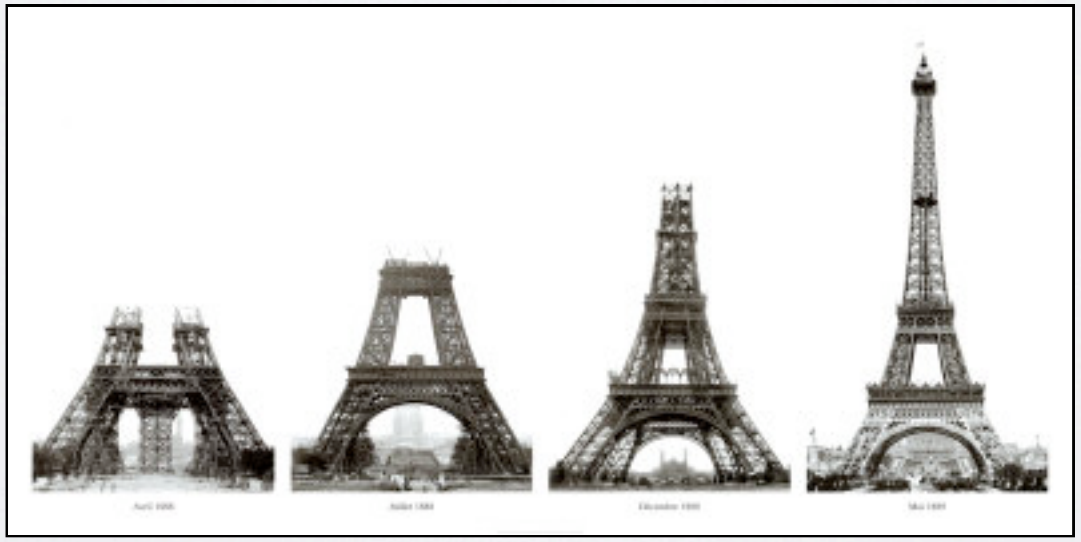
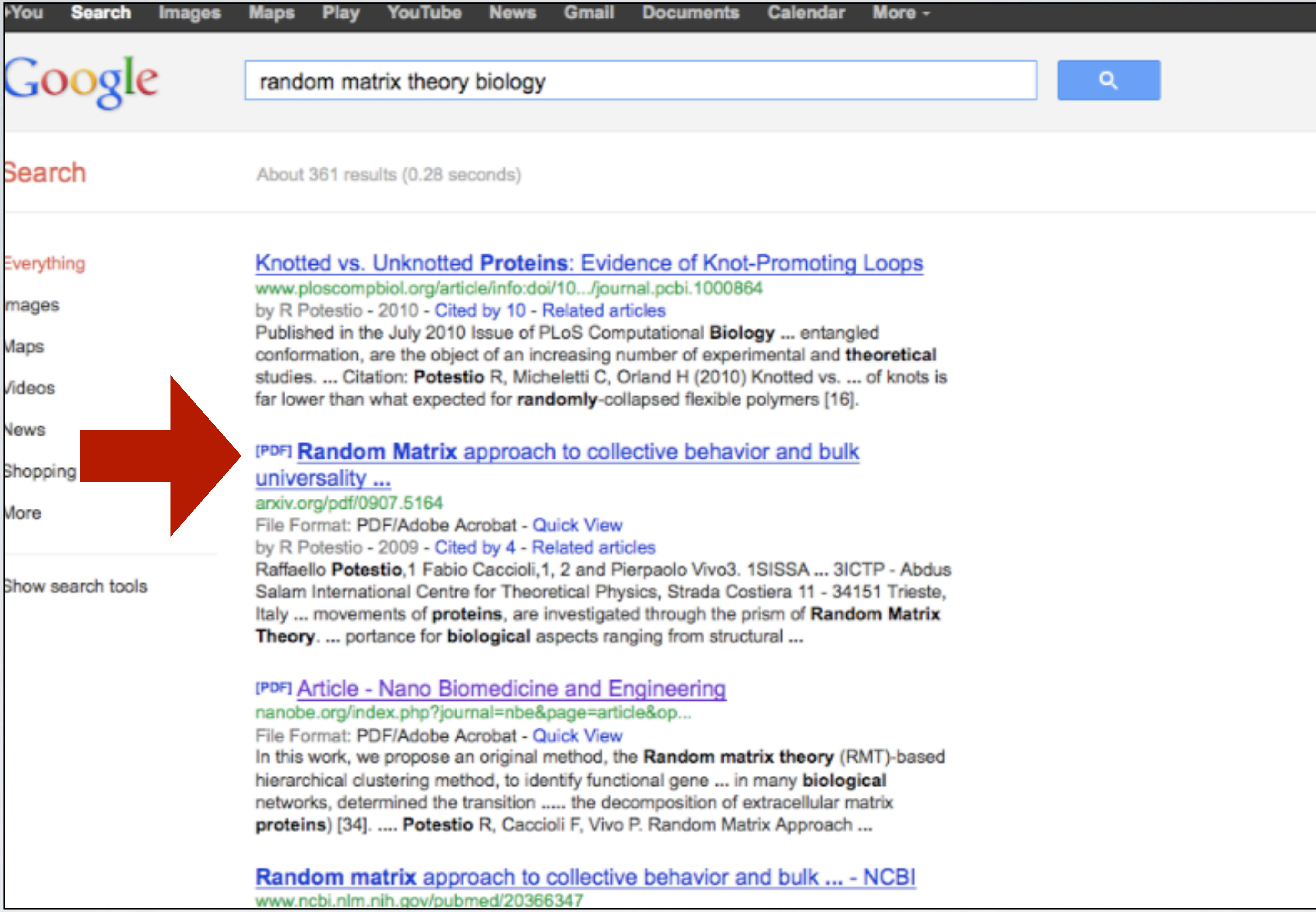
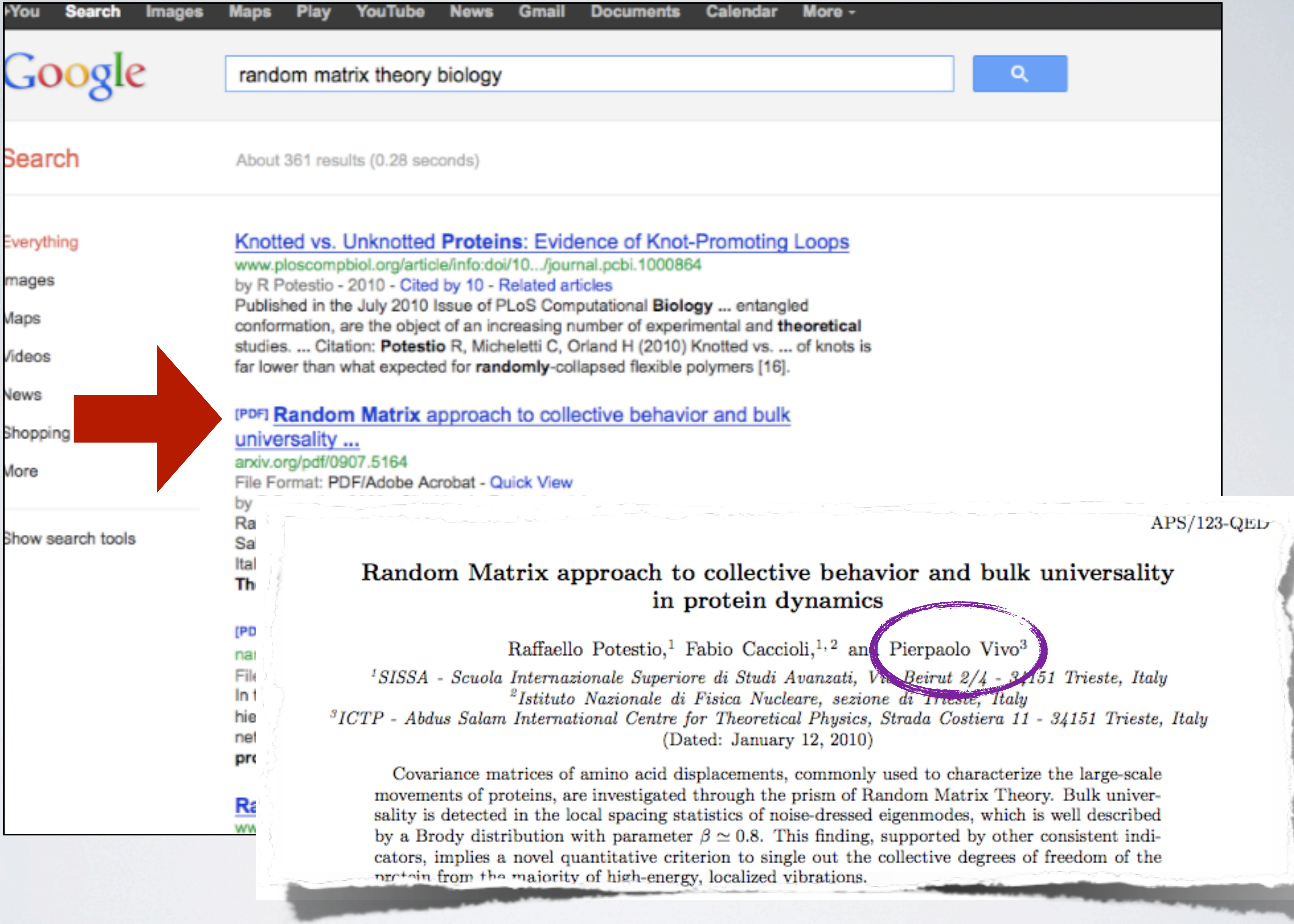
RANDOM MATRIX THEORY: OLD TRICKS FOR NEW DOGS

