Phase Transitions in the Quantum Transport Problem

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In collaboration with

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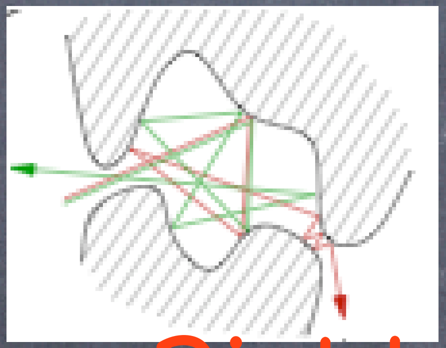
- “A cavity of sub-micron dimensions, etched in a semiconductor is called a quantum dot” [C.W.J. Beenakker]

- “...is essentially a mesoscopic electron billiard, consisting of a ballistic cavity connected by two small holes to two electron reservoirs.” [R.A. Jalabert et al.]

Left lead



Right lead

‘N’ electronic channels in each of the two leads

Sample-to-sample fluctuations of experimental observables

$$G = \lim_{V \to 0} \frac{\bar{I}}{V}$$

Conductance

The mathematical problem

Conductance

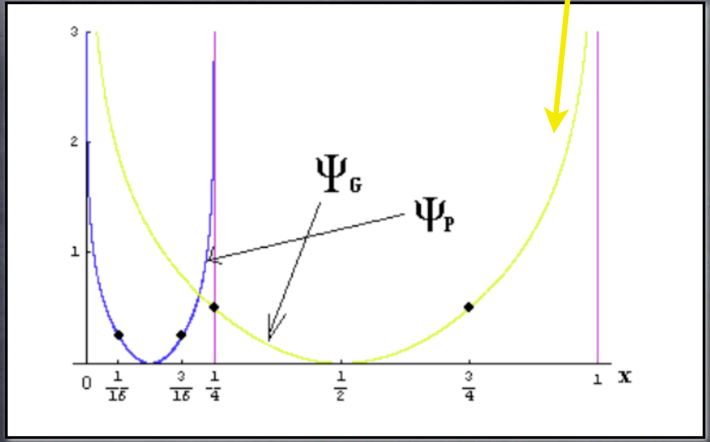
$$\mathcal{P}(G,N)=\frac{1}{Z}\int_0^1\cdots\int_0^1 dT_1\cdots dT_N\prod_{j<k}|T_j-T_k|^\beta\prod_{i=1}^N T_i^{\beta/2-1}\delta\left(\sum_{i=1}^N T_i-G\right)$$

Distribution of the sum of 'N' correlated random variables (for 'N' large)

Large Deviation Principle

$$\lim_{N\to\infty}\left[-\frac{2\log\mathcal{P}(G=Nx,N)}{\beta N^2}\right]=\Psi_G(x)$$

$$\mathcal{P}(G,N)\approx\exp\left(-\frac{\beta}{2}N^2\Psi_G\left(\frac{G}{N}\right)\right)$$



A simple example of large deviation tails

- Let $M \to$ no. of heads in N tosses of an unbiased coin
- Clearly $P(M,N) = \binom{N}{M} 2^{-N}$ $(M = 0, 1, \ldots, N)$ $\to$ binomial distribution

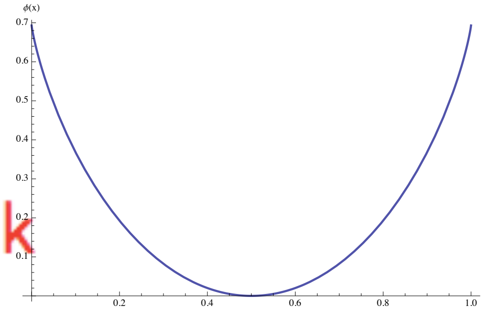
  with mean= $\langle M \rangle = \frac{N}{2}$ and variance=$\sigma^2 = \langle (M - \frac{N}{2})^2 \rangle = \frac{N}{4}$
- typical fluctuations $M - \frac{N}{2} \sim O(\sqrt{N})$ are well described by the Gaussian form: $P(M,N) \sim \exp\left[-\frac{2}{N}\left(M - \frac{N}{2}\right)^2\right]$
- Atypical large fluctuations $M - \frac{N}{2} \sim O(N)$ are not described by Gaussian form
- Setting $M/N = x$ and using Stirling's formula $N! \sim N^{N+1/2} e^{-N}$ gives

$$P(M = Nx, N) \sim \exp\left[-N\Phi(x)\right] \qquad \text{where}$$

$$\boxed{\Phi(x) = x \log(x) + (1-x)\log(1-x) + \log 2} \to \text{large deviation function}$$

