$$

[Paveri-Fontana and Zweifel, 1994]

$$\hat{\varrho}_w(\tau) = \frac{1}{\pi\sqrt{2\tau}}\begin{cases} \frac{(k_w+w)^{1/2}-(w-\tau)(k_w-w)^{1/2}}{1+(w-\tau)^2} & w \geq 0 \\ \frac{((\tau w-k_w^2)(w-\tau)+k_w(w-\tau)^2+1)(k_w+w)^{1/2}+\tau(w-\tau)(k_w-w)^{1/2}}{k_w(w-\tau)^2+1} & w \leq 0 \end{cases}$$

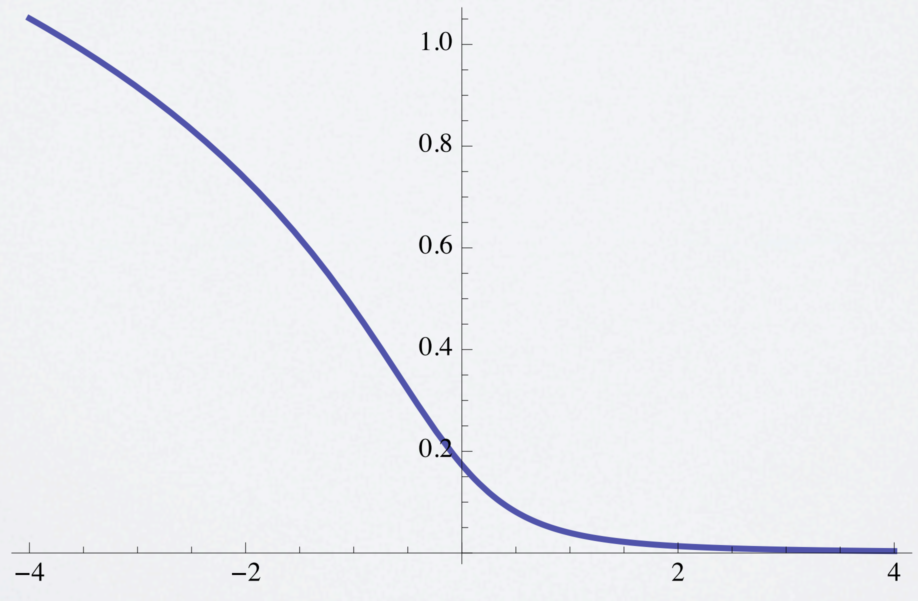


$$S_w[\hat{\varrho}_w] = \frac{1}{2}\left(\frac{1}{2}\log\left(w^2+1\right) - \sinh^{-1}(w)\right) + \frac{3\log(2)}{2}\,.$$

$$S_w[\hat{\varrho}_w] = \log 2 + \frac{1}{8w^2} + \mathcal{O}(w^{-5/2}).$$

$$\mathcal{P}[\lambda_{\max} < w] \propto \int \mathcal{D}[\varrho_w] \exp\left(-\frac{\beta N^2}{2} \mathcal{S}_w[\varrho_w]\right)$$

$$\begin{aligned} \mathcal{P}[\lambda_{\max} < w] &\approx \exp\left(-\beta N^2 \psi(w)\right) \\ \psi(w) &= \frac{1}{2}[\mathcal{S}_w[\hat{\varrho}_w] - \mathcal{S}_\infty[\hat{\varrho}_w]] \\ &= \frac{1}{4}\left(\frac{1}{2}\log\left(w^2+1\right) - \sinh^{-1}(w)\right) + \frac{\log(2)}{4} \,. \end{aligned}$$



Witte and Forrester studied the cumulative distribution $F(s) = \text{Prob}[\lambda_{\max} < s]$ of the largest eigenvalue for the case $\beta = 2$. They found that

$$F(s) = \exp\left[-\int_s^\infty ds' \frac{\sigma(s')}{1+s'^2}\right] \tag{50}$$

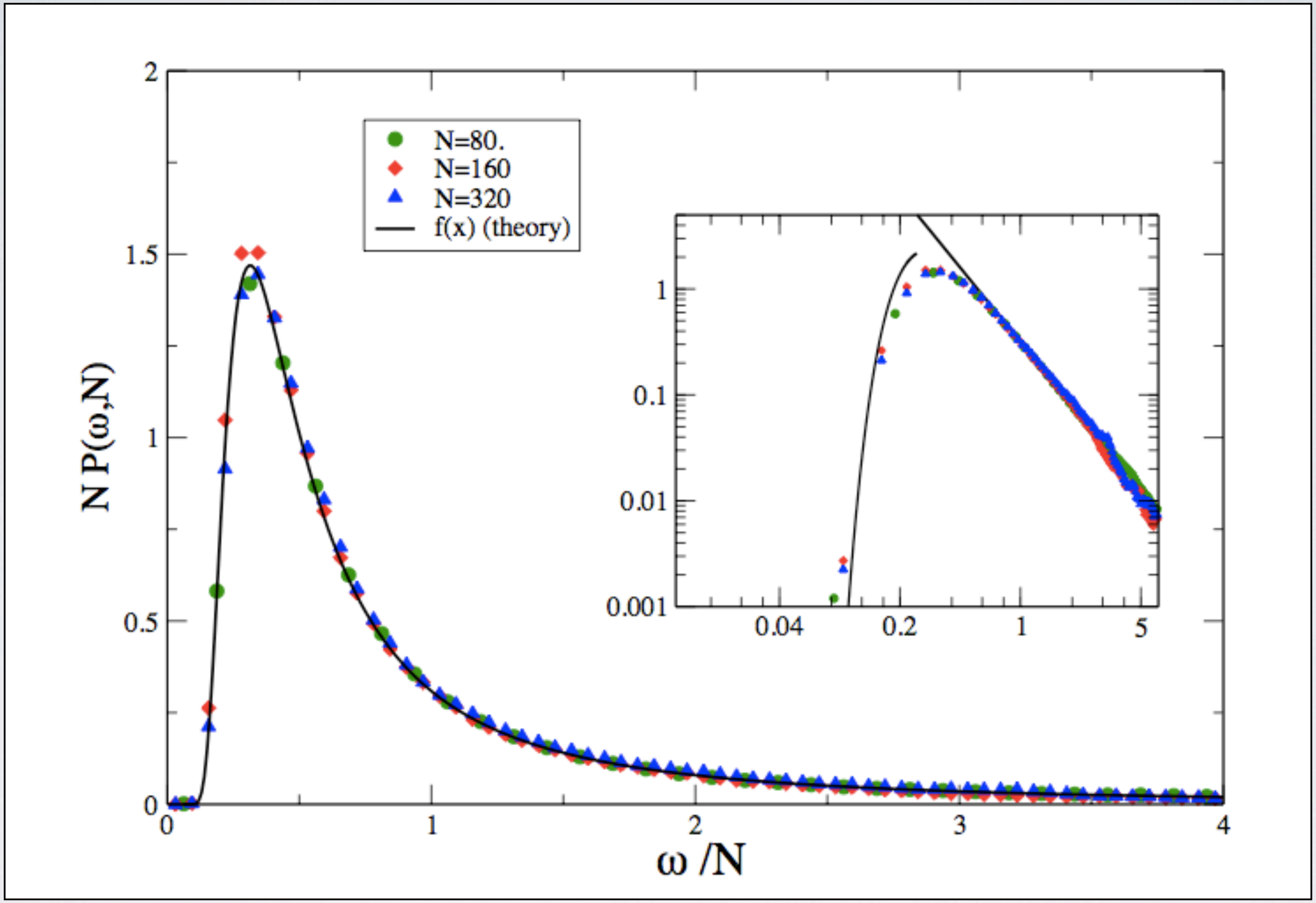
where $\sigma(s)$ satisfies

$$(1+s^2)^2(\sigma'')^2 + 4(1+s^2)(\sigma')^3 - 8s\sigma(\sigma')^2 + 4\sigma^2\sigma' + 4N^2(\sigma')^2 = 0\,, \tag{51}$$

$$\sigma(s) = N\tau(s/N)$$

$$f(x) = \frac{\tau(x)}{x^2}\exp\left[-\int_x^\infty \frac{\tau(y)}{y^2}dy\right]$$

$$x^4(\tau'')^2 + 4x^2(\tau')^3 - 8x\tau(\tau')^2 + 4\tau^2\tau' + 4(\tau')^2 = 0\,.$$



$$f(x) \approx \begin{cases} 1/(4x^3)\exp[-1/(8x^2)] & \text{as } x \to 0 \\ 1/[\pi x^2] & \text{as } x \to \infty \end{cases}$$

Consider N eigenvalues and for each of them, define a binary variable $\sigma_i = 1$ if the ith eigenvalue λ_i falls in the region $w \leq \lambda_i < \infty$ and $\sigma_i = 0$ if $\lambda_i < w$. Then, the probability that the region $[w, \infty]$ is free of eigenvalues, which is also the cumulative distribution of $\lambda_{\max}$, namely

$$\int_{-\infty}^{w} \mathcal{P}(w', N) dw' \tag{A1}$$

can be written as

$$\int_{-\infty}^{w} \mathcal{P}(w', N) dw' = \langle [1 - \sigma_1][1 - \sigma_2] \cdots [1 - \sigma_N] \rangle \tag{A2}$$

where the average $\langle \cdot \rangle$ is over the joint distribution of the eigenvalues. Expanding the product, one gets

$$\int_{-\infty}^{w} \mathcal{P}(w', N) dw' = 1 - N \int_{w}^{\infty} \rho(x, N) dx + \text{two-point} + \text{three-point} + \ldots \tag{A3}$$

When $w \to \infty$ (extreme right tail), all the higher order contributions vanish and one obtains in this limit

$$\int_{-\infty}^{w} \mathcal{P}(w', N) dw' \approx 1 - N \int_{w}^{\infty} \rho(x, N) dx \tag{A4}$$

Taking derivative w.r.t w gives,

$$\mathcal{P}(w, N) \approx N \rho(w, N) \tag{A5}$$

as claimed.

$$P(w,N) \approx \begin{cases} \exp[-\beta N^2 \psi(w)] & w \ll N \\ N^{-1} f(w/N) & w \sim N \\ N\phi(w) & w \gg N \end{cases}$$

$$\begin{aligned} \psi(w) &= \frac{1}{2}[\mathcal{S}_w[\hat{\varrho}_w] - \mathcal{S}_\infty[\hat{\varrho}_w]] \\ &= \frac{1}{4}\left(\frac{1}{2}\log\left(w^2+1\right) - \sinh^{-1}(w)\right) + \frac{\log(2)}{4} . \end{aligned}$$

$$f(x) = \frac{\tau(x)}{x^2} \exp\left[-\int_x^\infty \frac{\tau(y)}{y^2} dy\right]$$

$$x^4(\tau'')^2 + 4x^2(\tau')^3 - 8x\tau(\tau')^2 + 4\tau^2\tau' + 4(\tau')^2 = 0 .$$

$$\phi(w) = \frac{1}{\pi w^2}.$$

Conclusions

- Cauchy ensemble: density of eigenvalues falling off as a power law
- Few results available for largest eigenvalue of rotationally invariant ensembles
- 3 regimes: central (scaling) + 2 large deviation tails
- Central regime: scaling analysis of a result by Witte and Forrester
- Left large deviation tail: Coulomb gas approach
- Right large deviation tail: simple 'tail-of-the-density' argument

Thank you.

OUTLINE

First Part: Old Tricks

i) The old days... RMT in nuclear physics
ii) 4 applications

- Riemann hypothesis
- Vicious brownian walkers
- Covariance matrices of financial data
- The longest increasing subsequence problem

Second Part: New Dogs

i) Rare events and linear statistics
ii) How many eigenvalues of a random matrix are positive?

Why are random matrix eigenvalues cool?

Message

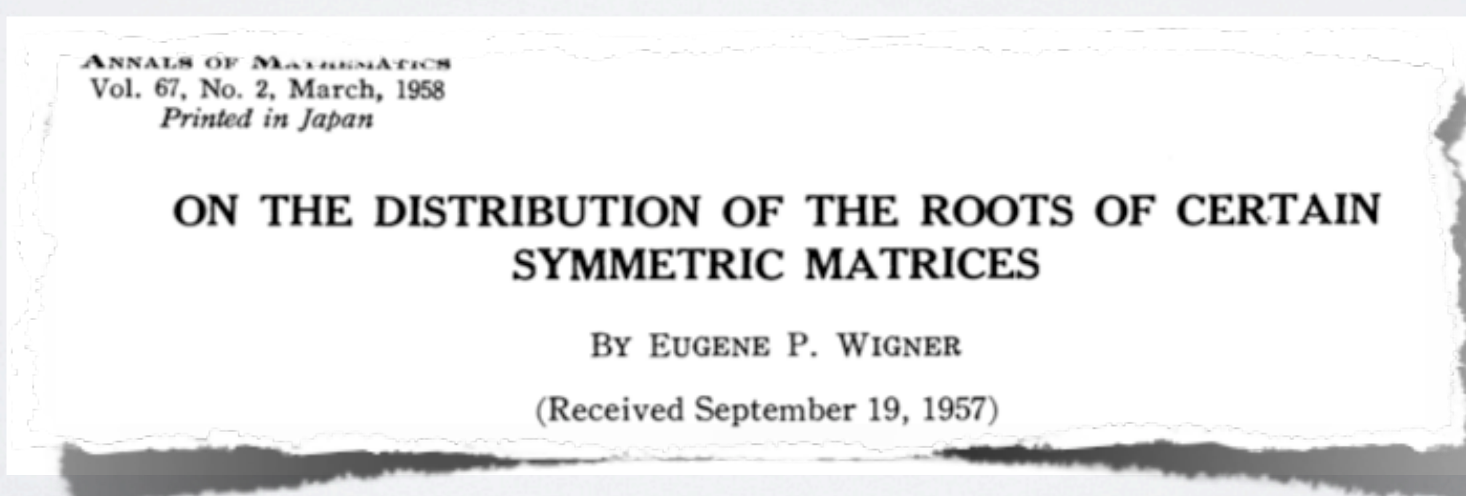
- Ingredient: Take Any important mathematics
- Then Randomize!
- This will have many applications!

from a talk by Alan Edelman (MIT)