Pierpaolo Vivo
(LPTMS - CNRS - Paris XI)




You Search Images Maps Play YouTube News Gmail Documents Calendar More
Google
random matrix theory biology
Search
About 361 results (0.28 seconds)
Everything
Images
Maps
Videos
News
Shopping
More
Show search tools
Knotted vs. Unknotted Proteins: Evidence of Knot-Promoting Loops
www.ploscompbiol.org/article/info:doi/10.../journal.pcbi.1000864
by R Potestio - 2010 - Cited by 10 - Related articles
Published in the July 2010 Issue of PLoS Computational Biology ... entangled conformation, are the object of an increasing number of experimental and theoretical studies. ... Citation: Potestio R, Micheletti C, Orland H (2010) Knotted vs. ... of knots is far lower than what expected for randomly-collapsed flexible polymers [16].
[PDF] Random Matrix approach to collective behavior and bulk universality ...
arxiv.org/pdf/0907.5164
File Format: PDF/Adobe Acrobat - Quick View
by R Potestio - 2009 - Cited by 4 - Related articles
Raffaello Potestio,1 Fabio Caccioli,1, 2 and Pierpaolo Vivo3. 1SISSA ... 3ICTP - Abdus Salam International Centre for Theoretical Physics, Strada Costiera 11 - 34151 Trieste, Italy ... movements of proteins, are investigated through the prism of Random Matrix Theory. ... portance for biological aspects ranging from structural ...
[PDF] Article - Nano Biomedicine and Engineering
nanobe.org/index.php?journal=nbe&page=article&op...
File Format: PDF/Adobe Acrobat - Quick View
In this work, we propose an original method, the Random matrix theory (RMT)-based hierarchical clustering method, to identify functional gene ... in many biological networks, determined the transition ..... the decomposition of extracellular matrix proteins) [34]. .... Potestio R, Caccioli F, Vivo P. Random Matrix Approach ...
Random matrix approach to collective behavior and bulk ... - NCBI
www.ncbi.nlm.nih.gov/pubmed/20366347


+You Search Images Maps Play YouTube News Gmail Documents Calendar More
Google
random matrix theory biology
Search
About 361 results (0.28 seconds)
Everything
Images
Maps
Videos
News
Shopping
More
Show search tools
Knotted vs. Unknotted Proteins: Evidence of Knot-Promoting Loops
www.ploscompbiol.org/article/info:doi/10.../journal.pcbi.1000864
by R Potestio - 2010 - Cited by 10 - Related articles
Published in the July 2010 Issue of PLoS Computational Biology ... entangled conformation, are the object of an increasing number of experimental and theoretical studies. ... Citation: Potestio R, Micheletti C, Orland H (2010) Knotted vs. ... of knots is far lower than what expected for randomly-collapsed flexible polymers [16].
[PDF] Random Matrix approach to collective behavior and bulk universality ...
arxiv.org/pdf/0907.5164
File Format: PDF/Adobe Acrobat - Quick View
APS/123-QED
Random Matrix approach to collective behavior and bulk universality in protein dynamics
Raffaello Potestio,[1] Fabio Caccioli,[1,2] and Pierpaolo Vivo[3]
[1]SISSA - Scuola Internazionale Superiore di Studi Avanzati, Via Beirut 2/4 - 34151 Trieste, Italy
[2]Istituto Nazionale di Fisica Nucleare, sezione di Trieste, Italy
[3]ICTP - Abdus Salam International Centre for Theoretical Physics, Strada Costiera 11 - 34151 Trieste, Italy
(Dated: January 12, 2010)
Covariance matrices of amino acid displacements, commonly used to characterize the large-scale movements of proteins, are investigated through the prism of Random Matrix Theory. Bulk universality is detected in the local spacing statistics of noise-dressed eigenmodes, which is well described by a Brody distribution with parameter β ≃ 0.8. This finding, supported by other consistent indicators, implies a novel quantitative criterion to single out the collective degrees of freedom of the protein from the majority of high-energy, localized vibrations.

OUTLINE

First Part: Old Tricks

i) The old days... RMT in nuclear physics
ii) 4 applications

- Riemann hypothesis
- Vicious brownian walkers
- Covariance matrices of financial data
- The longest increasing subsequence problem

Second Part: New Dogs

i) Rare events and linear statistics
ii) How many eigenvalues of a random matrix are positive?

Why are random matrix eigenvalues cool?

Message

- Ingredient: Take Any important mathematics
- Then Randomize!
- This will have many applications!

from a talk by Alan Edelman (MIT)

Eugene Wigner

John Wishart

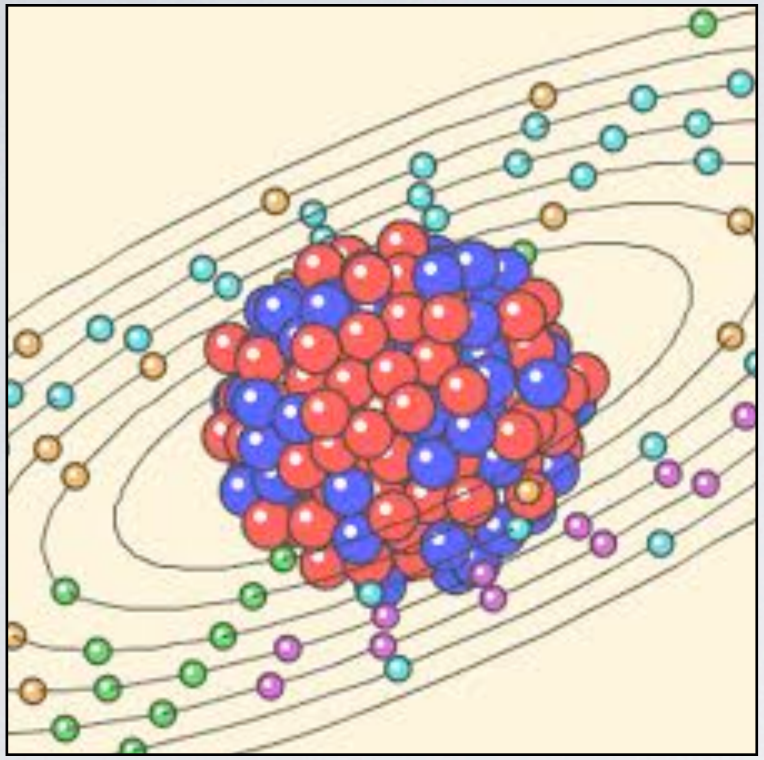
Freeman Dyson

ANNALS OF MATHEMATICS
Vol. 67, No. 2, March, 1958
Printed in Japan

ON THE DISTRIBUTION OF THE ROOTS OF CERTAIN SYMMETRIC MATRICES

BY EUGENE P. WIGNER

(Received September 19, 1957)