- $\Phi(x) \to$ symmetric with a minimum at $x = 1/2$ and for small arguments $|x - 1/2| << 1$, $\Phi(x) \approx 2(x - 1/2)^2$

  $\to$ recovers the Gaussian form near the peak



$$\mathcal{S} = \underbrace{\begin{pmatrix} \mathbf{r} & \mathbf{t}' \\ \mathbf{t} & \mathbf{r}' \end{pmatrix}}_{2N} \; 2N$$

Scattering Matrix of the cavity is drawn uniformly at random from the unitary group

Europhys. Lett., 27 (4), pp. 255-260 (1994)

Universal Quantum Signatures of Chaos in Ballistic Transport.

R. A. JALABERT (*), J.-L. PICHARD (**) and C. W. J. BEENAKKER (***)

VOLUME 73, NUMBER 1 PHYSICAL REVIEW LETTERS 4 JULY 1994

Mesoscopic Transport through Chaotic Cavities: A Random *S*-Matrix Theory Approach

Harold U. Baranger[1] and Pier A. Mello[2]

How to connect experimental quantities with properties of the scattering matrix?

Landauer-Imry-Büttiker Theory

$$S = \begin{array}{c} \\ N \\ \\ \end{array} \begin{pmatrix} \mathbf{r} & \mathbf{t}' \\ \mathbf{t} & \mathbf{r}' \end{pmatrix}$$
$$\qquad\qquad N$$

$$\mathbf{T} = \mathbf{t}\mathbf{t}^{\dagger}$$

$$0 \leq T_i \leq 1$$

Hermitian transmission matrix

Experimental observables are linear statistics on the random eigenvalues of the transmission matrix

$$\mathcal{A} = \sum_{i=1}^{N} f(T_i)$$

$$G = \sum_{i=1}^{N} T_i$$

Conductance

$$P = \sum_{i=1}^{N} T_i(1 - T_i)$$

Shot Noise

An Example for $\beta = 2$ and $N = 2$

$$\begin{pmatrix} -0.2277+0.0543i & 0.2360+0.4072i & -0.0093-0.1596i & -0.4345-0.7137i \\ -0.8358-0.0563i & -0.2372-0.1252i & -0.2902-0.3261i & 0.0002+0.1890i \\ -0.3391-0.1650i & 0.7410+0.0487i & -0.0378+0.4035i & 0.3733+0.0515i \\ 0.3176+0.0206i & 0.2871-0.2694i & -0.5986-0.5111i & 0.2836-0.2090i \end{pmatrix}$$

$$\mathbf{T} = tt^\dagger = \begin{pmatrix} 0.6937 & 0.0885+0.1682i \\ 0.0885-0.1682i & 0.2563 \end{pmatrix}$$

$$\mathrm{eig}(\mathbf{T}) = [0.1852 \quad , \quad 0.7647] \in [0,1] \times [0,1]$$

$$\boxed{G = 0.1852 + 0.7647 = 0.9499}$$

$$S = \begin{pmatrix} \mathbf{r} & \mathbf{t}' \\ \mathbf{t} & \mathbf{r}' \end{pmatrix}$$

N

N

+

'S' is drawn uniformly from the unitary group

$$\mathbf{T} = \mathbf{t}\mathbf{t}^\dagger$$

$$P(T_1,\ldots,T_N) = \frac{1}{Z}\prod_{j<k}|T_j - T_k|^{\beta}\prod_{i=1}^{N}T_i^{\beta/2-1}$$