John Wishart

Eugene Wigner

Freeman Dyson

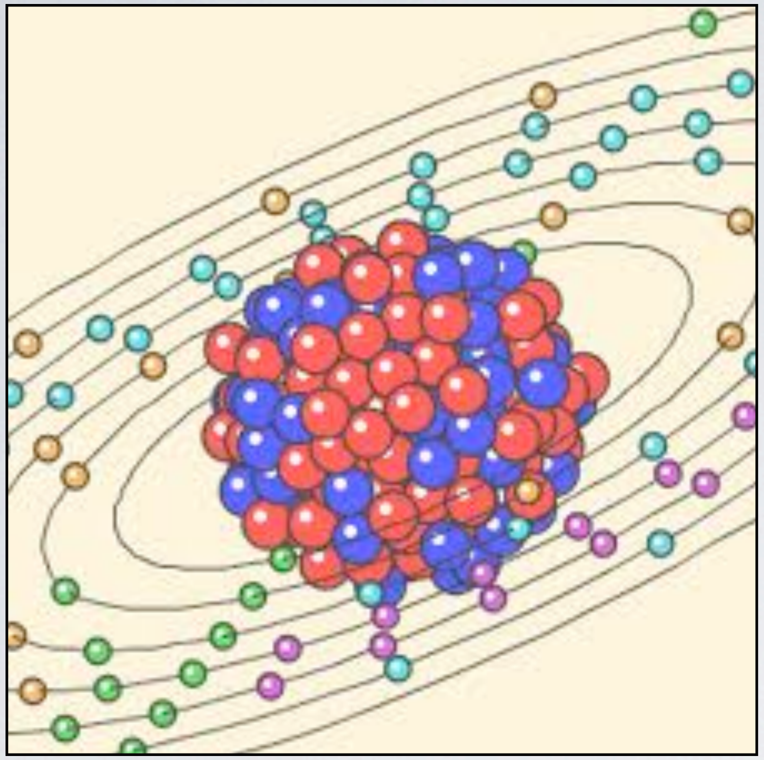


ANNALS OF MATHEMATICS
Vol. 67, No. 2, March, 1958
Printed in Japan

ON THE DISTRIBUTION OF THE ROOTS OF CERTAIN SYMMETRIC MATRICES

BY EUGENE P. WIGNER

(Received September 19, 1957)

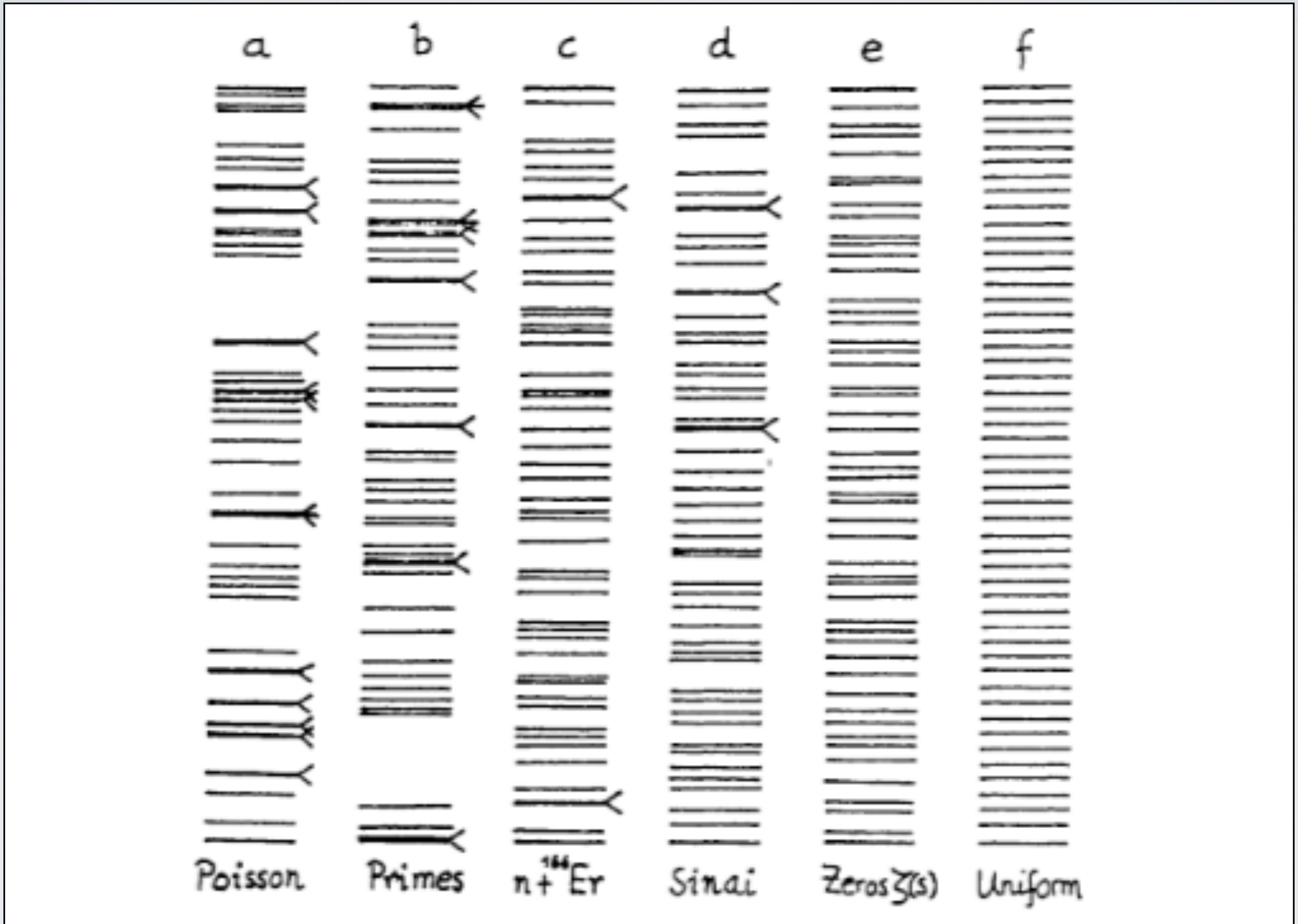


Hamiltonian (total energy) of heavy nuclei: hopeless task!

BUT.....

The Hamiltonian in a given basis is just a **HUGE** matrix....

Idea: take the matrix entries **at random...**



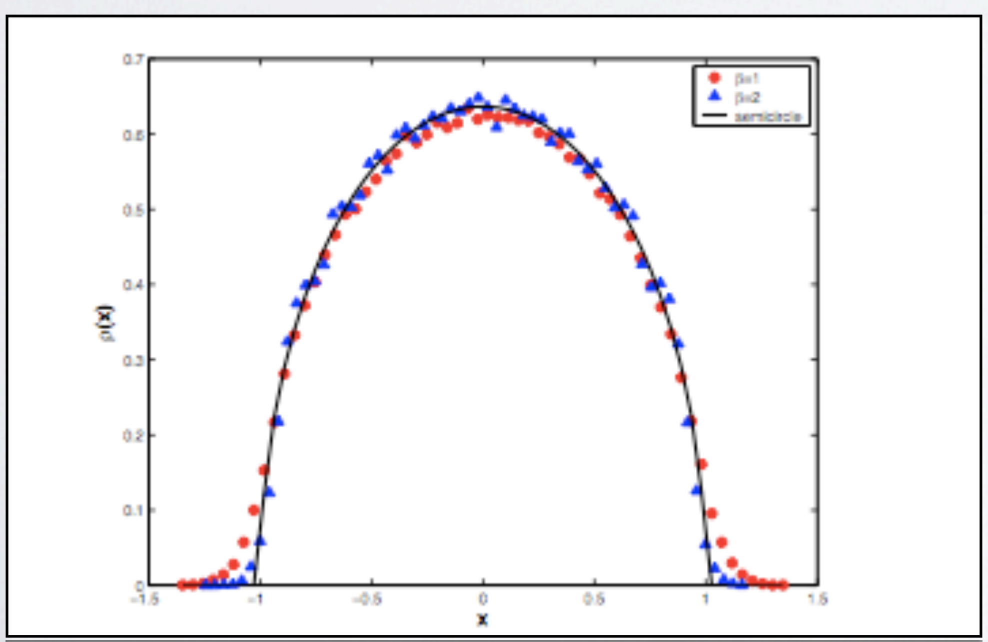
a
b
c
d
e
f
Poisson
Primes
n+^{166}Er
Sinai
Zeros ζ(s)
Uniform

Random Matrix Theory = Randomness + Symmetry

N = 5

$$\begin{pmatrix} 0.5377 & 0.2631 & -1.8044 & 0.3286 & 0.4951 \\ 0.2631 & -0.4336 & 1.6888 & 1.7271 & 0.7810 \\ -1.8044 & 1.6888 & 0.7254 & 0.7133 & 0.7160 \\ 0.3286 & 1.7271 & 0.7133 & 1.4090 & 1.5237 \\ 0.4951 & 0.7810 & 0.7160 & 1.5237 & 0.4889 \end{pmatrix}$$

$$\vec{\lambda} = \begin{bmatrix} -2.4341 & -0.8386 & -0.5203 & 2.2594 & 4.2610 \end{bmatrix}$$



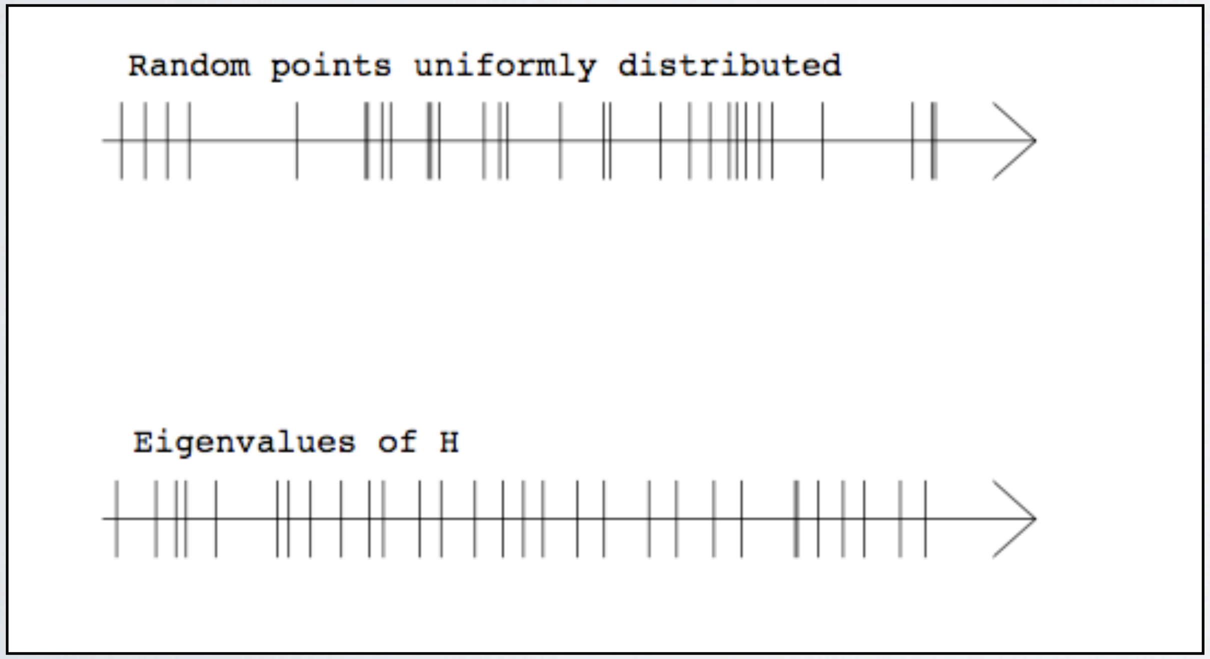
Semicircle Law

$$\rho(\lambda) = \frac{1}{2\sqrt{N}} f\left(\frac{\lambda}{2\sqrt{N}}\right)$$

$$f(x) = \frac{2}{\pi}\sqrt{1-x^2}$$

Typical questions:

(i) Density of eigenvalues
(ii) Gaps between adjacent eigenvalues
(iii) Distribution of individual eigenvalues (e.g. largest)
(iv) Probability of **rare** events in **linear statistics**



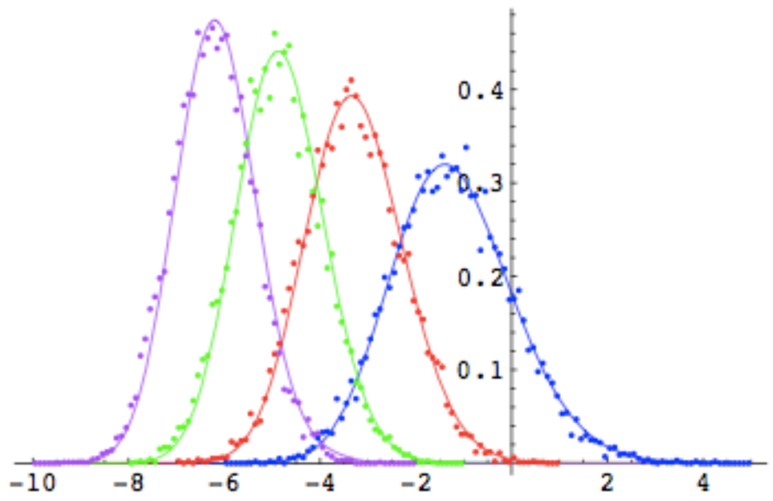
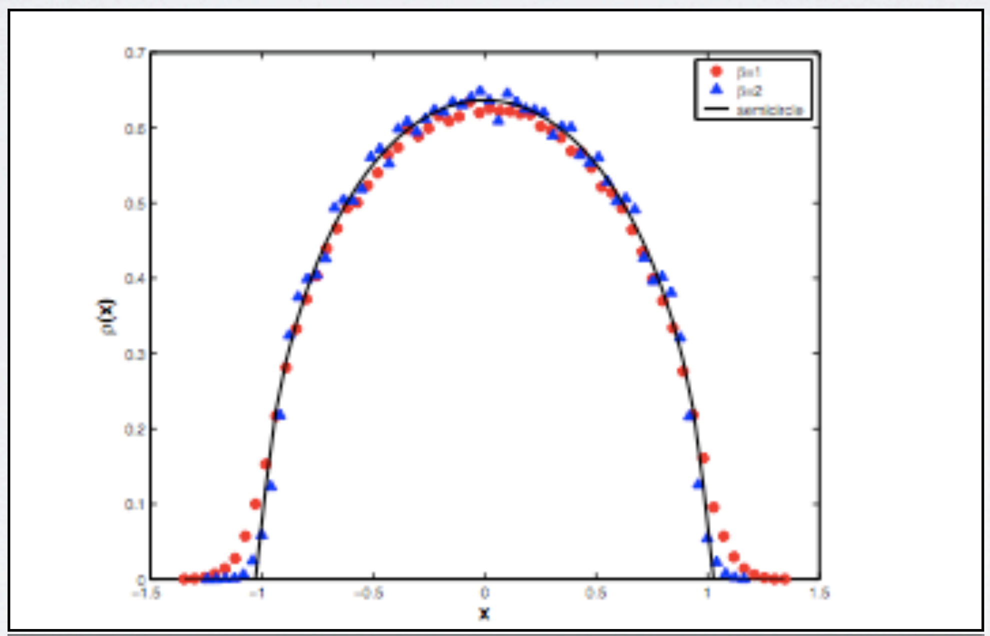


Figure 2: A histogram of the four largest (centered and normalized) eigenvalues for 10^4 realizations of $10^3 \times 10^3$ GOE matrices. Solid curves are the limiting distributions from [11]. Figure a courtesy of Momar Dieng.