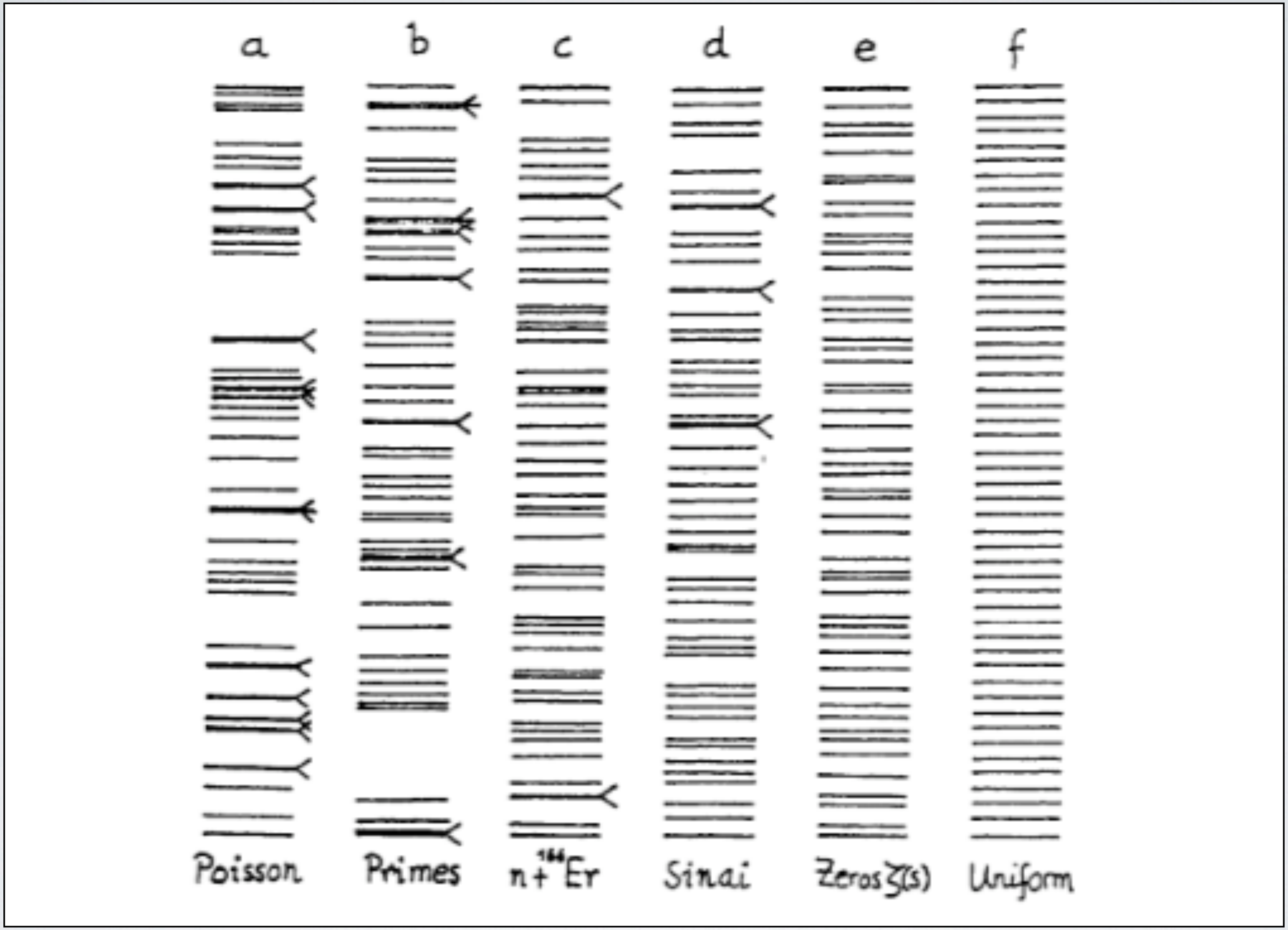


Hamiltonian (total energy) of heavy nuclei: hopeless task!

BUT.....

The Hamiltonian in a given basis is just a **HUGE** matrix....

Idea: take the matrix entries **at random...**

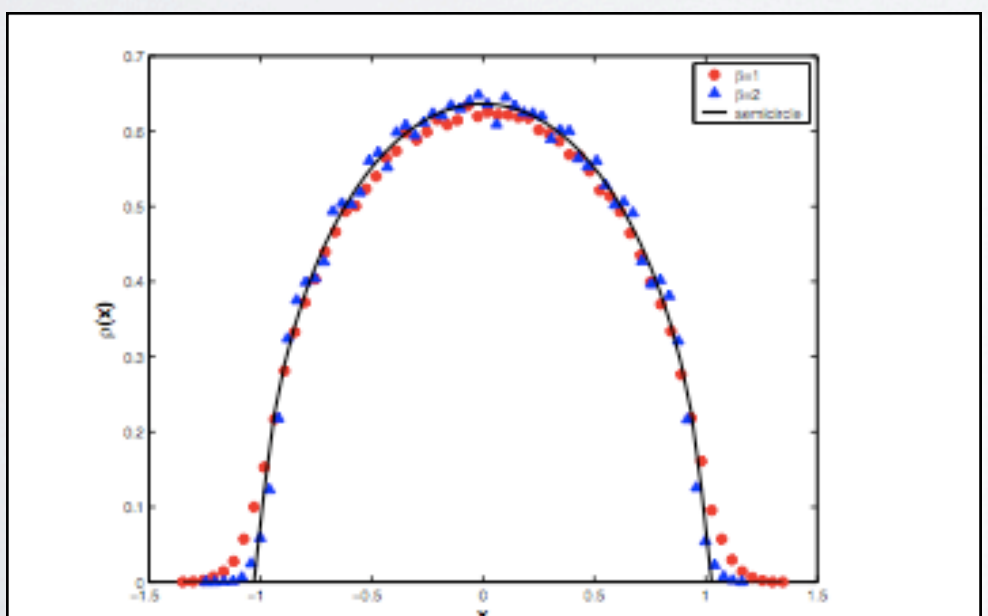

a
b
c
d
e
f
Poisson
Primes
$n + {}^{166}Er$
Sinai
Zeros $\zeta(s)$
Uniform

Random Matrix Theory = Randomness + Symmetry

N = 5

$$\begin{pmatrix} 0.5377 & 0.2631 & -1.8044 & 0.3286 & 0.4951 \\ 0.2631 & -0.4336 & 1.6888 & 1.7271 & 0.7810 \\ -1.8044 & 1.6888 & 0.7254 & 0.7133 & 0.7160 \\ 0.3286 & 1.7271 & 0.7133 & 1.4090 & 1.5237 \\ 0.4951 & 0.7810 & 0.7160 & 1.5237 & 0.4889 \end{pmatrix}$$

$$\vec{\lambda} = \begin{bmatrix} -2.4341 & -0.8386 & -0.5203 & 2.2594 & 4.2610 \end{bmatrix}$$