Joint Probability Density of Transmission Eigenvalues

A physical realization of Jacobi ensemble of random matrices

*Muttalib, Pichard and Stone (1987)

*Mello, Pereira and Kumar (1988)

*Forrester (2006)

Sketch of derivation (I)

$$\mathcal{P}(G,N) = \frac{1}{Z}\int_0^1 \cdots \int_0^1 dT_1 \cdots dT_N \prod_{j<k} |T_j - T_k|^\beta \prod_{i=1}^N T_i^{\beta/2-1} \delta\left(\sum_{i=1}^N T_i - G\right)$$

Laplace Transform

$$\langle e^{-\frac{\beta}{2}NpG} \rangle$$

Partition Function

$$e^{-\beta \mathcal{H}(\{T_i\};p,N)}$$

$$\int_0^\infty \mathcal{P}(G,N) e^{-\frac{\beta}{2}NpG} dG = \frac{1}{Z}\int_{[0,1]^N} \prod_{i=1}^N dT_i \exp\left(\frac{\beta}{2}\sum_{j\neq k} \log|T_j - T_k| + \left(\frac{\beta}{2}-1\right)\sum_{i=1}^N \log T_i - \frac{\beta}{2}pN\sum_{i=1}^N T_i\right)$$

= 'N'

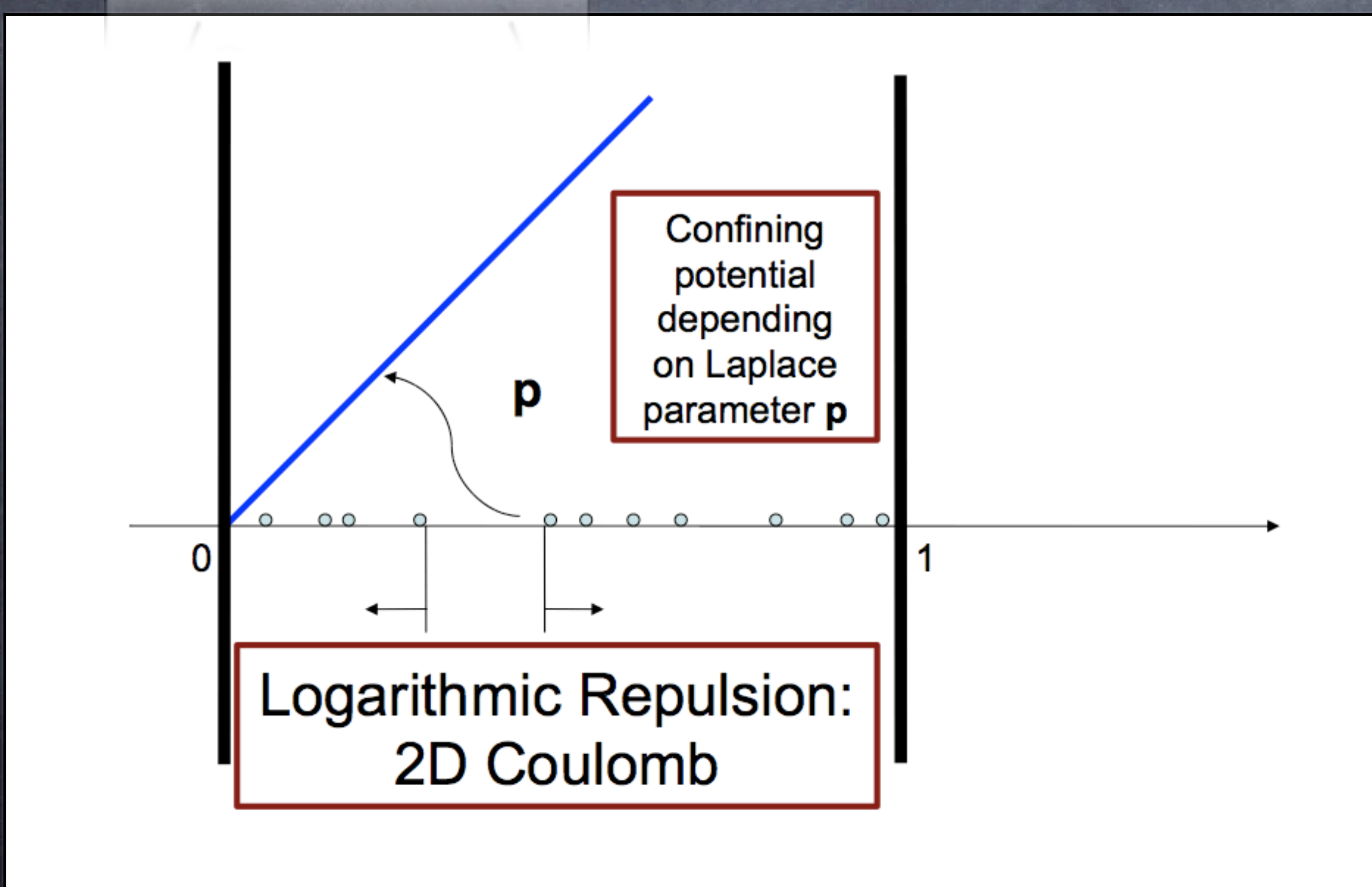
$$p \in \mathbb{R}$$

Sketch of derivation (II)

$$\int_0^\infty \mathcal{P}(G,N)\, e^{-\frac{\beta}{2}NpG}\, dG = \frac{1}{Z}\int_{[0,1]^N}\prod_{i=1}^N dT_i \exp\left(\frac{\beta}{2}\sum_{j\neq k}\log|T_j-T_k| + \cancel{\left(\frac{\beta}{2}-1\right)\sum_{i=1}^N \log T_i} - \frac{\beta}{2}pN\sum_{i=1}^N T_i\right)$$