$$\mathbb{P}_{\beta,N}(x_1,\ldots,x_N) = C_{N,\beta} \prod_{1\le i<j\le N} |x_i - x_j|^\beta \prod_{i=1}^{N} e^{-\beta x_i^2/2},\ \beta = 1,2,4,$$

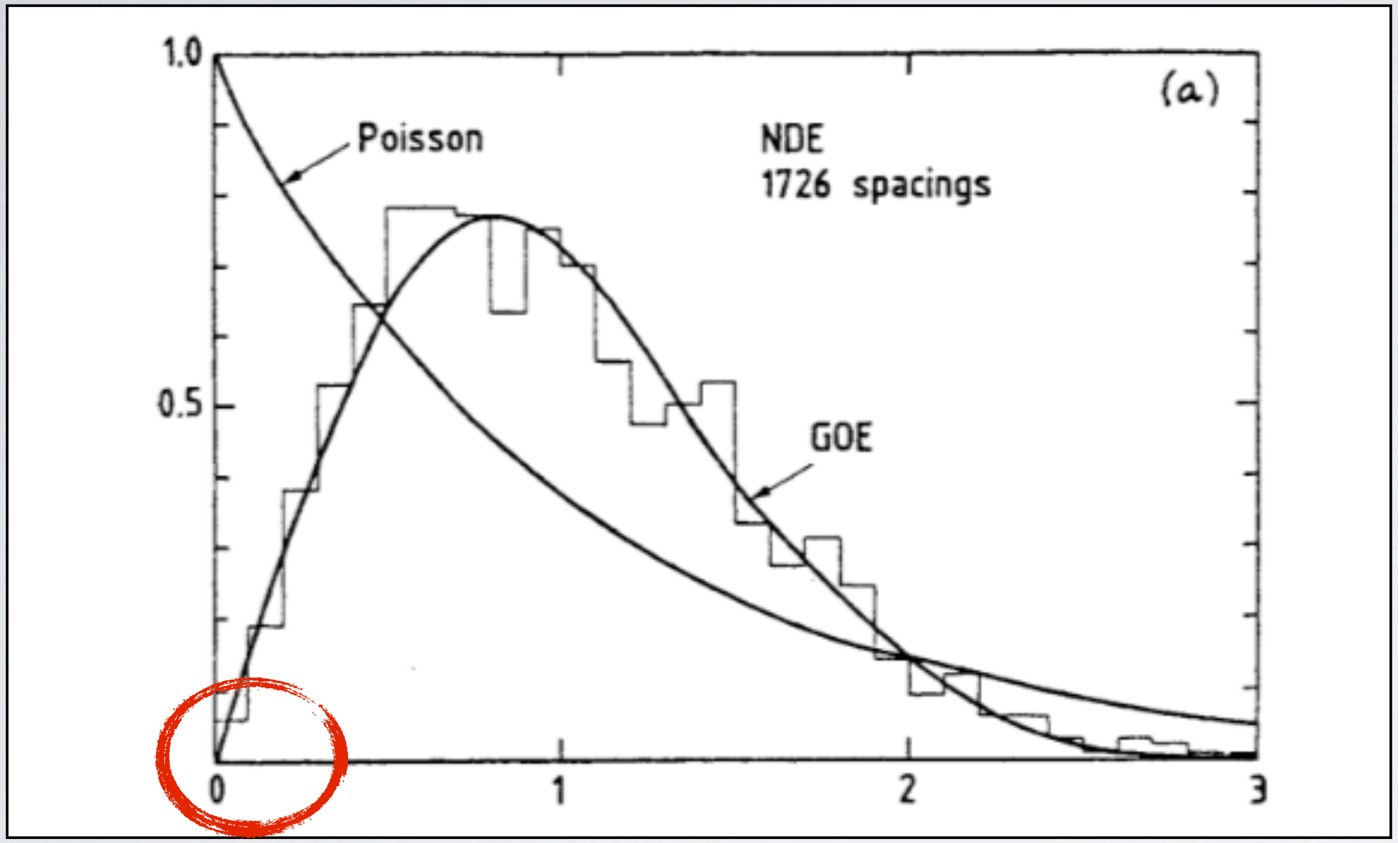
Level Repulsion

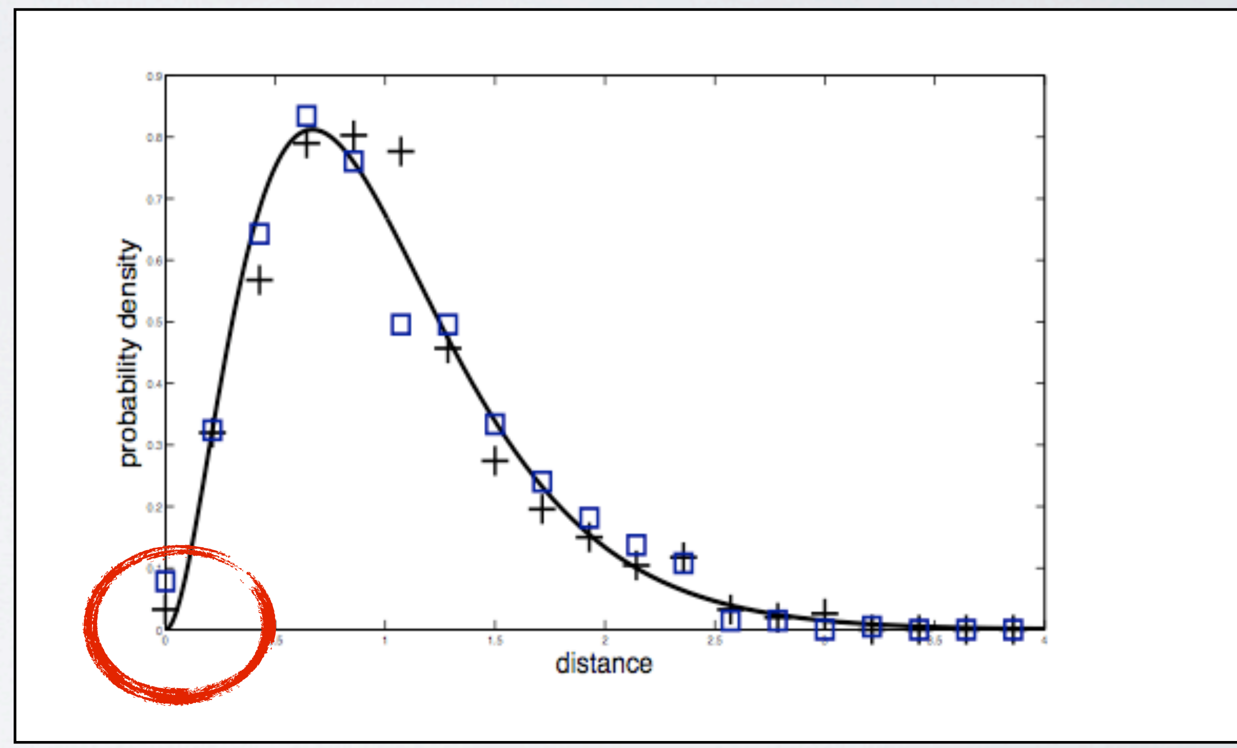
Confinement



Strongly Correlated Random Variables!!

Level Spacings: universality





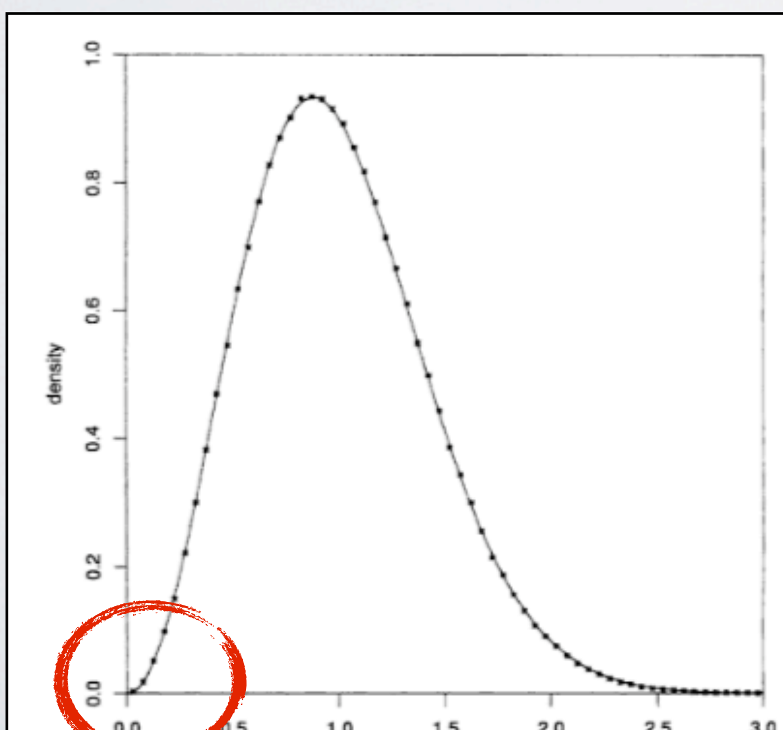


Figure 6. *A comparison of the statistics of spacings between conecutive eigenvalues of random matrices (solid curve) and nearest-neighbor spacings of zeroes of the Riemann zeta function (data points). The fit is not bad for the first million zeroes (left), but nearly perfect for the spacings among 1,041,600 zeroes near the 2×10^{20}th zero. (Figure*

$$\mathcal{P}(s) \propto s^{\beta} e^{-s^2}$$

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Warsaw, Poland, May 25–27, 2007

Parking in the City

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ELSEVIER

Physica A: Statistical Mechanics and its Applications

Volume 346, Issues 3–4, 15 February 2005, Pages 621–630

Modelling the gap size distribution of parked cars

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Received 19 July 2004; Available online ...

Modelling gap-size distribution of parked cars using random-matrix theory

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We apply the random-matrix theory to the car-parking problem. For this purpose, we adopt a Coulomb gas model that associates the coordinates of the gas particles with the eigenvalues of a random matrix. The nature of interaction between the particles is consistent with the tendency of the drivers to park their cars near to each other and in the same time keep a distance sufficient for manoeuvring. We show that the recently measured gap-size distribution of parked cars in a number of roads in central London is well represented by the spacing distribution of a Gaussian unitary ensemble.

PACS: 05.40; 05.20.Gg; 02.50.r; 68.43.-h
Keywords: Car parking; Coulomb gas; Gaussian unitary ensemble

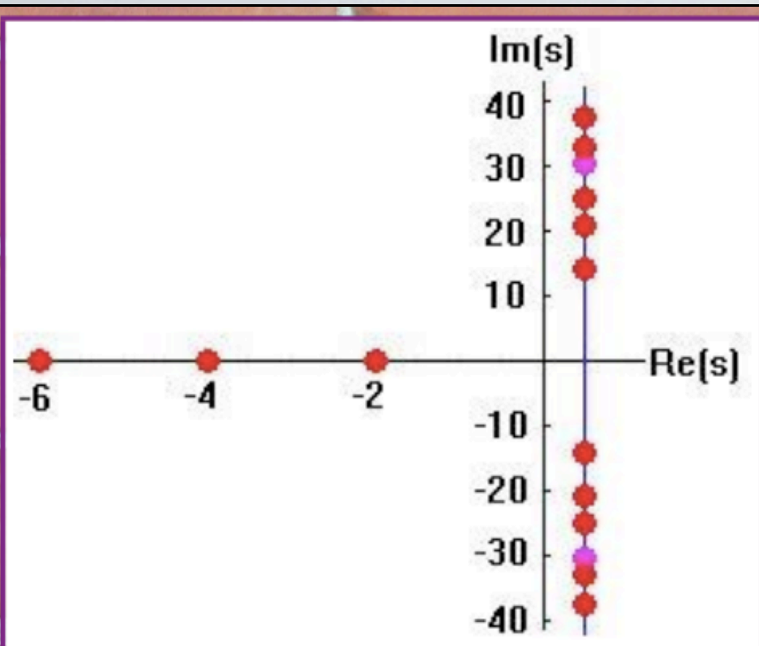
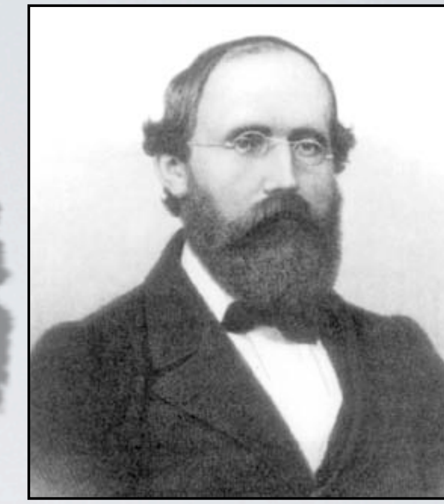


Figure 1: The zeros of the ζ-function.

$$\sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p \text{ prime}} \frac{1}{1-p^{-s}}$$

liegenden Wurzeln von $\xi(t) = 0$ multiplicirt mit $2\pi i$. Man findet nun in der That etwa so viel reelle Wurzeln innerhalb dieser Grenzen, und es ist sehr wahrscheinlich, dass alle Wurzeln reell sind. Hiervon wäre allerdings ein strenger Beweis zu wünschen; ich habe indess die Aufsuchung desselben nach einigen flüchtigen vergeblichen Versuchen vorläufig bei Seite gelassen, da er für den nächsten Zweck meiner Untersuchung entbehrlich schien.



...it is very probable that all roots are real. One would, however, wish for a strict proof of this; I have, though, after some fleeting futile attempts, provisionally put aside the search for such, as it appears unnecessary for the next objective of my investigation.

"Sometimes I think that we essentially have a complete proof of the Riemann Hypothesis except for a gap. The problem is, the gap occurs right at the beginning, and so it's hard to fill that gap because you don't see what's on the other side of it."

Montgomery's Pair Correlation Conjecture

Montgomery's pair correlation conjecture, published in 1973, asserts that the two-point correlation function $R_2(r)$ for the zeros of the Riemann zeta function $\zeta(z)$ on the critical line is

$$R_2(r) = 1 - \frac{\sin^2(\pi r)}{(\pi r)^2}.$$

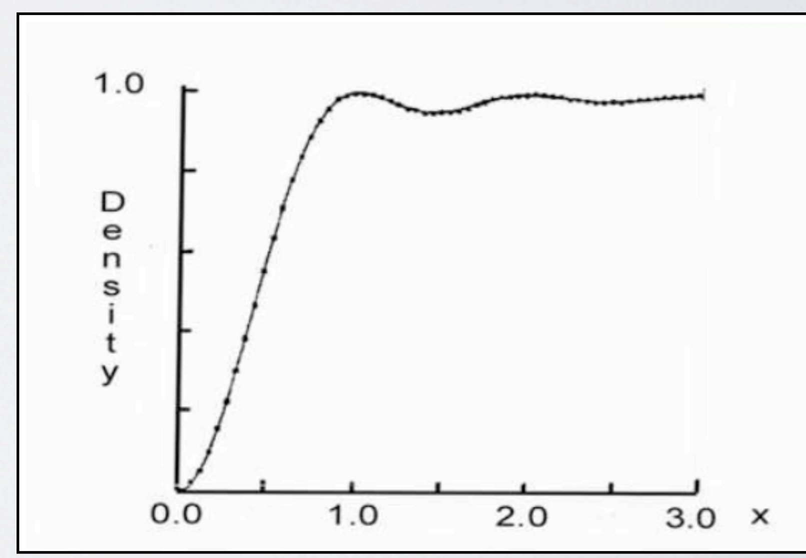
As first noted by Dyson, this is precisely the form expected for the pair correlation of random Hermitian matrices (Derbyshire 2004, pp. 287-291).