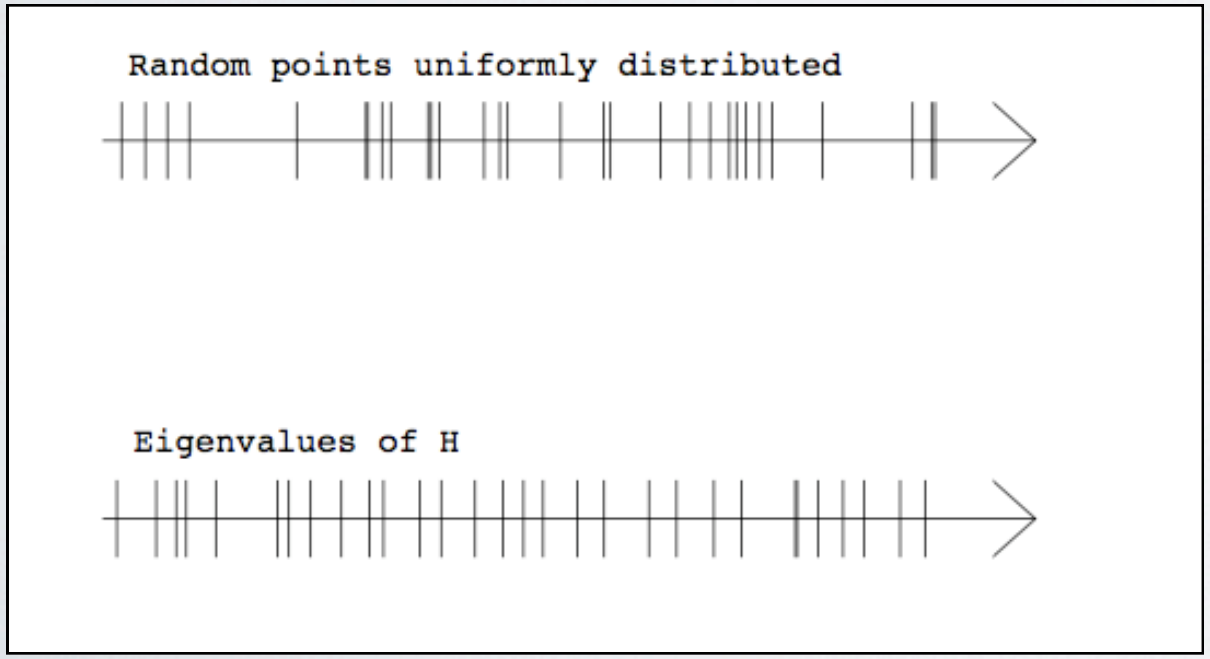
Semicircle Law

$$\rho(\lambda) = \frac{1}{2\sqrt{N}} f\left(\frac{\lambda}{2\sqrt{N}}\right)$$

$$f(x) = \frac{2}{\pi}\sqrt{1 - x^2}$$

Typical questions:

(i) Density of eigenvalues
(ii) Gaps between adjacent eigenvalues
(iii) Distribution of individual eigenvalues (e.g. largest)
(iv) Probability of **rare** events in **linear statistics**



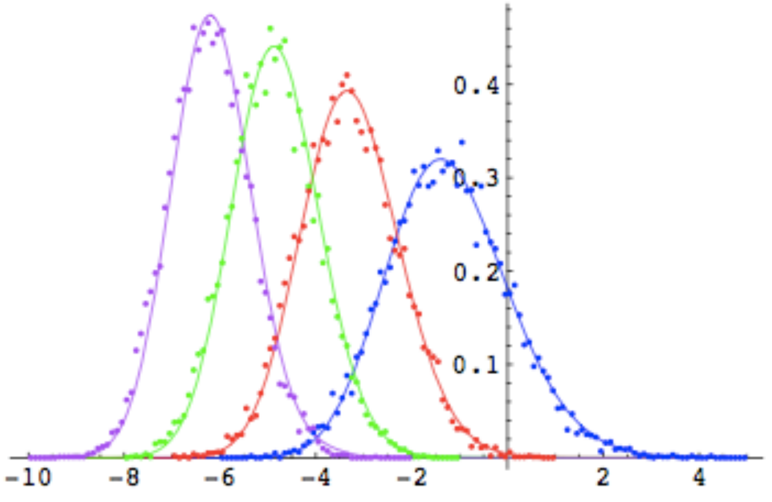
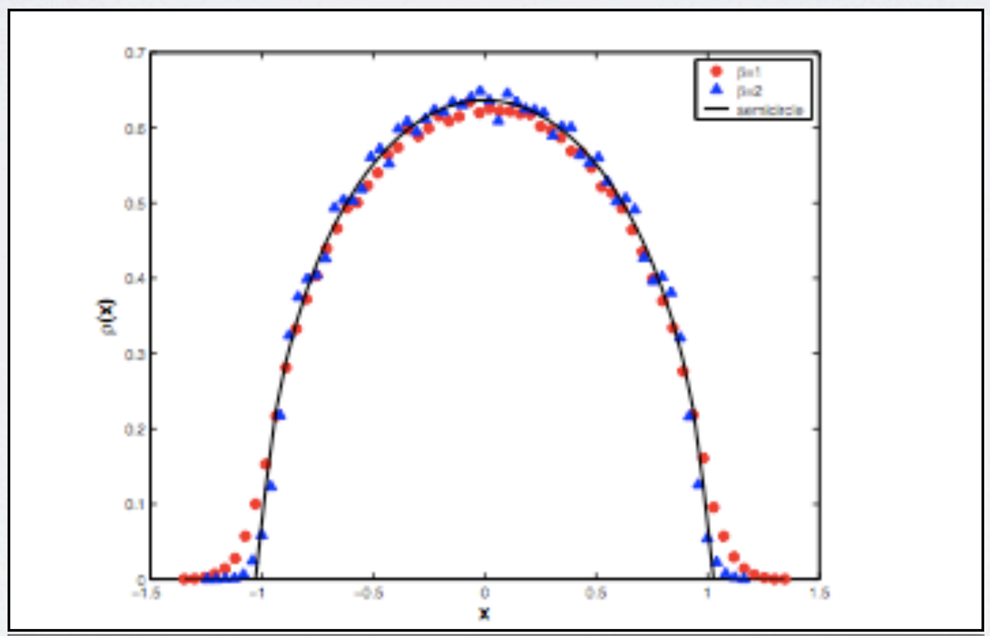


Figure 2: A histogram of the four largest (centered and normalized) eigenvalues for 10^4 realizations of $10^3 \times 10^3$ GOE matrices. Solid curves are the limiting distributions from [11]. Figure a courtesy of Momar Dieng.

$$\mathbb{P}_{\beta,N}(x_1,\dots,x_N) = C_{N,\beta} \prod_{1\leq i<j\leq N} |x_i - x_j|^\beta \prod_{i=1}^{N} e^{-\beta x_i^2/2}, \ \beta = 1,2,4,$$