$$\langle e^{-\frac{\beta}{2}NpG}\rangle$$

$$\mathcal{O}(N^2) \qquad \cancel{\mathcal{O}(N)} \qquad \mathcal{O}(N^2)$$



Mapping: Laplace transform of sought probability --> canonical partition function of an auxiliary problem

Sketch of derivation (III)

Dyson (1962); Dean & Majumdar (2006,2008)

$$\int_{[0,1]^N} \prod_{i=1}^N dT_i \to \int \mathcal{D}[\varrho]$$

$$\sum_{i=1}^N \phi(T_i) \to \int dT\, \varrho(T)\phi(T)$$

$$\int_0^\infty \mathcal{P}(G,N) e^{-\frac{\beta}{2}NpG} dG \simeq \frac{Z_p(N)}{Z_0(N)}$$

$$Z_p(N) \propto \int \mathcal{D}[\varrho_p] e^{-\frac{\beta}{2}N^2 S[\varrho_p]}$$

Saddle Point!

$$S[\varrho_p] = p\int_0^1 \varrho_p(T)\,T dT + B\left[\int_0^1 \varrho_p(T) dT - 1\right] - \int_0^1\int_0^1 dT dT' \varrho_p(T)\varrho_p(T')\log|T-T'|.$$

$$\frac{\delta S[\varrho_p]}{\delta \varrho_p} = 0 \Rightarrow \varrho_p^\star(x)$$

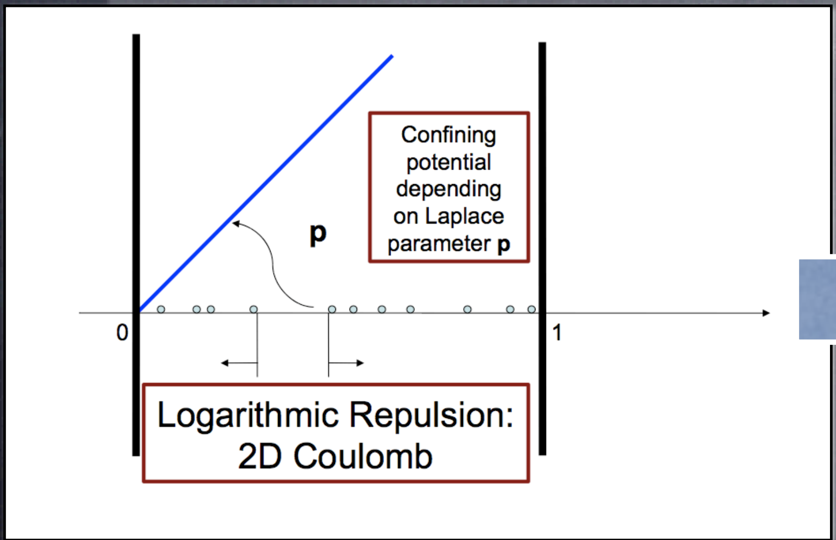
Sketch of derivation (IV)