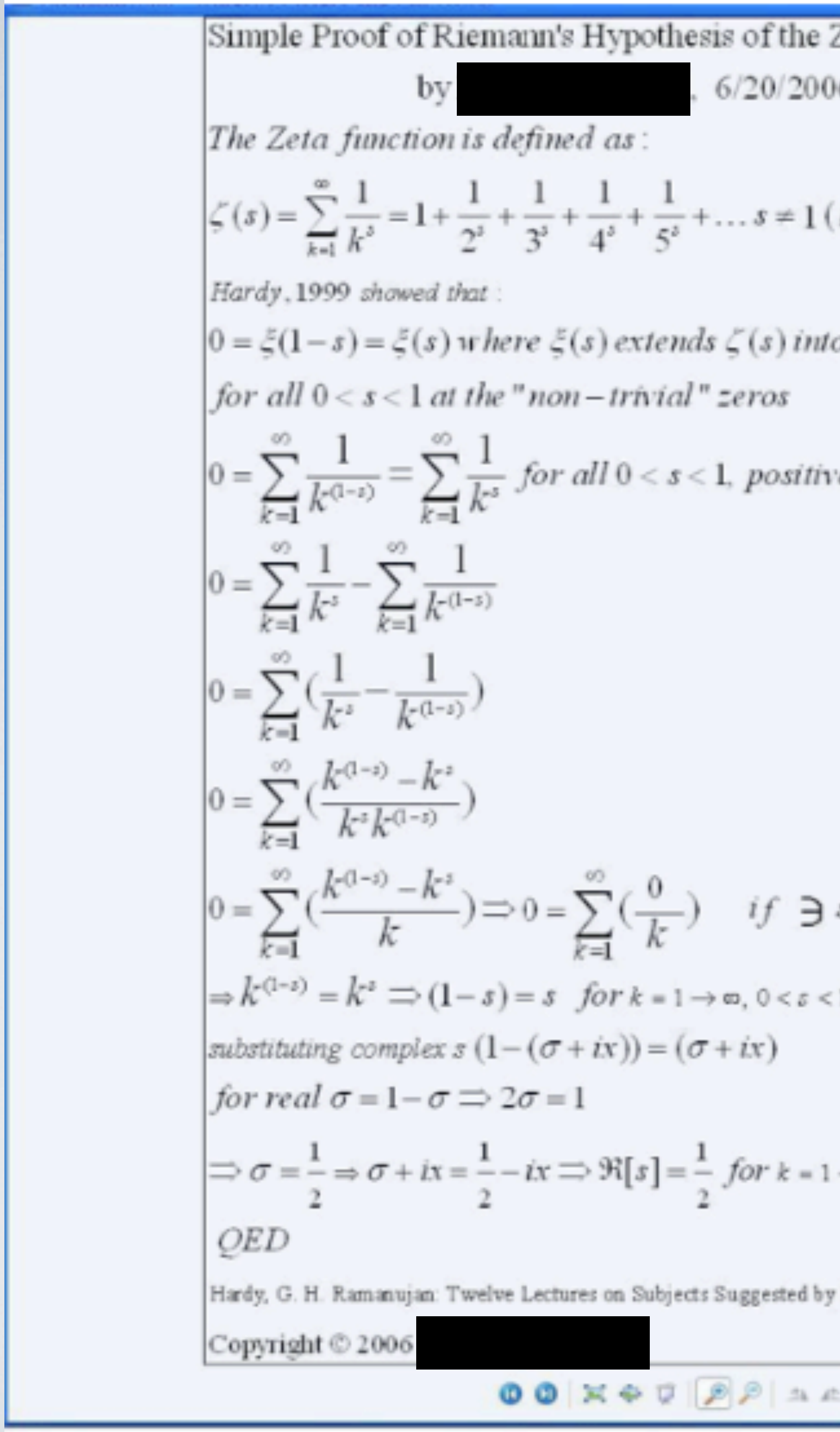
In 1972, Hugh Montgomery, a number theorist at the University of Michigan, was visiting the Institute for Advanced Study. Montgomery had been studying the distribution of zeroes of the zeta function, in hopes of gaining insight into the Riemann Hypothesis. He was able to prove that the Riemann Hypothesis had implications for the spacing of zeroes along the critical line, but his key discovery was an additional property that the zeroes seemed to have, one which implied a particularly nice formula for the average spacing between zeroes.

During tea one day at the Institute, Montgomery was introduced to Dyson and described his conjecture. Dyson immediately recognized it as the same result as had been obtained for random matrices.

"It just so happened that he was one of the two or three physicists in the world who had worked all of these things out, so I was actually talking to the greatest expert in exactly this!" Montgomery recalls.

Odlyzko's computations agree amazingly well with Montgomery's conjecture.





Simple Proof of Riemann's Hypothesis of the Zeta function

by [redacted], 6/20/2006

The Zeta function is defined as:

$$\zeta(s) = \sum_{k=1}^{\infty} \frac{1}{k^s} = 1 + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \frac{1}{5^s} + \ldots \quad s \neq 1 \; (k = 1 \to \infty)$$

Hardy, 1999 showed that:

$$0 = \xi(1-s) = \xi(s) \text{ where } \xi(s) \text{ extends } \zeta(s) \text{ into}$$

for all $0 < s < 1$ *at the "non-trivial" zeros*

$$0 = \sum_{k=1}^{\infty} \frac{1}{k^{(1-s)}} = \sum_{k=1}^{\infty} \frac{1}{k^s} \text{ for all } 0 < s < 1, \text{ positive}$$

$$0 = \sum_{k=1}^{\infty} \frac{1}{k^s} - \sum_{k=1}^{\infty} \frac{1}{k^{(1-s)}}$$

$$0 = \sum_{k=1}^{\infty} \left(\frac{1}{k^s} - \frac{1}{k^{(1-s)}}\right)$$

$$0 = \sum_{k=1}^{\infty} \left(\frac{k^{(1-s)} - k^s}{k^s k^{(1-s)}}\right)$$

$$0 = \sum_{k=1}^{\infty} \left(\frac{k^{(1-s)} - k^s}{k}\right) \Rightarrow 0 = \sum_{k=1}^{\infty} \left(\frac{0}{k}\right) \quad \text{if } \exists$$

$$\Rightarrow k^{(1-s)} = k^s \Rightarrow (1-s) = s \quad \text{for } k = 1 \to \infty, 0 < s < 1$$

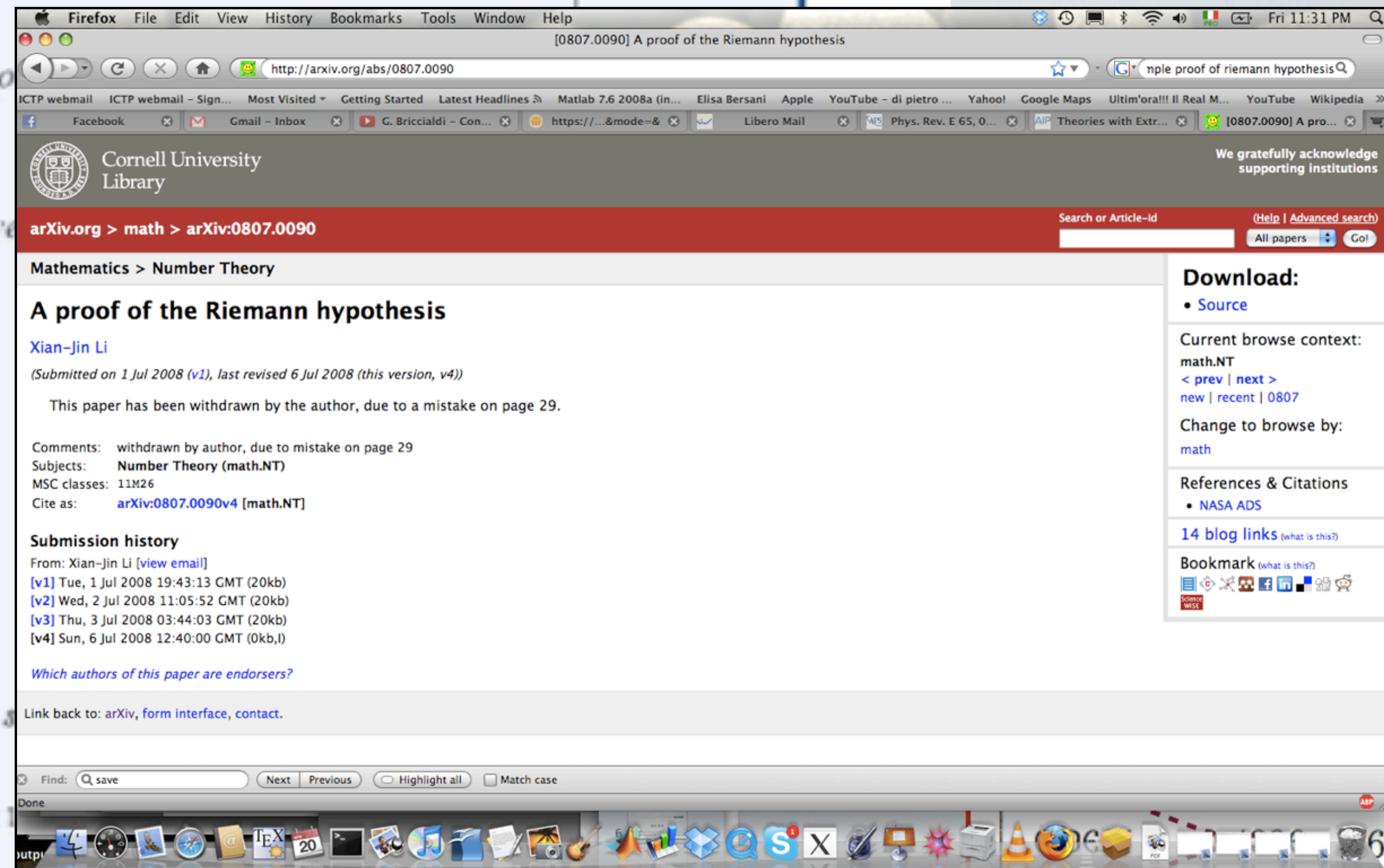
substituting complex s $(1 - (\sigma + ix)) = (\sigma + ix)$

for real $\sigma = 1 - \sigma \Rightarrow 2\sigma = 1$

$$\Rightarrow \sigma = \frac{1}{2} \Rightarrow \sigma + ix = \frac{1}{2} - ix \Rightarrow \Re[s] = \frac{1}{2} \quad \text{for } k = 1 \to \infty, 0 < s < 1, \text{'critical strip'}$$

QED

Hardy, G. H. Ramanujan: Twelve Lectures on Subjects Suggested by His Life and Work, 3rd ed. New York: Chelsea, 1999.



[0807.0090] A proof of the Riemann hypothesis

http://arxiv.org/abs/0807.0090

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arXiv.org > math > arXiv:0807.0090

Mathematics > Number Theory

A proof of the Riemann hypothesis

Xian-Jin Li

(Submitted on 1 Jul 2008 (v1), last revised 6 Jul 2008 (this version, v4))

This paper has been withdrawn by the author, due to a mistake on page 29.

Comments: withdrawn by author, due to mistake on page 29
Subjects: **Number Theory (math.NT)**
MSC classes: 11M26
Cite as: arXiv:0807.0090v4 [math.NT]

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From: Xian-Jin Li [view email]
[v1] Tue, 1 Jul 2008 19:43:13 GMT (20kb)
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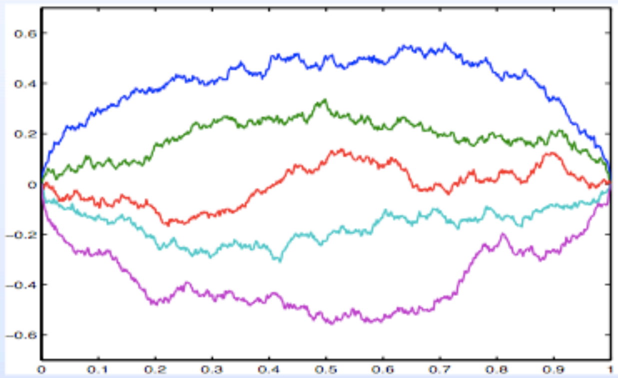
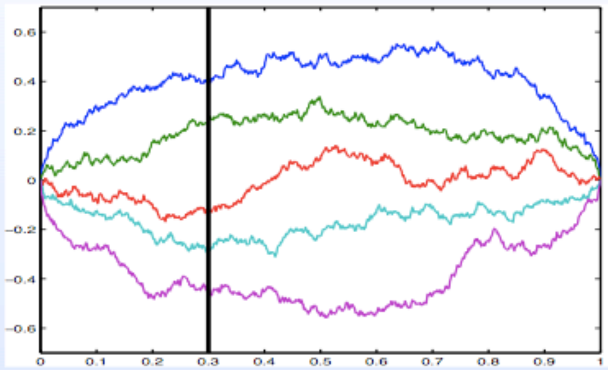
Non-intersecting Brownian motion paths

- Take n independent 1-dimensional Brownian motions with time in $[0,1]$ conditioned so that:
  - All paths start and end at the same point.
  - The paths do not intersect at any intermediate time.



Five non-intersecting Brownian bridges

- **Remarkable fact**: At any intermediate time the positions of the paths have exactly the same distribution as the eigenvalues of an $n \times n$ GUE matrix (up to a scaling factor).



Positions of five non-intersecting Brownian paths behave the same as the eigenvalues of a 5×5 GUE matrix

- This interpretation is basic for the connection of random matrix theory with growth models of statistical physics.