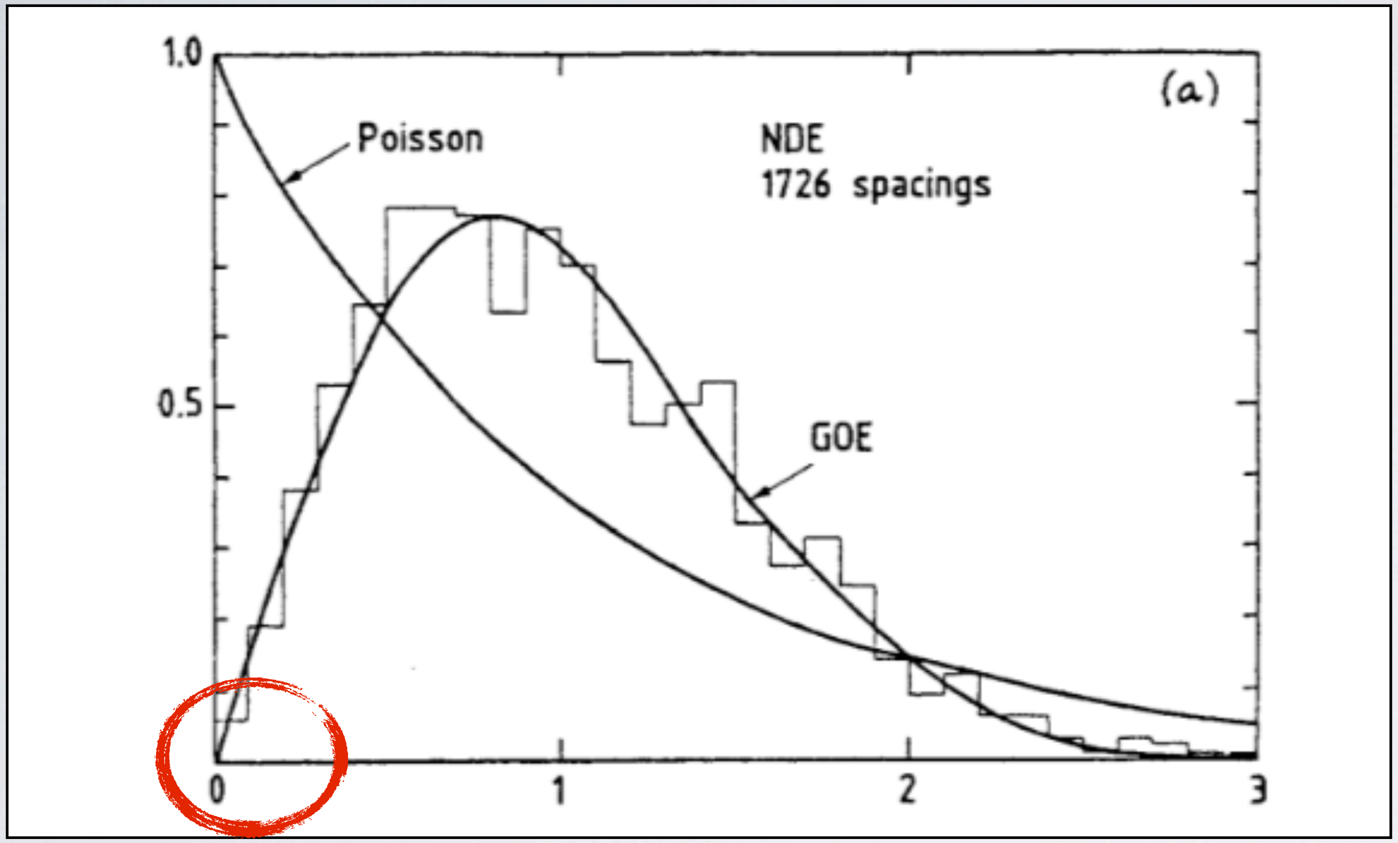
Level Repulsion

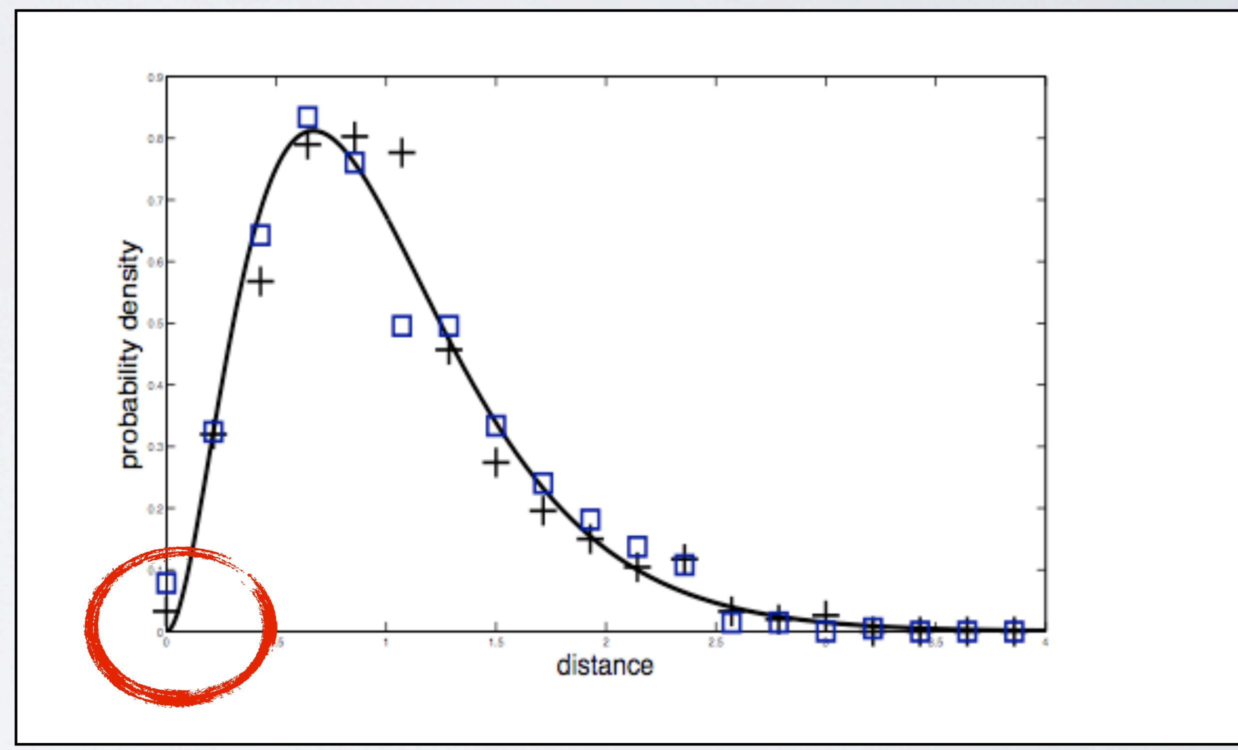
Confinement



Strongly Correlated Random Variables!!

Level Spacings: universality





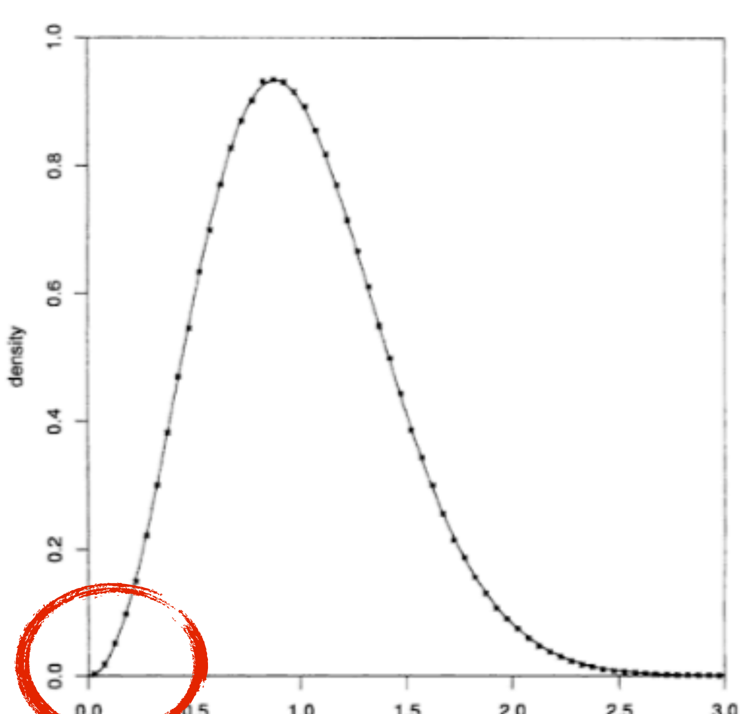


Figure 6. *A comparison of the statistics of spacings between conecutive eigenvalues of random matrices (solid curve) and nearest-neighbor spacings of zeroes of the Riemann zeta function (data points). The fit is not bad for the first million zeroes (left), but nearly perfect for the spacings among 1,041,600 zeroes near the 2×10^{20}th zero. (Figure*

$$\mathcal{P}(s) \propto s^{\beta} e^{-s^2}$$

Vol. 112 (2007) *ACTA PHYSICA POLONICA A* No. 4

Proceedings of the 3rd Workshop on Quantum Chaos and Localisation Phenomena
Warsaw, Poland, May 25–27, 2007