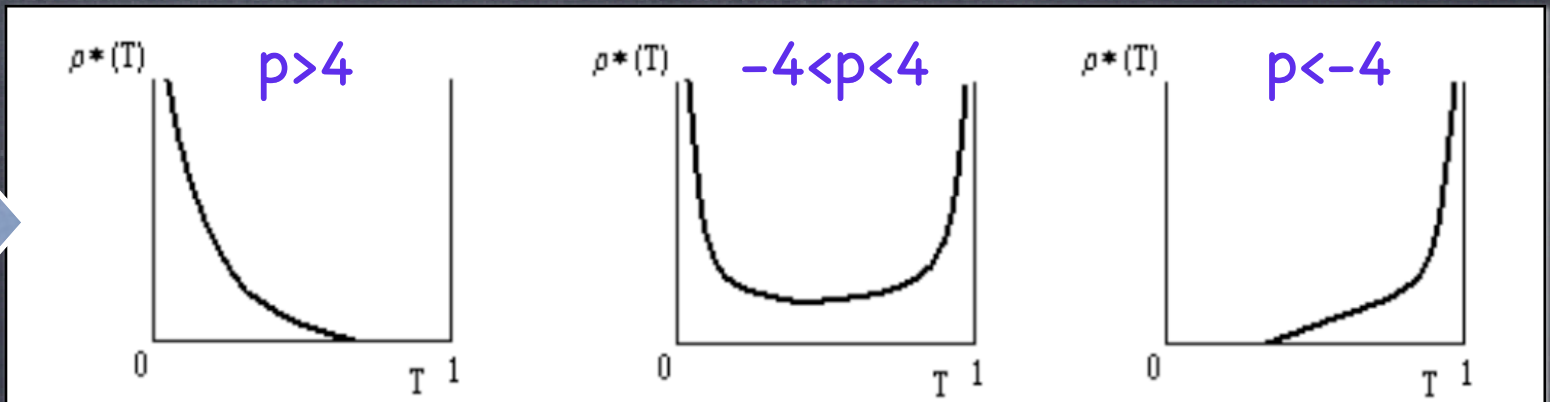
$$\frac{p}{2} = \mathrm{Pr}\int_0^1 \frac{\varrho_p^\star(T')}{T-T'}dT'$$

Inverse Electrostatic Problem

Flowchart:

$$\varrho_p^\star(x) \Rightarrow S[\varrho_p^\star] \Rightarrow \langle e^{-\frac{\beta}{2}NpG}\rangle \approx e^{-\frac{\beta}{2}N^2 \overbrace{[S[\varrho_p^\star]-S[\varrho_0^\star]]}^{J_G(p)}}$$