Introduction. Since the pioneering work of de Gennes [1], followed up by Fisher [2], the subject of vicious (non-intersecting) random walkers has attracted a lot of interest among physicists. It has been studied in the context of wetting and melting [2], networks of polymers [3] and fibrous structures [1], persistence properties in nonequilibrium systems [4] and stochastic growth models [5, 6]. There also exist connections between the

PHYSICAL REVIEW E VOLUME 52, NUMBER 6 DECEMBER 1995

Vicious walkers and directed polymer networks in general dimensions

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Random Covariance Matrices

phys. math

$$X = \begin{array}{c} 1 \\ 2 \\ 3 \end{array} \begin{vmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \\ x_{31} & x_{33} \end{vmatrix}$$

in general

(MxN)

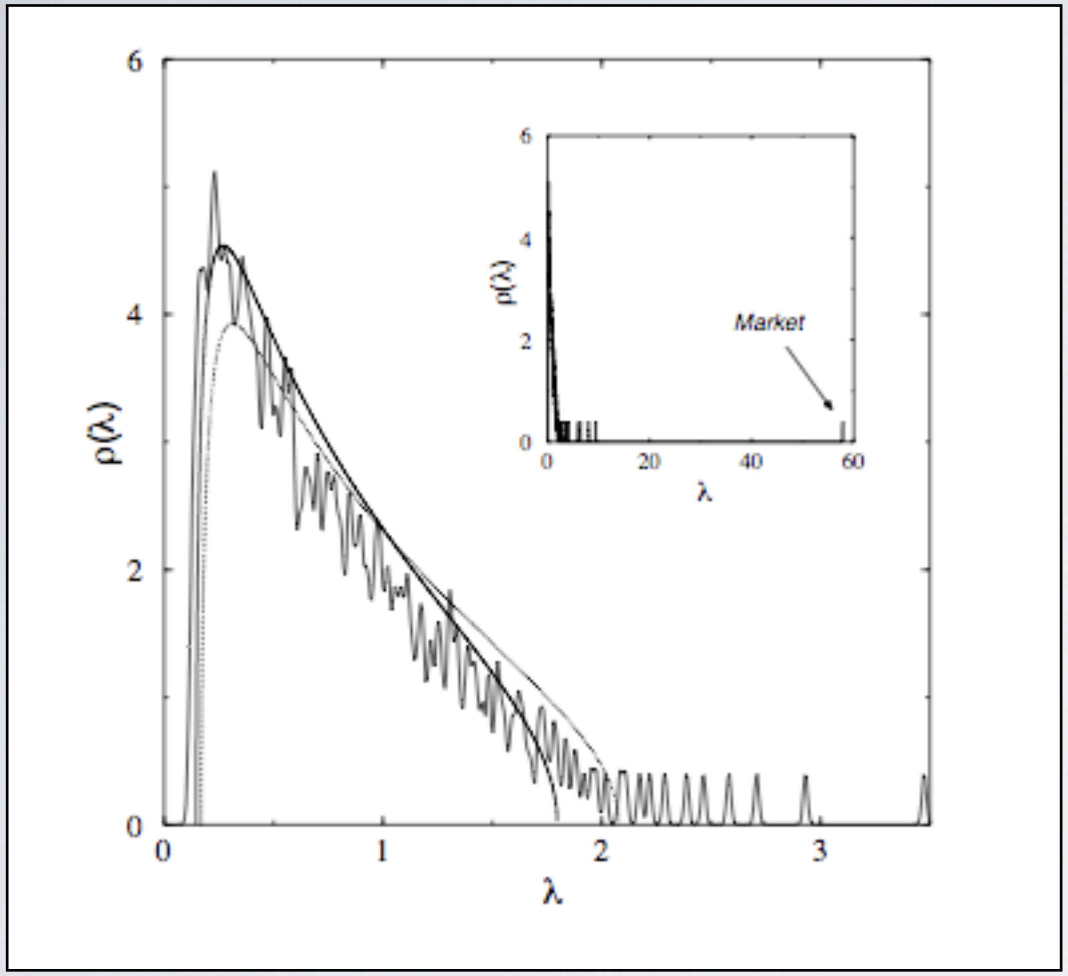
$$X^t = \begin{vmatrix} x_{11} & x_{21} & x_{31} \\ x_{12} & x_{22} & x_{33} \end{vmatrix}$$

in general

(NxM)

$$W = X^t X = \begin{vmatrix} x_{11}^2 + x_{21}^2 + x_{31}^2 & x_{11}x_{12} + x_{21}x_{22} + x_{31}x_{33} \\ x_{12}x_{11} + x_{22}x_{21} + x_{33}x_{31} & x_{12}^2 + x_{22}^2 + x_{33}^2 \end{vmatrix}$$

→ (NxN) COVARIANCE MATRIX (unnormalized)



VOLUME 83, NUMBER 7 PHYSICAL REVIEW LETTERS 16 AUGUST 1999

Noise Dressing of Financial Correlation Matrices

Laurent Laloux,[1,*] Pierre Cizeau,[1] Jean-Philippe Bouchaud,[1,2] and Marc Potters[1]
[1]*Science & Finance, 109-111 rue Victor Hugo, 92532 Levallois Cedex, France*
[2]*Service de Physique de l'État Condensé, Centre d'études de Saclay, Orme des Merisiers, 91191 Gif-sur-Yvette Cedex, France*

time. From this point of view, it is interesting to compare the properties of an empirical correlation matrix **C** to a null hypothesis purely *random* matrix as one could obtain from a finite time series of strictly independent assets. Deviations from the random matrix case might then suggest the presence of true information. The theory of random matri-

Debate: is the bulk of the stock market correlation matrix just pure noise?

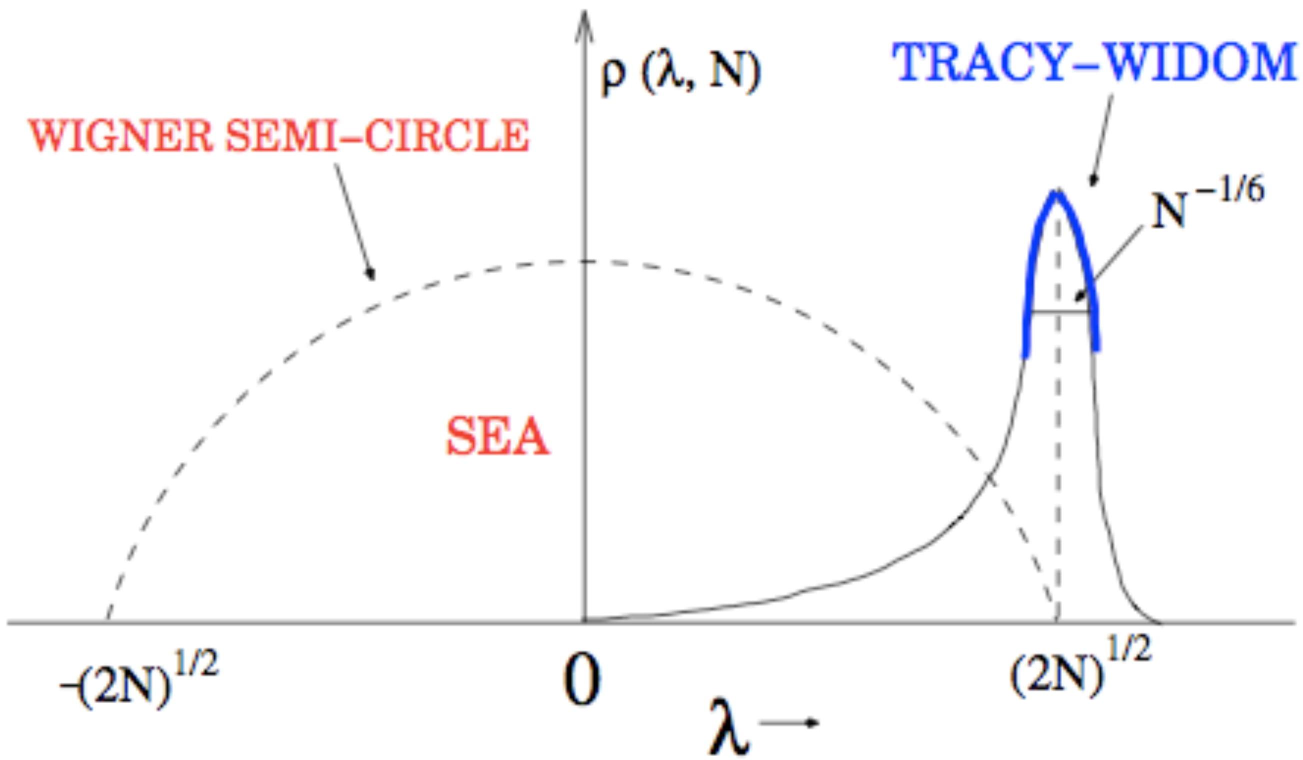
INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL
J. Phys. A: Math. Gen. **36** (2003) 3009–3032 PII: S0305-4470(03)37794-7

A new method to estimate the noise in financial correlation matrices

Thomas Guhr[1] and Bernd Kälber[2,3]

LARGEST EIGENVALUE

Tracy-Widom distribution for $\lambda_{\max}$



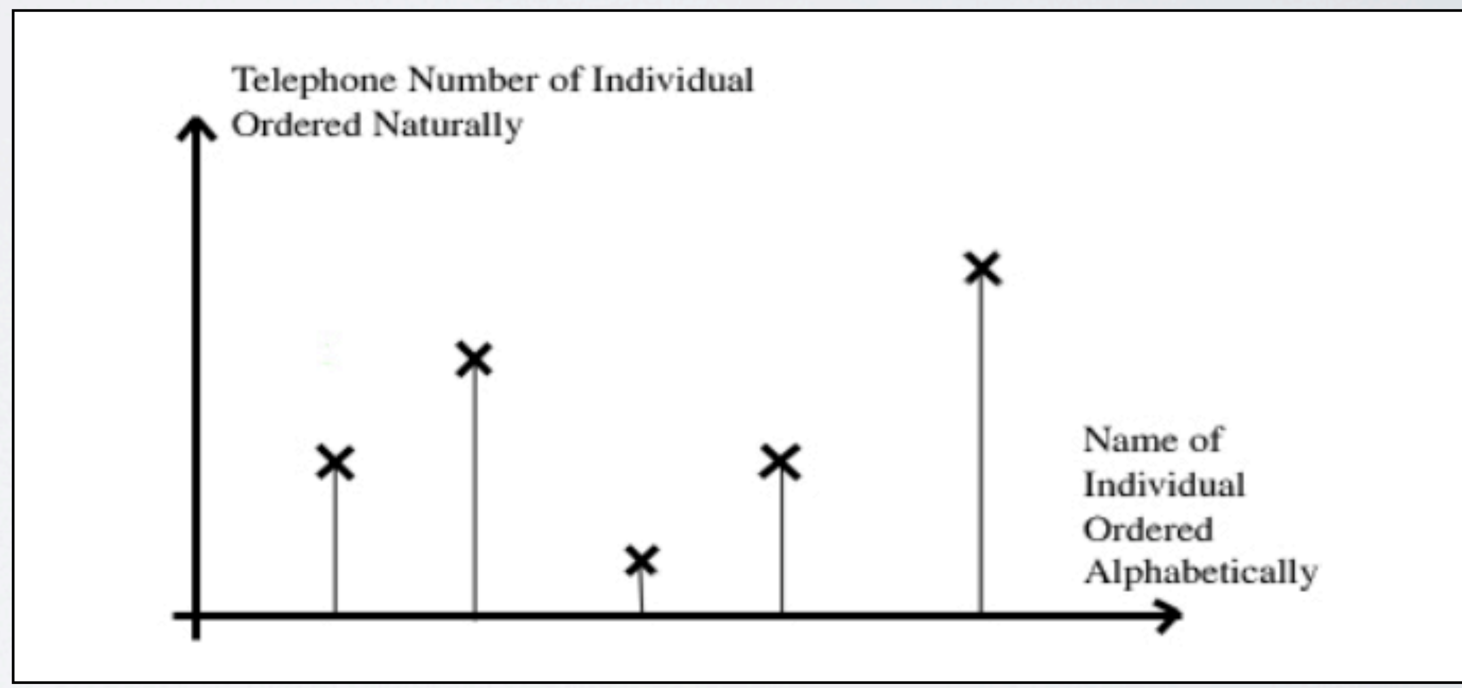
- $\langle \lambda_{\max} \rangle = \sqrt{2N}$; typical fluctuation: $|\lambda_{\max} - \sqrt{2N}| \sim N^{-1/6}$ (small)
- typical fluctuations are distributed via Tracy-Widom (1994):
- cumulative distribution:

$$\mathrm{Prob}[\lambda_{\max} \leq t, N] \to F_\beta\left(\sqrt{2}N^{1/6}(t - \sqrt{2N})\right)$$

- Prob. density (pdf): $f_\beta(z) = dF_\beta(z)/dz$
- $F_\beta(z) \to$ obtained from solution of Painlevé-II equation

“Are Tracy and Widom in Your Local Telephone Directory?”

Ryan Witko
Advisor: Percy Deift



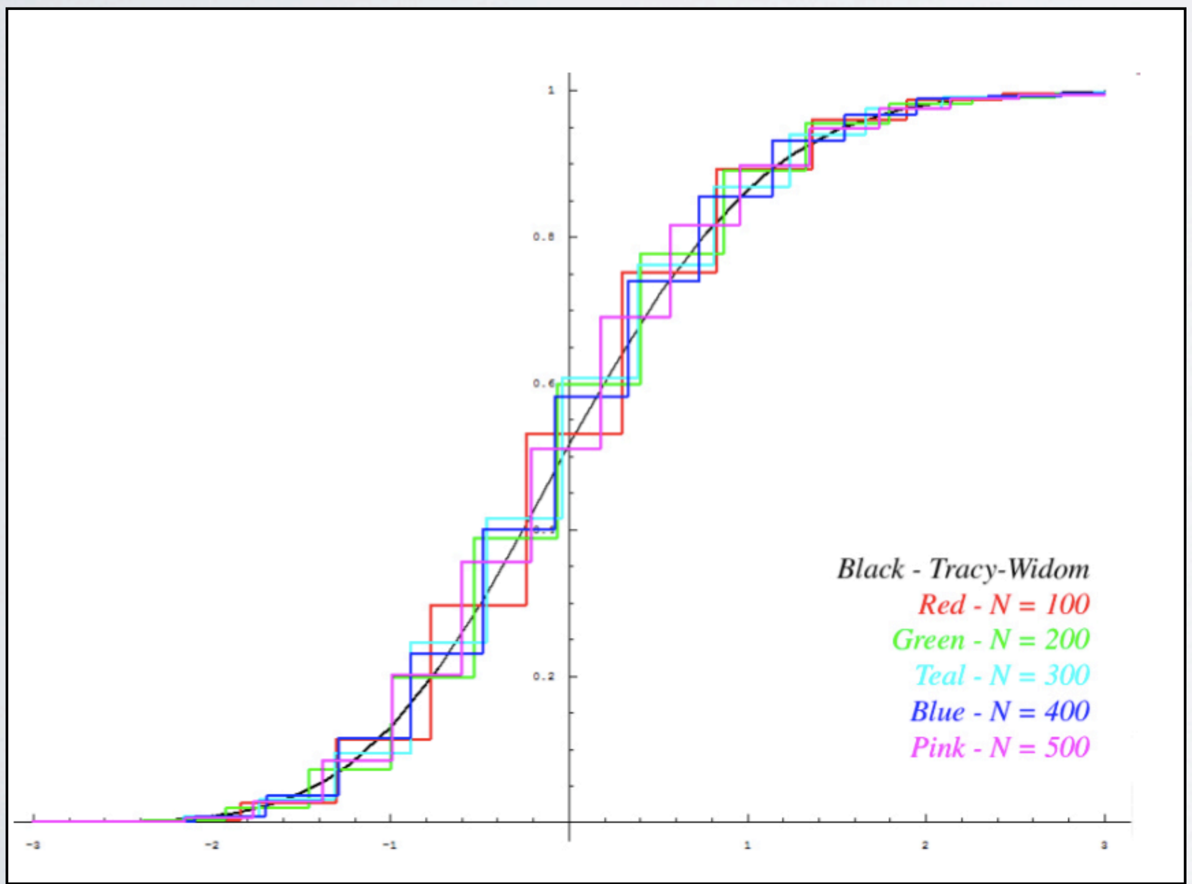
Definition:

The longest increasing (contiguous) subsequence of a given sequence is the subsequence of increasing terms containing the largest number of elements. For example, the longest increasing subsequence of the permutation {6, 3, 4, 8, 10, 5, 7, 1, 9, 2} is {3, 4, 8, 10}.