Parking in the City

P. Šeba[a,b,c]

[a]University of Hradec Králové, Hradec Kr

ELSEVIER

Physica A: Statistical Mechanics and its Applications

Volume 346, Issues 3–4, 15 February 2005, Pages 621–630

Modelling the gap size distribution of parked cars

S. Rawal, G.J. Rodgers

Department of Mathematical Sciences, Brunel University, Uxbridge, Middlesex UB8 3PH, UK

Received 19 July 2004, Available online

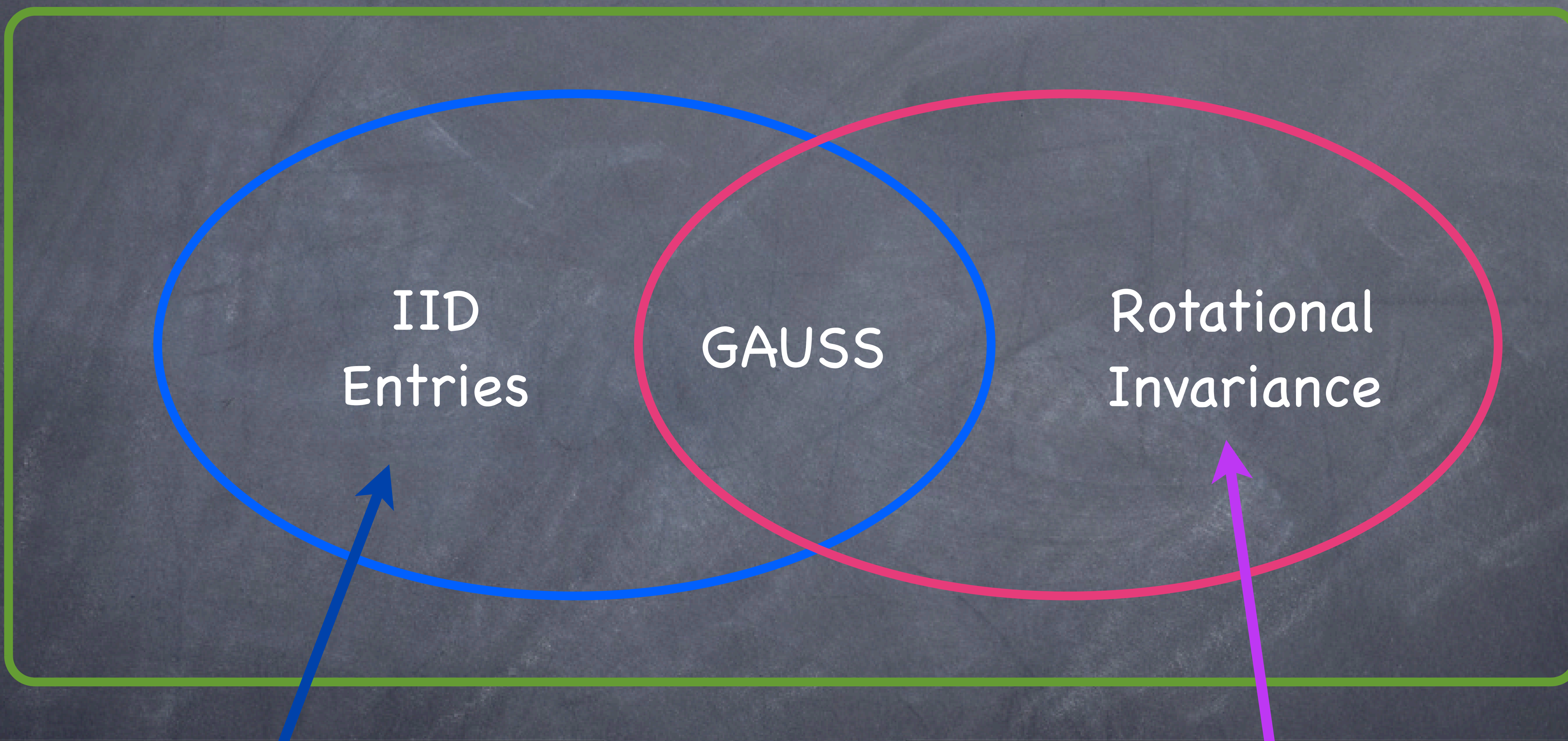
Modelling gap-size distribution of parked cars using random-matrix theory

A.Y. Abul-Magd
Department of Mathematics, Faculty of Science, Zagazig University, Zagazig, Egypt

We apply the random-matrix theory to the car-parking problem. For this purpose, we adopt a Coulomb gas model that associates the coordinates of the gas particles with the eigenvalues of a random matrix. The nature of interaction between the particles is consistent with the tendency of the drivers to park their cars near to each other and in the same time keep a distance sufficient for manoeuvring. We show that the recently measured gap-size distribution of parked cars in a number of roads in central London is well represented by the spacing distribution of a Gaussian unitary ensemble.

PACS: 05.40; 05.20.Gg; 02.50.r; 68.43.-h
Keywords: Car parking; Coulomb gas; Gaussian unitary ensemble

The fundamental diagram (not in textbooks)



$$\mathcal{P}(x_{11}, \ldots, x_{NN}) = \prod_{ij} p(x_{ij})$$

$$\mathcal{P}(\mathbf{X}) = \mathcal{P}(\mathbf{U}\mathbf{X}\mathbf{U}^{-1})$$

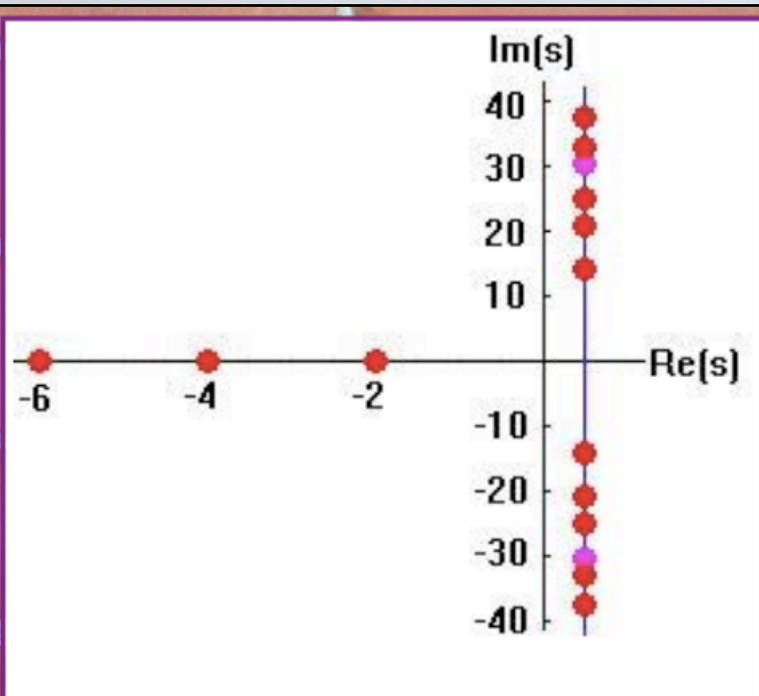
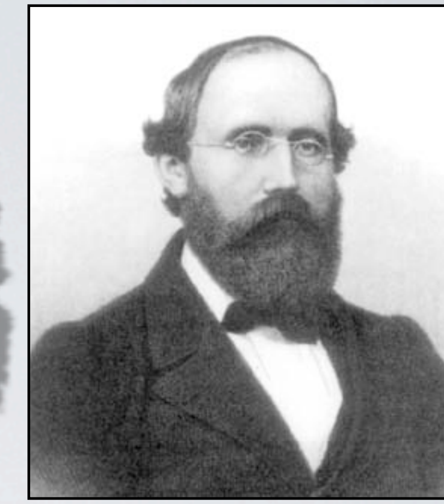


Figure 1: The zeros of the ζ-function.

$$\sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p \text{ prime}} \frac{1}{1-p^{-s}}$$

liegenden Wurzeln von $\xi(t) = 0$ multiplicirt mit $2\pi i$. Man findet nun in der That etwa so viel reelle Wurzeln innerhalb dieser Grenzen, und es ist sehr wahrscheinlich, dass alle Wurzeln reell sind. Hiervon wäre allerdings ein strenger Beweis zu wünschen; ich habe indess die Aufsuchung desselben nach einigen flüchtigen vergeblichen Versuchen vorläufig bei Seite gelassen, da er für den nächsten Zweck meiner Untersuchung entbehrlich schien.