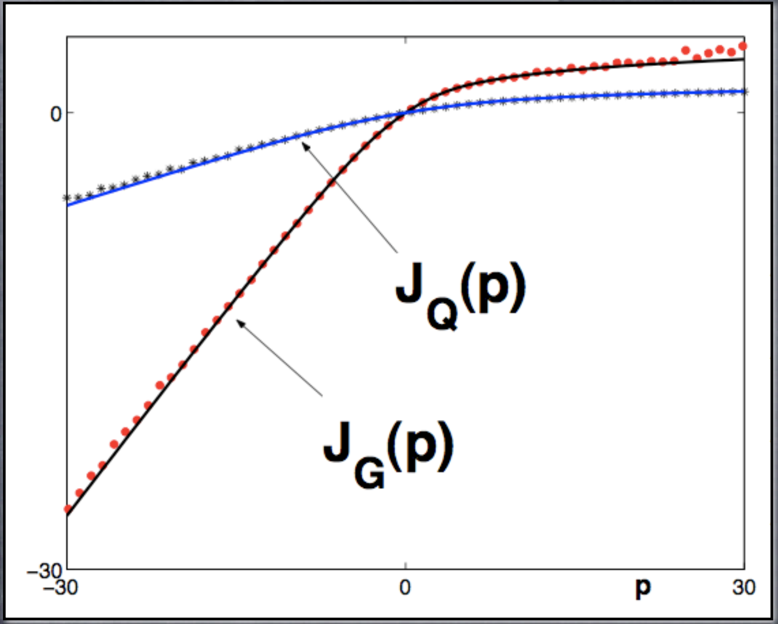


F. G. Tricomi, Integral Equations (1957)

Sketch of derivation (V)

$$\langle e^{-\frac{\beta}{2}NpG}\rangle \approx e^{-\frac{\beta}{2}N^2 \overbrace{[S[\varrho_p^\star]-S[\varrho_0^\star]]}^{J_G(p)}}$$

$$J_G(p) = \begin{cases} -\frac{p^2}{32}+\frac{p}{2} & -4 \le p \le 4 \\ 3/2+\log(p/4) & p \ge 4 \\ 3/2+p+\log(-p/4) & p \le -4 \end{cases}$$