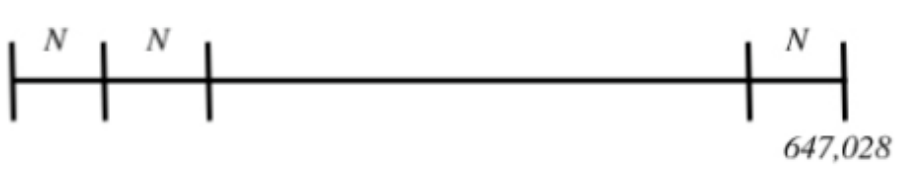
It can be coded in *Mathematica* as follows.

```
<<Combintorica`
LongestContinguousIncreasingSubsequence[p_] :=
  Last[
  Split[Sort[Runs[p]], Length[#1] >= Length[#2]&]
  ]
```

More information »



We broke the 647,028 entries into successive samples each containing *N* entries.



Jinho Baik, Kurt Johansson and Percy Deift showed that as $N \to \infty$

(3) $$\text{Prob}\left(\frac{\ell_N - 2\sqrt{N}}{N^{1/6}} \le t\right) \to F(t)$$

The function *F(t)* was shown by Craig Tracy and Harold Widom to be the distribution of the largest eigenvalue of a random matrix in the Gaussian Unitary Ensemble (GUE). It

SUMMARY

- Eigenvalues of random matrices: strongly correlated
- Level Repulsion
- Tracy-Widom distribution: analogue of Gaussian distribution for correlated random variables
- Zeros of Riemann zeta have the same statistical properties as the eigenvalues of Gaussian matrices
- Non-intersecting Brownian bridges
- Wishart matrices: covariance matrices of random data

SECOND PART

Probability of **rare** events in **linear statistics**

“Lies, damned lies, and statistics.”

A simple example of large deviation tails

- Let $M \to$ no. of heads in N tosses of an unbiased coin
- Clearly $P(M,N) = \binom{N}{M} 2^{-N}$ $(M = 0, 1, \ldots, N) \to$ binomial distribution

  with mean= $\langle M \rangle = \frac{N}{2}$ and variance=$\sigma^2 = \langle (M - \frac{N}{2})^2 \rangle = \frac{N}{4}$
- typical fluctuations $M - \frac{N}{2} \sim O(\sqrt{N})$ are well described by the Gaussian form: $P(M,N) \sim \exp\left[-\frac{2}{N}\left(M - \frac{N}{2}\right)^2\right]$
- Atypical large fluctuations $M - \frac{N}{2} \sim O(N)$ are not described by Gaussian form
- Setting $M/N = x$ and using Stirling's formula $N! \sim N^{N+1/2} e^{-N}$ gives

$$P(M = Nx, N) \sim \exp\left[-N\Phi(x)\right] \qquad \text{where}$$

$$\boxed{\Phi(x) = x\log(x) + (1-x)\log(1-x) + \log 2} \to \text{large deviation function}$$

- $\Phi(x) \to$ symmetric with a minimum at $x = 1/2$ and for small arguments $|x - 1/2| << 1$, $\Phi(x) \approx 2(x - 1/2)^2$

  $\to$ recovers the Gaussian form near the peak

LINEAR STATISTICS

$\{x_1, \ldots, x_N\}$ Random Variables

$$A = \sum_{i=1}^{N} f(x_i)$$

Question: what is the distribution of A for large N?

Independent → Central Limit Theorems

Correlated → ?

The first rigorous results concerning large deviations are due to the Swedish mathematician Harald Cramér, who applied them to model the insurance business. From the point of view of an insurance company, the earning is at a constant rate per month (the monthly premium) but the claims X_i come randomly.
For the company to be successful over a certain period of time (preferably many months), the total earning should exceed the total claim.
Thus to estimate the premium you have to ask the following question : "What should we choose as the premium q such that over N months the total claim C = \Sum_i X_i should be less than Nq ? "
Cramér gave a solution to this question for i.i.d. random variables

What if the random variables are strongly correlated?

A Trivial Problem

DIAGONAL MATRIX

$$\begin{bmatrix} x_{11} & & & & \\ & x_{22} & & 0 & \\ & & x_{33} & & \\ & 0 & & \ddots & \\ & & & & x_{NN} \end{bmatrix}$$

$\Pr[x_{ii} = x]$

$= (2\pi)^{-1/2} \exp[-x^2/2]$

$\downarrow$

GAUSSIAN

N Eigenvalues: $\lambda_i = x_{ii} \longrightarrow$ Independent

- $P_N = \text{Prob}[\lambda_1 \leq 0,\ \lambda_2 \leq 0,\ \ldots,\ \lambda_N \leq 0] = 2^{-N} = \exp[-(\ln 2)\, N]$

A Nontrivial Problem

REAL SYMMETRIC MATRIX (NxN)

$$X = \begin{vmatrix} x_{11} & x_{12} & \cdots & x_{1N} \\ x_{21} & x_{22} & \cdots & x_{2N} \\ \vdots & \vdots & & \vdots \\ x_{N1} & \cdots & \cdots & x_{NN} \end{vmatrix}$$

GAUSSIAN

↓

$\Pr[X] \propto \exp[-\frac{1}{2}\mathrm{Tr}(X^2)]$

N eigenvalues : $\lambda_1, \lambda_2, \ldots, \lambda_N$

→ strongly correlated

- $P_N = \mathrm{Prob}[\lambda_1 \le 0,\ \lambda_2 \le 0,\ \ldots,\ \lambda_N \le 0] = \mathrm{Prob}[\lambda_{\max} \le 0] = ?$

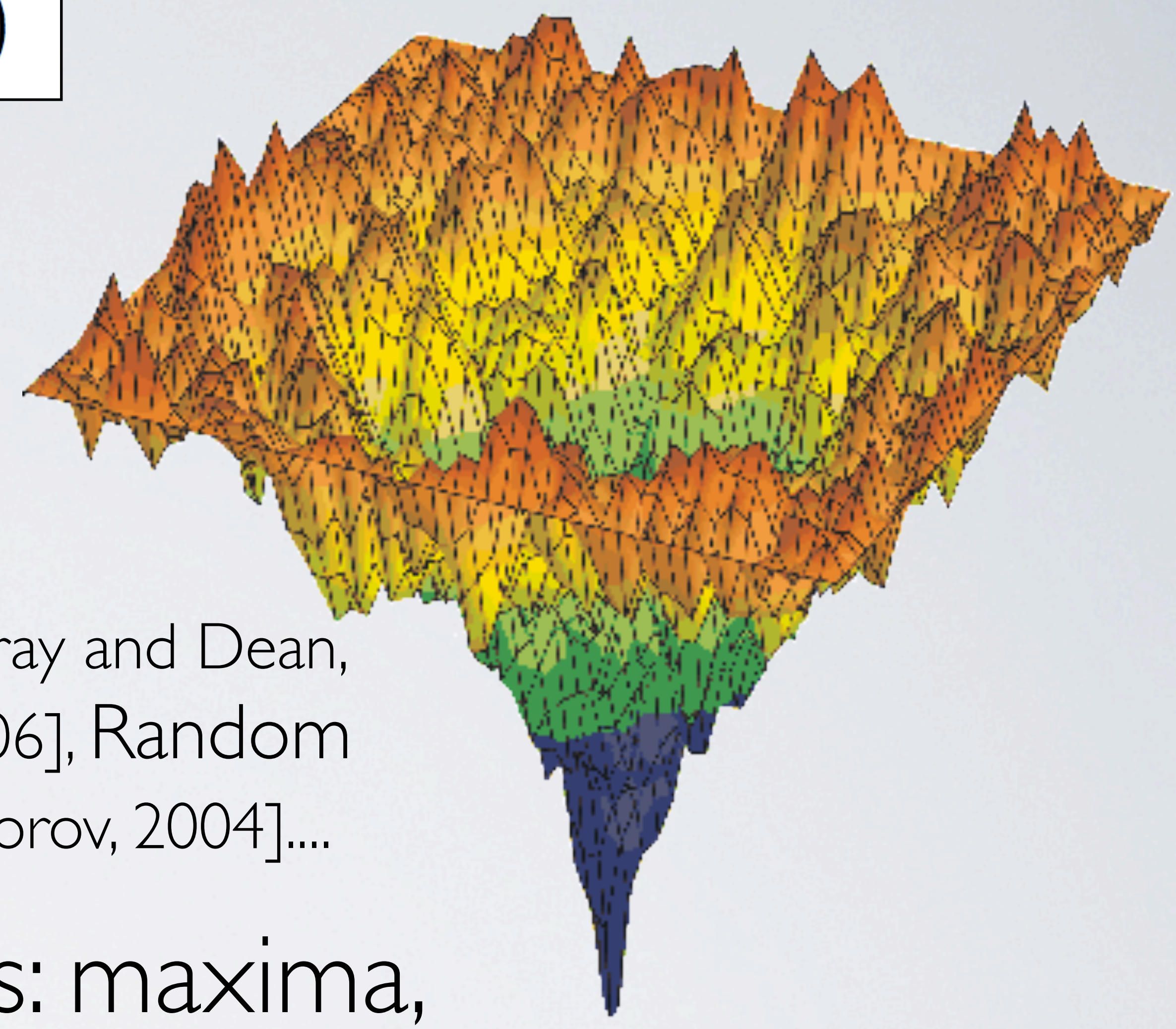
[R.M. May, Nature, 238, 413 (1972)—Ecosystems]

[Cavagna et. al. 2000, Fyodorov 2004, —— Glassy systems]

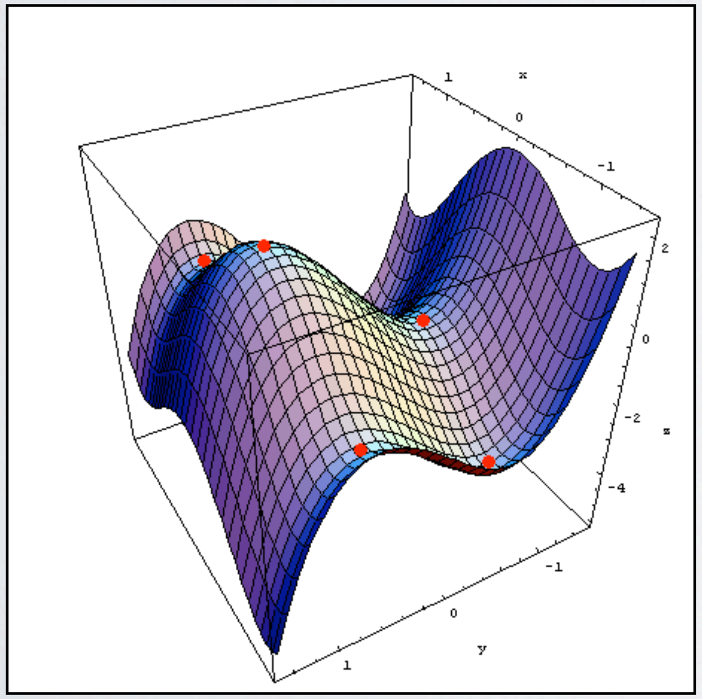
[Susskind 2003, Douglas et. al. 2004, Aazami & Easther 2006— String theory].....

A particle moving in a N-dim. landscape $V(y_1, \ldots, y_N)$

$$\frac{dy_i}{dt} = -\nabla_{y_i} V$$



- Spin and structural glasses, Gaussian fields [Bray and Dean, 2006], String landscapes [Aazami and Easther, 2006], Random Energy Landscapes and Glass Transition [Fyodorov, 2004]....



Stationary points: maxima, minima and saddles

$$H_{i,j} = \left[\frac{\partial^2 V}{\partial y_i \partial y_j} \right]$$

Hessian matrix

Eigenvalues of Hessian matrix determine the nature of the stationary point

RANDOM HESSIAN MODEL

Draw the elements of the Hessian matrix independently at random

$$H_{i,j} = \left[\frac{\partial^2 V}{\partial y_i \partial y_j} \right]$$

→ It belongs to the **GOE** of random matrices

The index distribution (number of positive eigenvalues) provides information about the ***typical*** stability pattern of a Random Hessian model

↓

Most of the stationary points are saddles!

Journal of Cosmology and Astroparticle Physics
An IOP and SISSA journal

Cosmology from random multifield potentials

Amir Aazami and Richard Easther

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E-mail: amir.aazami@yale.edu and richard.easther@yale.edu

Despite the approximation used to obtain equation (8), we have confirmed that the likelihood that all the eigenvalues of an $N \times N$ symmetric matrix have the same sign scales as e^{-cN^2}. The measured constant differs slightly from -0.25, although given the simplicity of our approximation the agreement is perhaps surprisingly good.

- Based on numerics, Aazami & Easther (2006) predicted for large N:

$$P_N \sim \exp[-\theta N^2] \text{ with } \theta_{\rm num} \approx 0.27$$

$\rightarrow$ very small probability $\rightarrow$ RARE EVENT

- Exact result: $\theta = \frac{1}{4}\ln(3) = 0.274653..$ (Dean and S.M., 2006)

More generally, for $\beta = 1$ (GOE), $\beta = 2$ (GUE) and $\beta = 4$ (GSE)

$$P_N \sim \exp[-\beta\theta N^2] \text{ for large } N$$

GAUSSIAN MATRIX NxN

$N = 5$

$$\begin{pmatrix} 0.5377 & 0.2631 & -1.8044 & 0.3286 & 0.4951 \\ 0.2631 & -0.4336 & 1.6888 & 1.7271 & 0.7810 \\ -1.8044 & 1.6888 & 0.7254 & 0.7133 & 0.7160 \\ 0.3286 & 1.7271 & 0.7133 & 1.4090 & 1.5237 \\ 0.4951 & 0.7810 & 0.7160 & 1.5237 & 0.4889 \end{pmatrix}$$

Real Symmetric or Complex Hermitian or Quaternion self-dual :
eigenvalues are real

$$\vec{\lambda} = [-2.4341 \ -0.8386 \ -0.5203 \ 2.2594 \ 4.2610]$$

$$\mathcal{N}_+ = \text{number of positive eigenvalues}$$

The index

$$P_\beta(\lambda_1, \ldots, \lambda_N)$$

Joint probability density of eigenvalues

$$\mathcal{P}(\mathcal{N}_+, N) = \int_{-\infty}^{\infty} d\lambda_1 \cdots d\lambda_N P_\beta(\lambda_1, \ldots, \lambda_N) \delta\left(\mathcal{N}_+ - \sum_{i=1}^{N} \theta(\lambda_i)\right)$$

Probability distribution of linear statistics

[Chen '94 - '98
Forrester '98
Beenakker '93
..........]