...it is very probable that all roots are real. One would, however, wish for a strict proof of this; I have, though, after some fleeting futile attempts, provisionally put aside the search for such, as it appears unnecessary for the next objective of my investigation.

"Sometimes I think that we essentially have a complete proof of the Riemann Hypothesis except for a gap. The problem is, the gap occurs right at the beginning, and so it's hard to fill that gap because you don't see what's on the other side of it."

Montgomery's Pair Correlation Conjecture

Montgomery's pair correlation conjecture, published in 1973, asserts that the two-point correlation function $R_2(r)$ for the zeros of the Riemann zeta function $\zeta(z)$ on the critical line is

$$R_2(r) = 1 - \frac{\sin^2(\pi r)}{(\pi r)^2}.$$

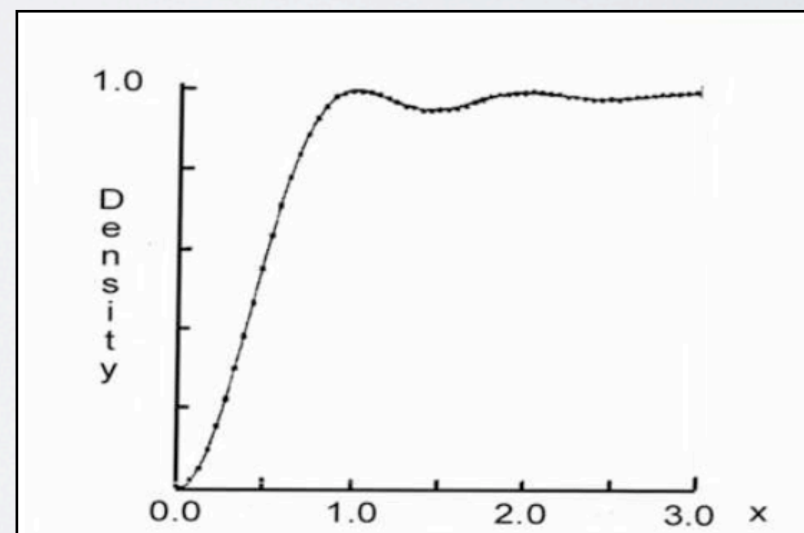
As first noted by Dyson, this is precisely the form expected for the pair correlation of random Hermitian matrices (Derbyshire 2004, pp. 287-291).

In 1972, Hugh Montgomery, a number theorist at the University of Michigan, was visiting the Institute for Advanced Study. Montgomery had been studying the distribution of zeroes of the zeta function, in hopes of gaining insight into the Riemann Hypothesis. He was able to prove that the Riemann Hypothesis had implications for the spacing of zeroes along the critical line, but his key discovery was an additional property that the zeroes seemed to have, one which implied a particularly nice formula for the average spacing between zeroes.

Odlyzko's computations agree amazingly well with Montgomery's conjecture.

During tea one day at the Institute, Montgomery was introduced to Dyson and described his conjecture. Dyson immediately recognized it as the same result as had been obtained for random matrices.

"It just so happened that he was one of the two or three physicists in the world who had worked all of these things out, so I was actually talking to the greatest expert in exactly this!" Montgomery recalls.



Simple Proof of Riemann's Hypothesis of the Zeta function

by [redacted], 6/20/2006

The Zeta function is defined as:

$$\zeta(s)=\sum_{k=1}^{\infty}\frac{1}{k^s}=1+\frac{1}{2^s}+\frac{1}{3^s}+\frac{1}{4^s}+\frac{1}{5^s}+\ldots\; s\neq 1\;(k=1\to\infty)$$

Hardy, 1999 showed that:

$0=\xi(1-s)=\xi(s)$ *where* $\xi(s)$ *extends* $\zeta(s)$ *into*

for all $0<s<1$ *at the "non-trivial" zeros*

$$0=\sum_{k=1}^{\infty}\frac{1}{k^{(1-s)}}=\sum_{k=1}^{\infty}\frac{1}{k^s}\quad \textit{for all } 0<s<1,\ \textit{positive}$$

$$0=\sum_{k=1}^{\infty}\frac{1}{k^s}-\sum_{k=1}^{\infty}\frac{1}{k^{(1-s)}}$$

$$0=\sum_{k=1}^{\infty}\left(\frac{1}{k^s}-\frac{1}{k^{(1-s)}}\right)$$

$$0=\sum_{k=1}^{\infty}\left(\frac{k^{(1-s)}-k^s}{k^s k^{(1-s)}}\right)$$

$$0=\sum_{k=1}^{\infty}\left(\frac{k^{(1-s)}-k^s}{k}\right)\Rightarrow 0=\sum_{k=1}^{\infty}\left(\frac{0}{k}\right)\quad \textit{if}\ \exists\ s$$

$$\Rightarrow k^{(1-s)}=k^s\Rightarrow(1-s)=s\quad \textit{for}\ k=1\to\infty,\ 0<s<1$$

substituting complex s $(1-(\sigma+ix))=(\sigma+ix)$

for real $\sigma=1-\sigma\Rightarrow 2\sigma=1$

$$\Rightarrow\sigma=\frac{1}{2}\Rightarrow\sigma+ix=\frac{1}{2}-ix\Rightarrow\Re[s]=\frac{1}{2}\quad \textit{for}\ k=1\to\infty,\ 0<s<1,\ \textit{'critical strip'}$$

QED

Hardy, G. H. Ramanujan: Twelve Lectures on Subjects Suggested by His Life and Work, 3rd ed. New York: Chelsea, 1999.

[0807.0090] A proof of the Riemann hypothesis

http://arxiv.org/abs/0807.0090

Cornell University Library

We gratefully acknowledge supporting institutions

arXiv.org > math > arXiv:0807.0090

Mathematics > Number Theory

A proof of the Riemann hypothesis

Xian-Jin Li

(Submitted on 1 Jul 2008 (v1), last revised 6 Jul 2008 (this version, v4))

This paper has been withdrawn by the author, due to a mistake on page 29.

Comments: withdrawn by author, due to mistake on page 29
Subjects: **Number Theory (math.NT)**
MSC classes: 11M26
Cite as: **arXiv:0807.0090v4 [math.NT]**

Submission history

From: Xian-Jin Li [view email]
[v1] Tue, 1 Jul 2008 19:43:13 GMT (20kb)
[v2] Wed, 2 Jul 2008 11:05:52 GMT (20kb)
[v3] Thu, 3 Jul 2008 03:44:03 GMT (20kb)
[v4] Sun, 6 Jul 2008 12:40:00 GMT (0kb,I)

Which authors of this paper are endorsers?