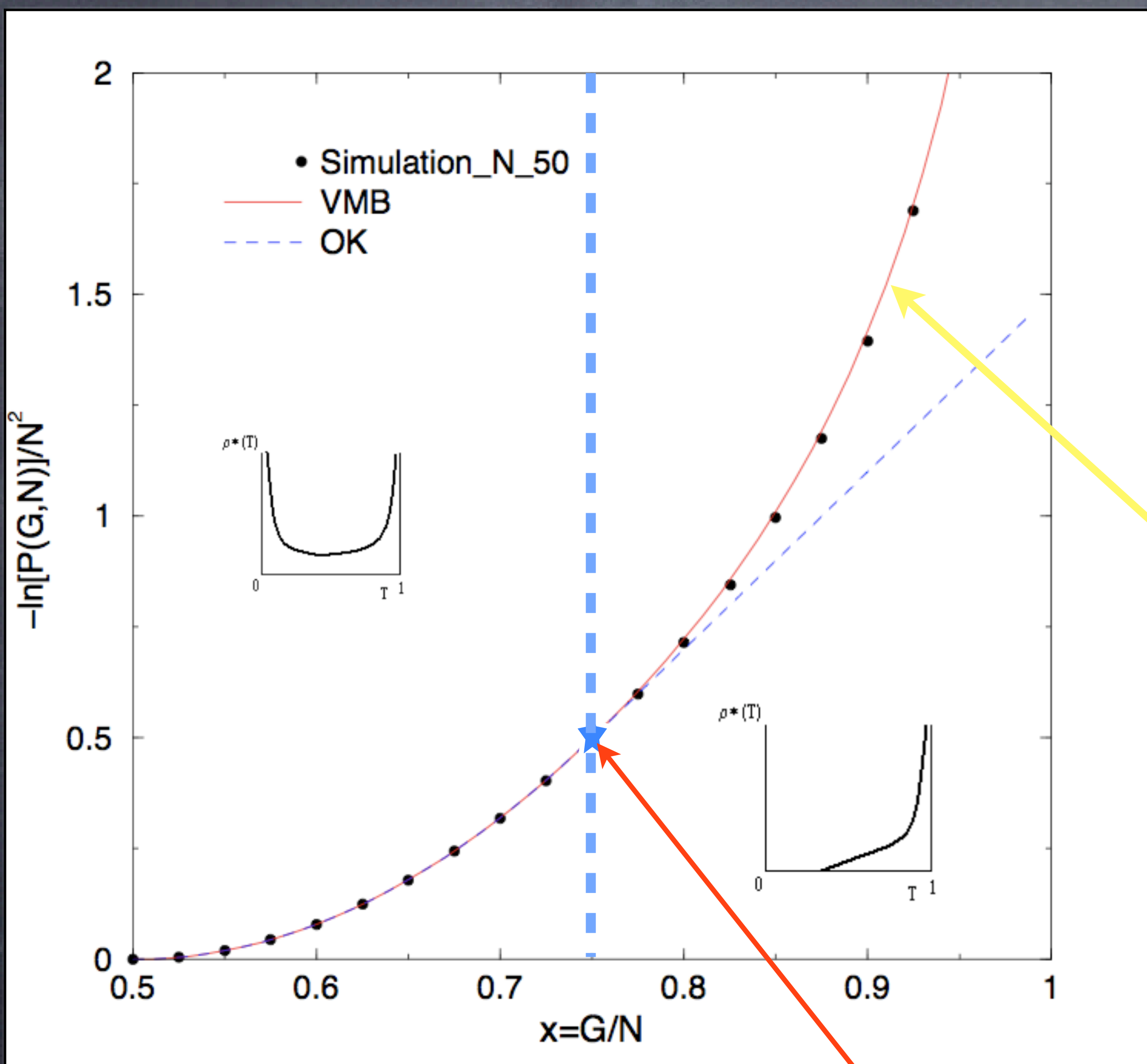


From Laplace space to real space... ---> Gärtner-Ellis Theorem

$$\mathcal{P}(G,N) \approx e^{-\frac{\beta}{2}N^2\Psi_G\left(\frac{G}{N}\right)}$$

$$\Psi_G(x) = \max_p\left[J_G(p) - p\,x\right]$$

Numerical Simulations



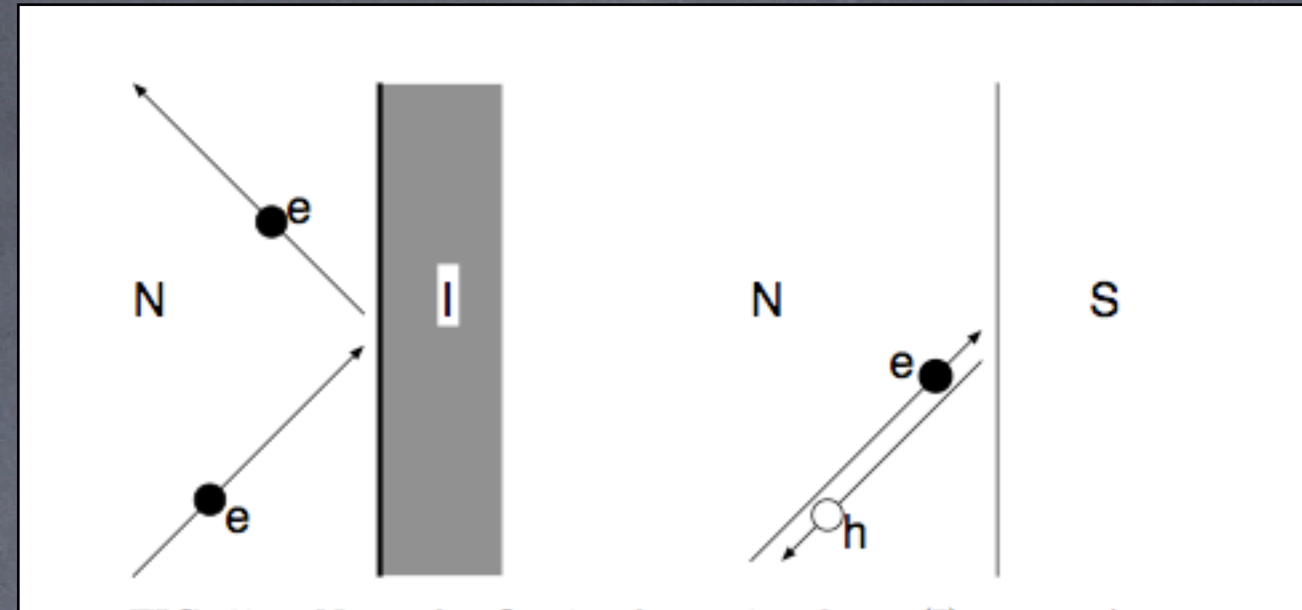
$$\Psi_G(x) = \begin{cases} \frac{1}{2} - \log(4x) & \text{for} \quad 0 \leq x \leq \frac{1}{4} \\ 8\left(x - \frac{1}{2}\right)^2 & \text{for} \quad \frac{1}{4} \leq x \leq \frac{3}{4} \\ \frac{1}{2} - \log[4(1-x)] & \text{for} \quad \frac{3}{4} \leq x \leq 1 \end{cases}$$

3rd derivative is discontinuous at critical points!