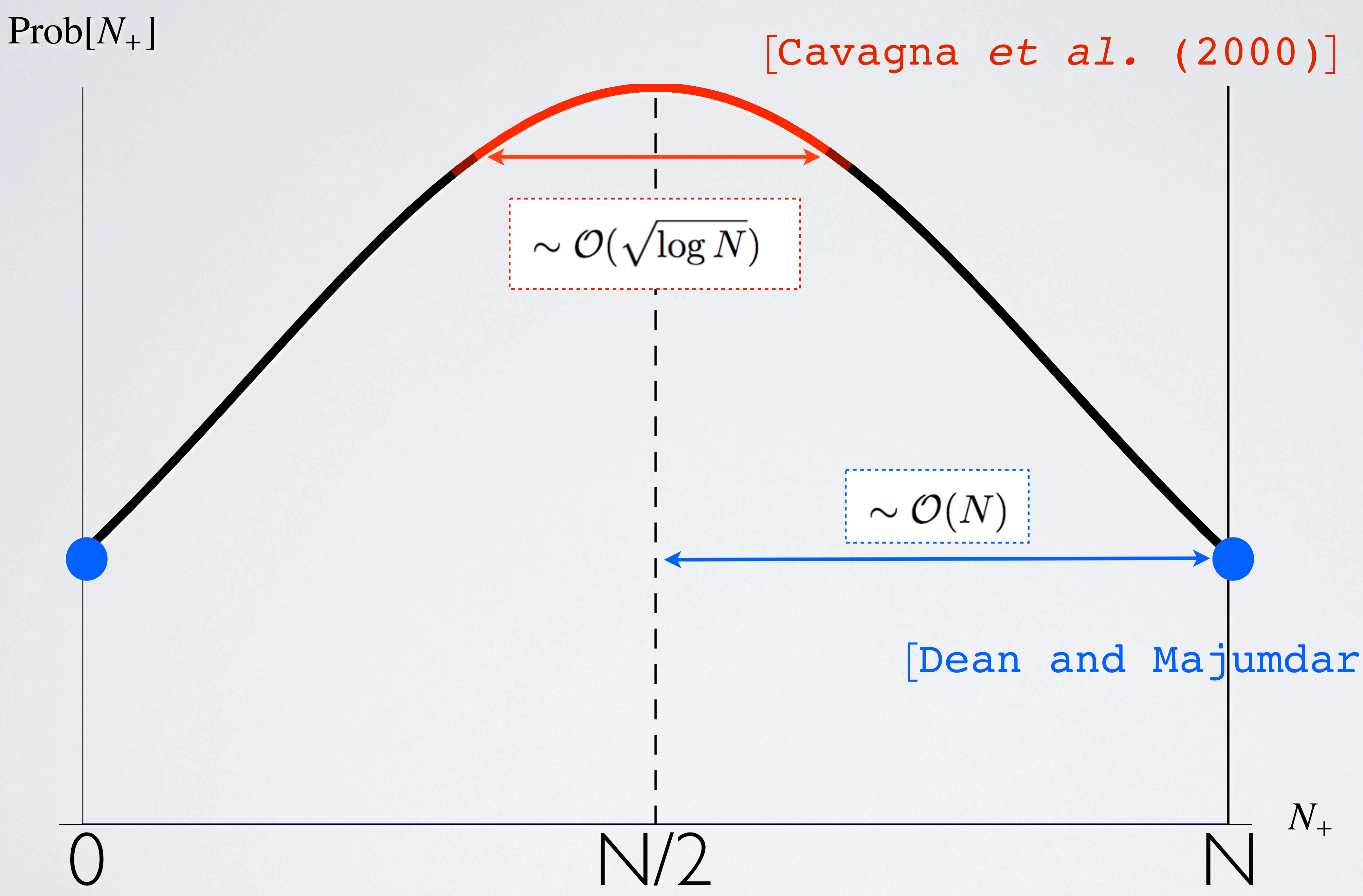
$$P_\beta(\lambda_1, \ldots, \lambda_N) = \frac{1}{Z_N} \mathrm{e}^{-\frac{\beta}{2}\sum_{i=1}^{N}\lambda_i^2} \prod_{j<k} |\lambda_j - \lambda_k|^\beta = \frac{1}{Z_N} \mathrm{e}^{-\beta \mathcal{H}(\vec{\lambda})}$$

$$\mathcal{H}(\vec{\lambda}) = \frac{1}{2}\sum_{i=1}^{N}\lambda_i^2 - \frac{1}{2}\sum_{j \neq k} \log|\lambda_j - \lambda_k|$$

Canonical weight of an auxiliary thermodynamical system

WHAT IS KNOWN? 2 SCALES IN THIS PROBLEM

$\mathcal{N}_+ =$ number of positive eigenvalues



TYPICAL VS. ATYPICAL FLUCTUATIONS: A PUZZLE?

Peak

PHYSICAL REVIEW B VOLUME 61, NUMBER 6 1 FEBRUARY 2000-II

Index distribution of random matrices with an application to disordered systems

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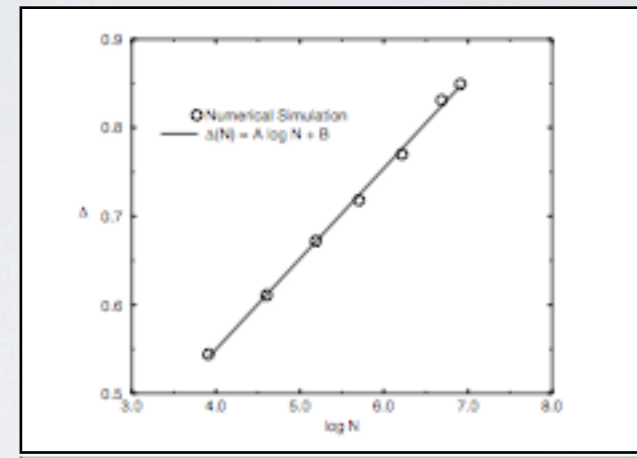
Theoretical Physics, University of Oxford, 1 Keble Road, Oxford, OX1 3NP, United Kingdom
(Received 21 July 1999; revised manuscript received 15 October 1999)

$$c = \frac{\mathcal{N}_+}{N}$$

$$\mathcal{P}(\mathcal{N}_+ = cN, N) \simeq \exp\left(-\frac{\pi^2 N^2}{2\ln N}(c-1/2)^2\right)$$

$$\Delta(N) = \left\langle \left(\mathcal{N}_+ - \frac{N}{2}\right)^2 \right\rangle \approx \frac{\ln N}{\pi^2}$$



Tails

PRL 97, 160201 (2006) PHYSICAL REVIEW LETTERS 20 OCTOBER 2006

Large Deviations of Extreme Eigenvalues of Random Matrices

David S. Dean[1] and Satya N. Majumdar[2]
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(Received 3 August 2006; published 20 October 2006)

$$\mathcal{P}(\mathcal{N}_+ = N, N) \simeq \exp\left(-\beta N^2 \theta\right), \qquad \theta = (\ln 3)/4$$

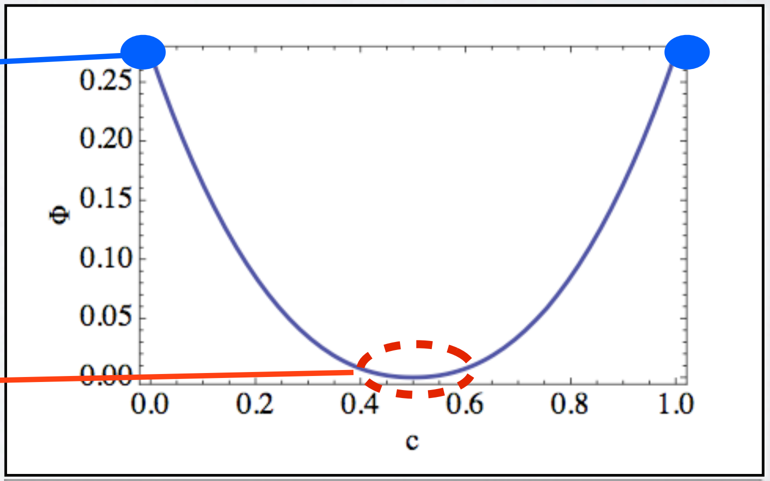
MAIN RESULT FOR LARGE N

$$\mathcal{P}(\mathcal{N}_+ = cN, N) \approx \exp\left(-\beta N^2 \Phi(c)\right)$$

$$\lim_{N\to\infty}\left[\frac{-\log \mathcal{P}(\mathcal{N}_+ = cN, N)}{\beta N^2}\right] = \Phi(c)$$

in agreement with
Dean & Majumdar

$$\Phi(0) = \Phi(1) = \theta = \log 3/4 \approx 0.27..$$



$$\Phi(c = 1/2 + \delta) = -\frac{\pi^2}{2}\frac{\delta^2}{\log \delta}$$

$$\mathcal{P}(\mathcal{N}_+, N) \approx \exp\left[-\frac{\beta\pi^2}{2\ln(N)}\left(\mathcal{N}_+ - N/2\right)^2\right]$$

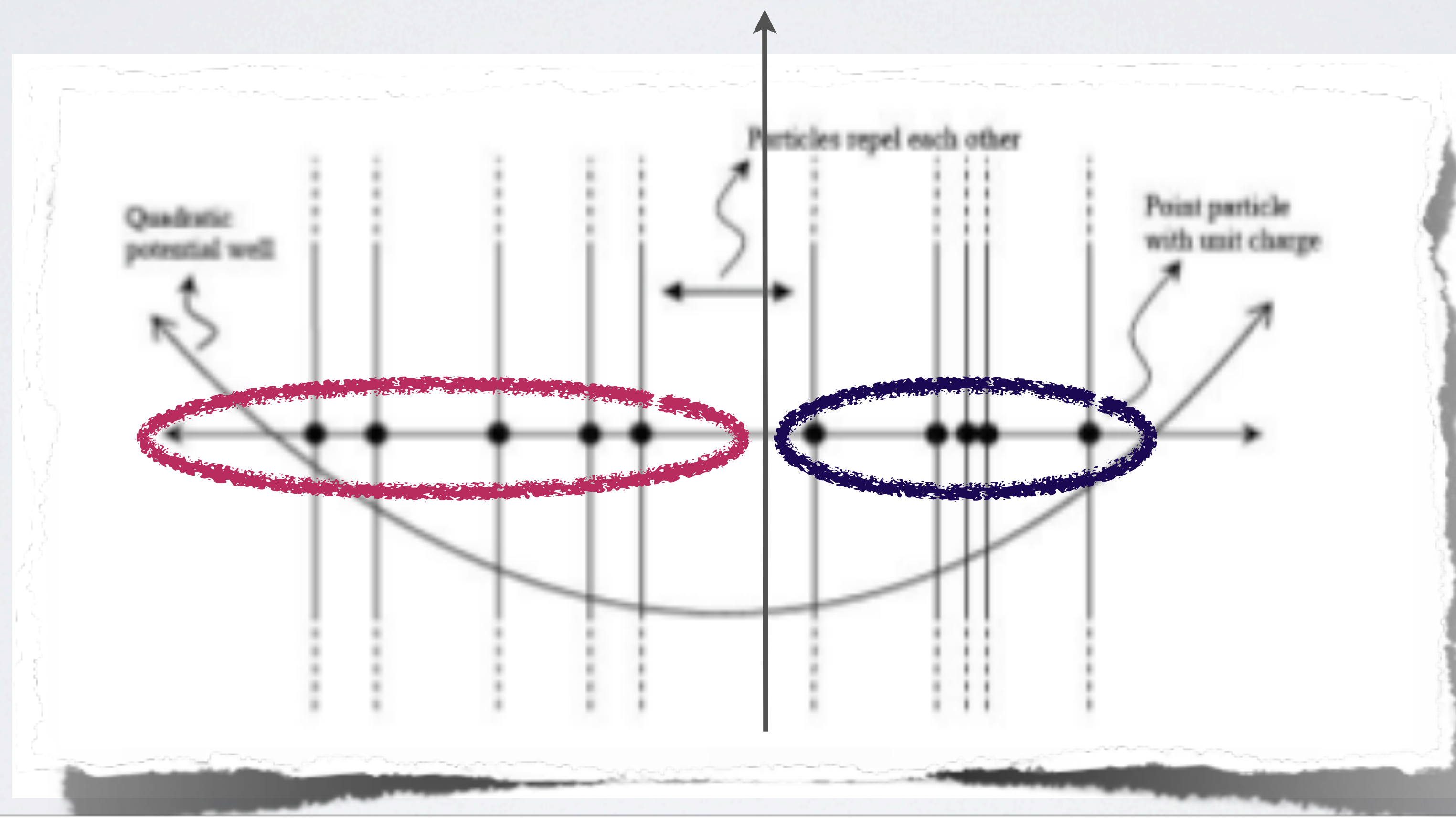
in agreement with
Cavagna *et al.*

$$\mathcal{P}(\mathcal{N}_+, N) = \int_{-\infty}^{\infty} d\lambda_1 \cdots d\lambda_N P_\beta(\lambda_1, \ldots, \lambda_N) \delta\left(\mathcal{N}_+ - \sum_{i=1}^{N} \theta(\lambda_i)\right)$$

$$e^{-\beta \mathcal{H}(\vec{\lambda})}$$

$$\mathcal{H}(\vec{\lambda}) = \frac{1}{2}\sum_{i=1}^{N} \lambda_i^2 - \frac{1}{2}\sum_{j \neq k} \log|\lambda_j - \lambda_k|$$

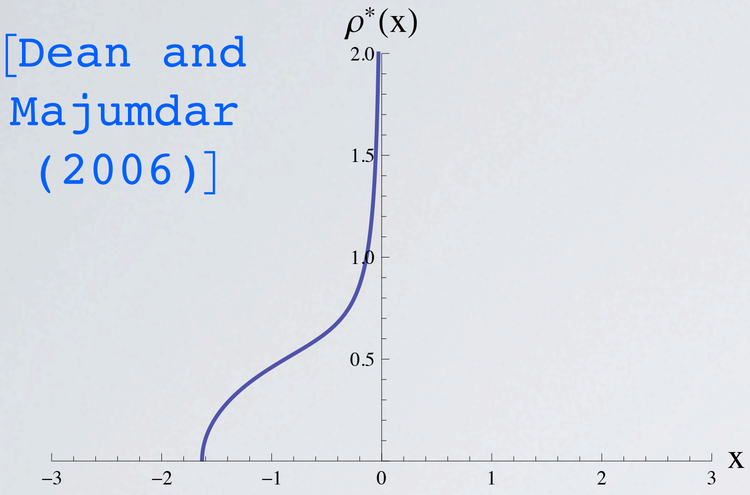
Task: evaluate this integral for large N by mapping it to a Coulomb gas problem

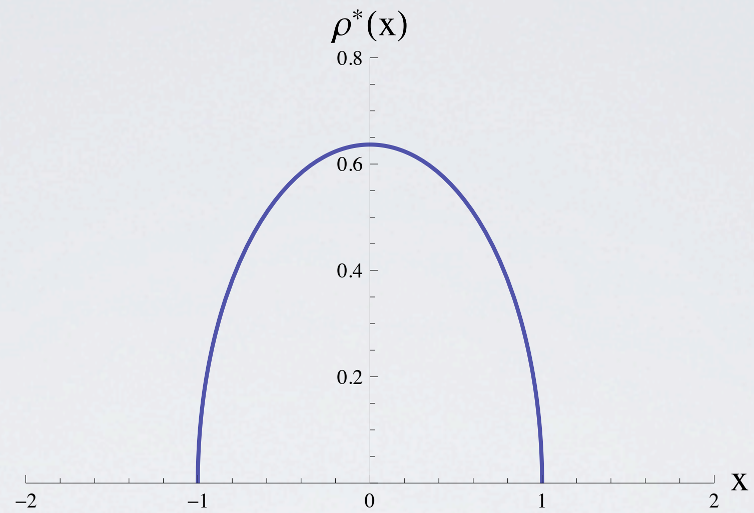


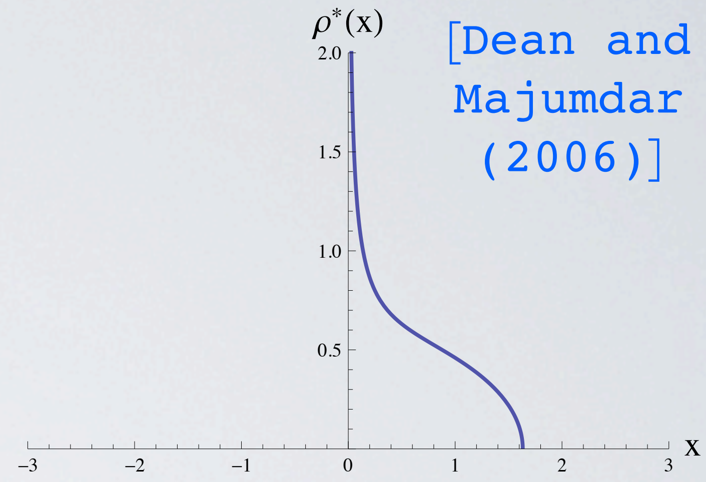
$\mathcal{P}(\mathcal{N}_+, N)$ is the canonical partition function of an auxiliary Coulomb gas with an extra hard constraint

PHASE TRANSITIONS IN THE CONSTRAINED GAS

[Dean and Majumdar (2006)]







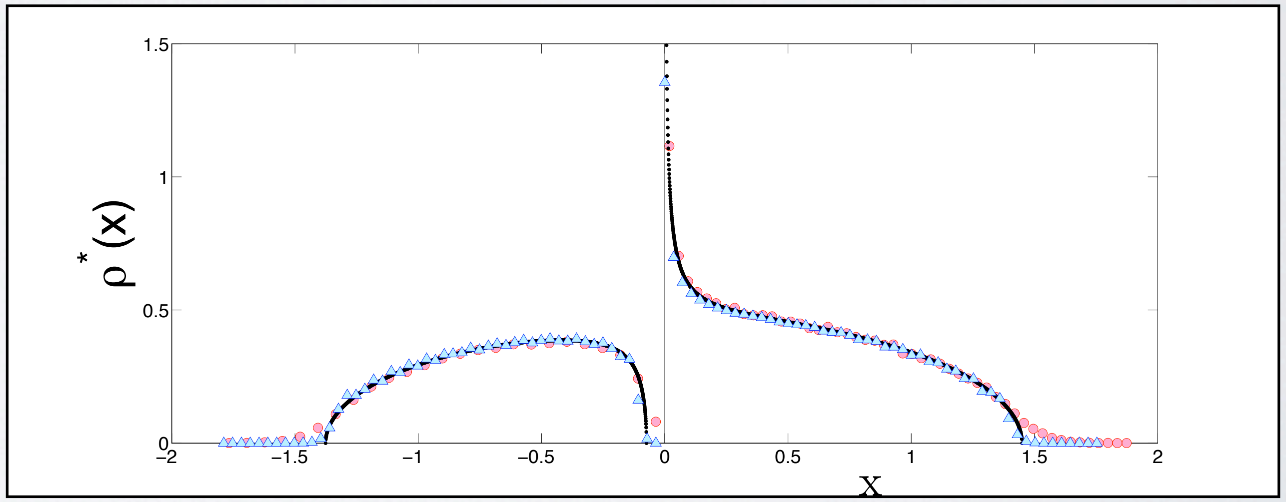
[Dean and Majumdar (2006)]

$c = 0$

$c = \frac{1}{2}$

$c = 1$

?



$$\rho^\star(x) = \frac{1}{\pi}\sqrt{\frac{L-x}{x}(x+L/a)(x+(1-1/a)L)}$$

$\rho^\star(x)$ → Partition Function → $\mathcal{P}(\mathcal{N}_+, N)$

$$\mathcal{P}(\mathcal{N}_+, N) = \int_{-\infty}^{\infty} d\lambda_1 \cdots d\lambda_N P_\beta(\lambda_1, \ldots, \lambda_N)\delta\left(\mathcal{N}_+ - \sum_{i=1}^{N}\theta(\lambda_i)\right)$$

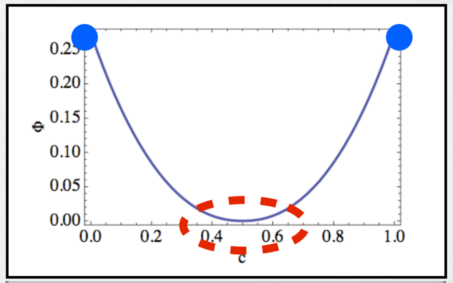
$$\mathcal{P}(\mathcal{N}_+, N) \propto \int \mathcal{D}[\rho]\mathrm{e}^{-\beta N^2 \mathcal{F}_c[\rho]}$$

where

$$\mathcal{F}_c[\rho] = \frac{1}{2}\int_{-\infty}^{\infty} dx\ x^2\rho(x) - \frac{1}{2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dx dx' \rho(x)\rho(x')\ln|x - x'| + $$
$$+ A_1\left(\int_{-\infty}^{\infty} dx\theta(x)\rho(x) - c\right) + A_2\left(\int_{-\infty}^{\infty} dx\rho(x) - 1\right)$$

Saddle point of the free energy: equilibrium density of the fluid $\rho^\star(x)$

$$\mathcal{P}(\mathcal{N}_+ = cN, N) \simeq \exp\left[-\beta N^2\left(\underbrace{\mathcal{F}_c[\rho^\star] - \mathcal{F}_{1/2}[\rho^\star]}_{\Phi(c)}\right)\right]$$



SOLVING THE SADDLE-POINT EQUATION

$$\frac{\delta \mathcal{F}_c[\rho]}{\delta \rho} = 0 \qquad \Rightarrow \qquad \rho^\star(x)$$

Inverse Electrostatic Problem

$$x^2 + A_1\theta(x) + A_2 = 2\int_{-\infty}^{\infty} \rho^\star(y) \ln|x-y| dy$$

$$x = \mathrm{Pr}\int dy \frac{\rho^\star(y)}{x-y}.$$

Equazioni integrali singolari del tipo di Carleman.

Francesco G. Tricomi (a Torino).

A Mauro Picone nel suo 70mo compleanno.

Commun. math. Phys. 59, 35—51 (1978)

Communications in Mathematical Physics

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Planar Diagrams

E. Brézin, C. Itzykson, G. Parisi*, and J. B. Zuber

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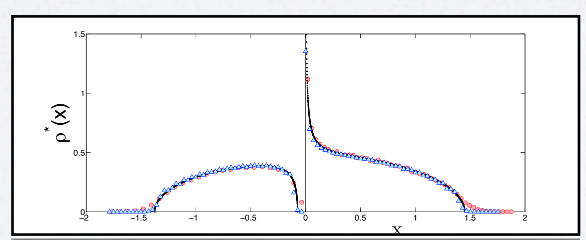
[Brezin *et al.* 1978]



New!

Iterated single-support solution by Tricomi (1957)

Phys. Rev. E **83**, 041105 (2011)

IN SUMMARY...

$$\mathcal{P}(\mathcal{N}_+ = cN, N) \simeq \exp\left[-\beta N^2 \left(\underbrace{\mathcal{F}_c[\rho^\star] - \mathcal{F}_{1/2}[\rho^\star]}_{\Phi(c)}\right)\right]$$

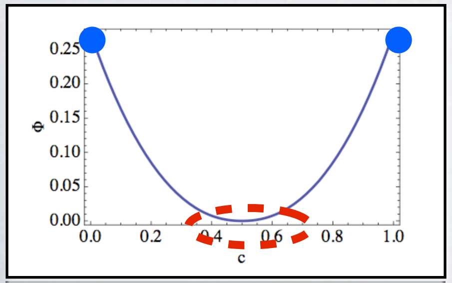
where

$$\rho^\star(x) = \frac{1}{\pi}\sqrt{\frac{L-x}{x}(x+L/a)(x+(1-1/a)L)}$$

and

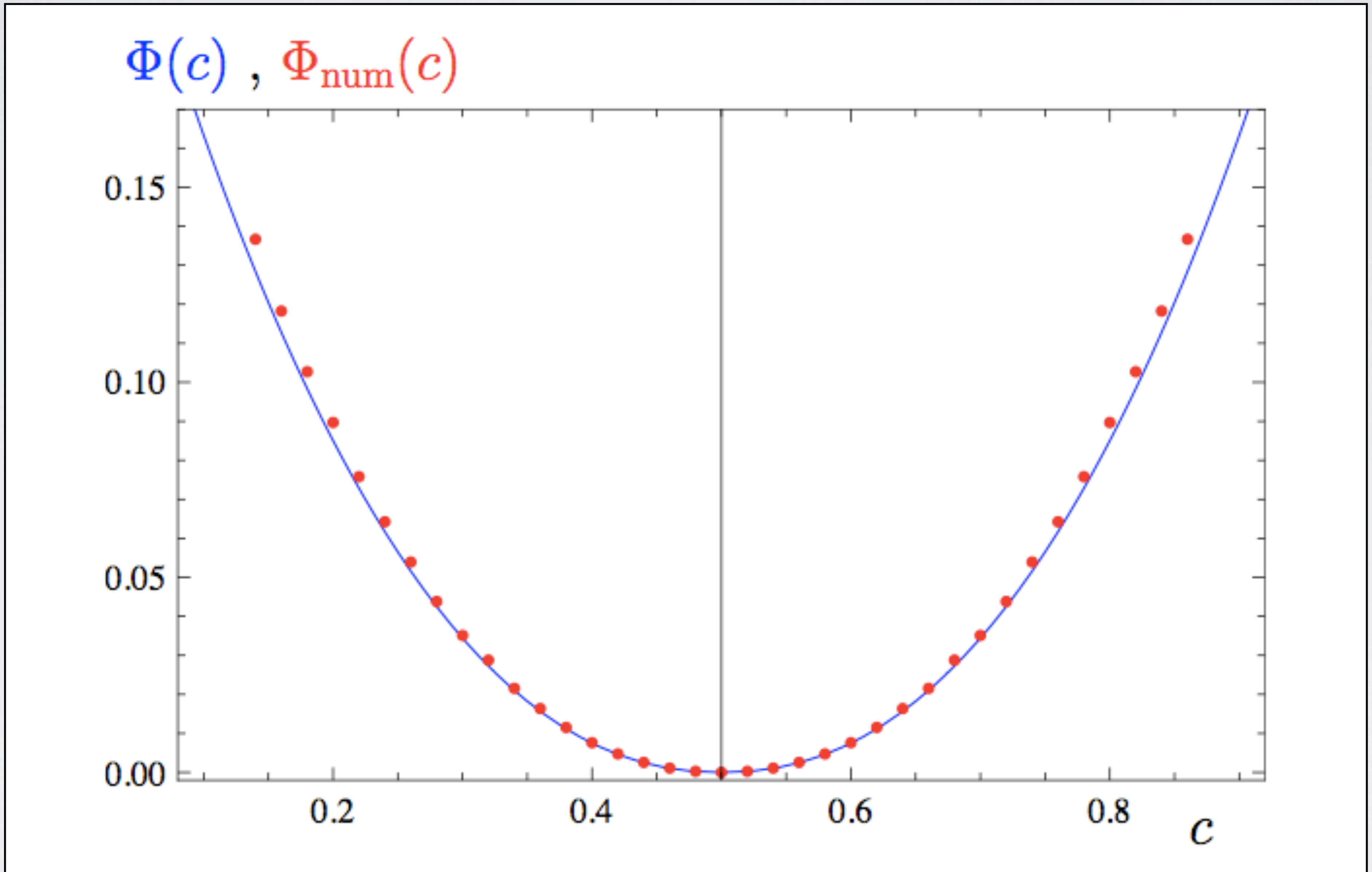
$$\mathcal{F}_c[\rho] = \frac{1}{2}\int_{-\infty}^{\infty} dx\ x^2\rho(x) - \frac{1}{2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dx dx' \rho(x)\rho(x') \ln|x-x'| + \\ + A_1\left(\int_{-\infty}^{\infty} dx \theta(x)\rho(x) - c\right) + A_2\left(\int_{-\infty}^{\infty} dx \rho(x) - 1\right)$$

$$\Phi(c) = \frac{1}{4}[L^2 - 1 - \log(2L^2)] + \frac{(1-c)}{2}\log(a) - \frac{(1-c)(a^2-1)}{4a^2}L^2 + \frac{c}{2}\int_L^\infty W_1(x)dx + \frac{(1-c)}{2}\int_{L/a}^\infty W_2(x)dx$$



$$W_1(x) = F(x) - \frac{1}{x} = x - \frac{1}{x} - \sqrt{\frac{(x-L)}{x}\left(x+\frac{L}{a}\right)\left(x+\left(1-\frac{1}{a}\right)L\right)}$$

Some numerics...



SUMMARY

- Any matrix coming up in your research? Randomize! (and come to my office later...)

- Strongly correlated random variables
- Ubiquity - Universality of local statistics
- Rare events for strongly correlated random variables ---> exactly solvable cases!

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Thank you.