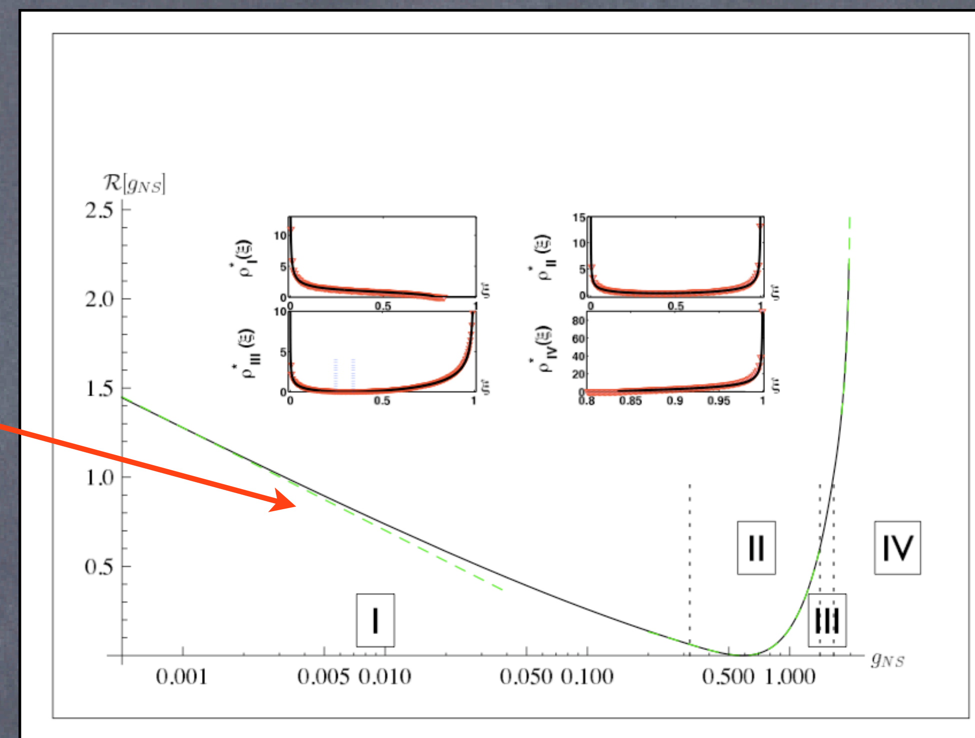
Andreev Reflection

$$G_{\rm NS} = 2\sum_{n=1}^{N_c}\left(\frac{T_n}{2-T_n}\right)^2$$

[Beenakker, 1992]



$$\mathcal{P}(G_{\rm NS}, N_c) \approx \exp\left(-N_c^2 \mathcal{R}(g_{\rm NS})\right)$$



Phase transitions in the distribution of the Andreev conductance of superconductor-metal junctions with many transverse modes.

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Summary

- Random Scattering Matrix approach to quantum transport in chaotic cavities
- Full probability distribution of experimental observables (conductance, shot noise, moments) when number of electronic channels is large
- Solution: canonical partition function of an auxiliary thermodynamical system (Coulomb gas)
- Phase Transitions in the equilibrium gas density

  → Weak non-analytic points in the distribution

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Distributions of Conductance and Shot Noise and Associated Phase Transitions

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Probability distributions of linear statistics in chaotic cavities and associated phase transitions

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FAST TRACK COMMUNICATION

Transmission eigenvalue densities and moments in chaotic cavities from random matrix theory

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Phase transitions in the distribution of the Andreev conductance of superconductor-metal junctions with many transverse modes.

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