Download:
- Source

Current browse context: math.NT
< prev | next >
new | recent | 0807

Change to browse by: math

References & Citations
- NASA ADS

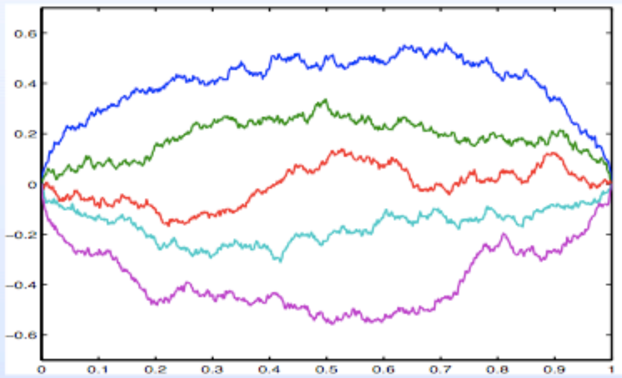
14 blog links

Bookmark

Link back to: arXiv, form interface, contact.

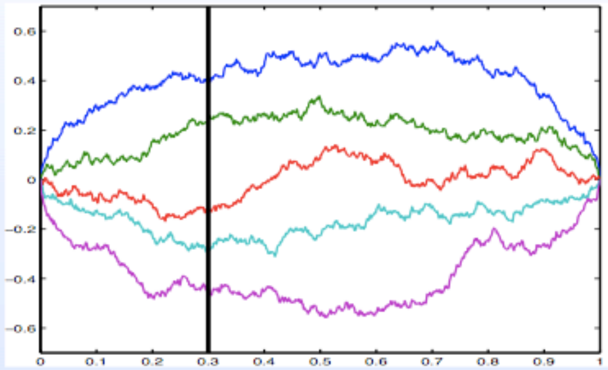
Non-intersecting Brownian motion paths

- Take n independent 1-dimensional Brownian motions with time in $[0, 1]$ conditioned so that:
  - All paths start and end at the same point.
  - The paths **do not intersect** at any intermediate time.



Five non-intersecting Brownian bridges

- **Remarkable fact**: At any intermediate time the positions of the paths have **exactly the same distribution** as the eigenvalues of an $n \times n$ GUE matrix (up to a scaling factor).



Positions of five non-intersecting Brownian paths behave the same as the eigenvalues of a 5×5 GUE matrix

- This interpretation is basic for the connection of random matrix theory with growth models of statistical physics.

Introduction. Since the pioneering work of de Gennes [1], followed up by Fisher [2], the subject of vicious (non-intersecting) random walkers has attracted a lot of interest among physicists. It has been studied in the context of wetting and melting [2], networks of polymers [3] and fibrous structures [1], persistence properties in nonequilibrium systems [4] and stochastic growth models [5, 6]. There also exist connections between the

PHYSICAL REVIEW E VOLUME 52, NUMBER 6 DECEMBER 1995

Vicious walkers and directed polymer networks in general dimensions

J. W. Essam
Department of Mathematics, Royal Holloway, University of London, Egham Hill, Egham, Surrey TW20 0EX, United Kingdom

A. J. Guttmann
Department of Mathematics, The University of Melbourne, Parkville, Victoria 3052, Australia

Random Covariance Matrices

phys. math

$$X = \begin{array}{c} 1 \\ 2 \\ 3 \end{array} \begin{vmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \\ x_{31} & x_{33} \end{vmatrix}$$

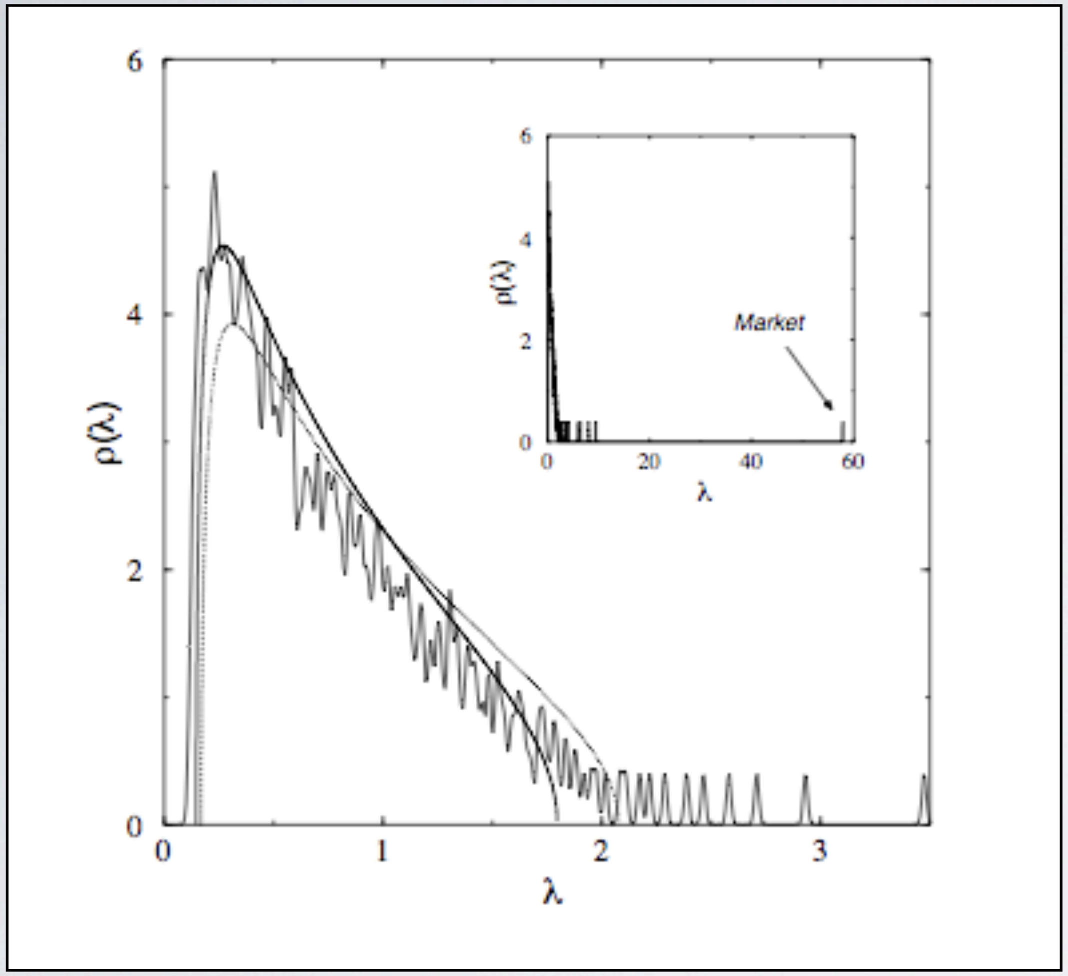
in general (MxN)

$$X^t = \begin{vmatrix} x_{11} & x_{21} & x_{31} \\ x_{12} & x_{22} & x_{33} \end{vmatrix}$$

in general (NxM)

$$W = X^t X = \begin{vmatrix} x_{11}^2 + x_{21}^2 + x_{31}^2 & x_{11} x_{12} + x_{21} x_{22} + x_{31} x_{33} \\ x_{12} x_{11} + x_{22} x_{21} + x_{33} x_{31} & x_{12}^2 + x_{22}^2 + x_{33}^2 \end{vmatrix}$$

(NxN) COVARIANCE MATRIX (unnormalized)



VOLUME 83, NUMBER 7 PHYSICAL REVIEW LETTERS 16 AUGUST 1999

Noise Dressing of Financial Correlation Matrices

Laurent Laloux,[1,*] Pierre Cizeau,[1] Jean-Philippe Bouchaud,[1,2] and Marc Potters[1]
[1]*Science & Finance, 109-111 rue Victor Hugo, 92532 Levallois Cedex, France*
[2]*Service de Physique de l'État Condensé, Centre d'études de Saclay, Orme des Merisiers, 91191 Gif-sur-Yvette Cedex, France*

time. From this point of view, it is interesting to compare the properties of an empirical correlation matrix **C** to a null hypothesis purely *random* matrix as one could obtain from a finite time series of strictly independent assets. Deviations from the random matrix case might then suggest the presence of true information. The theory of random matri-

Debate: is the bulk of the stock market correlation matrix just pure noise?

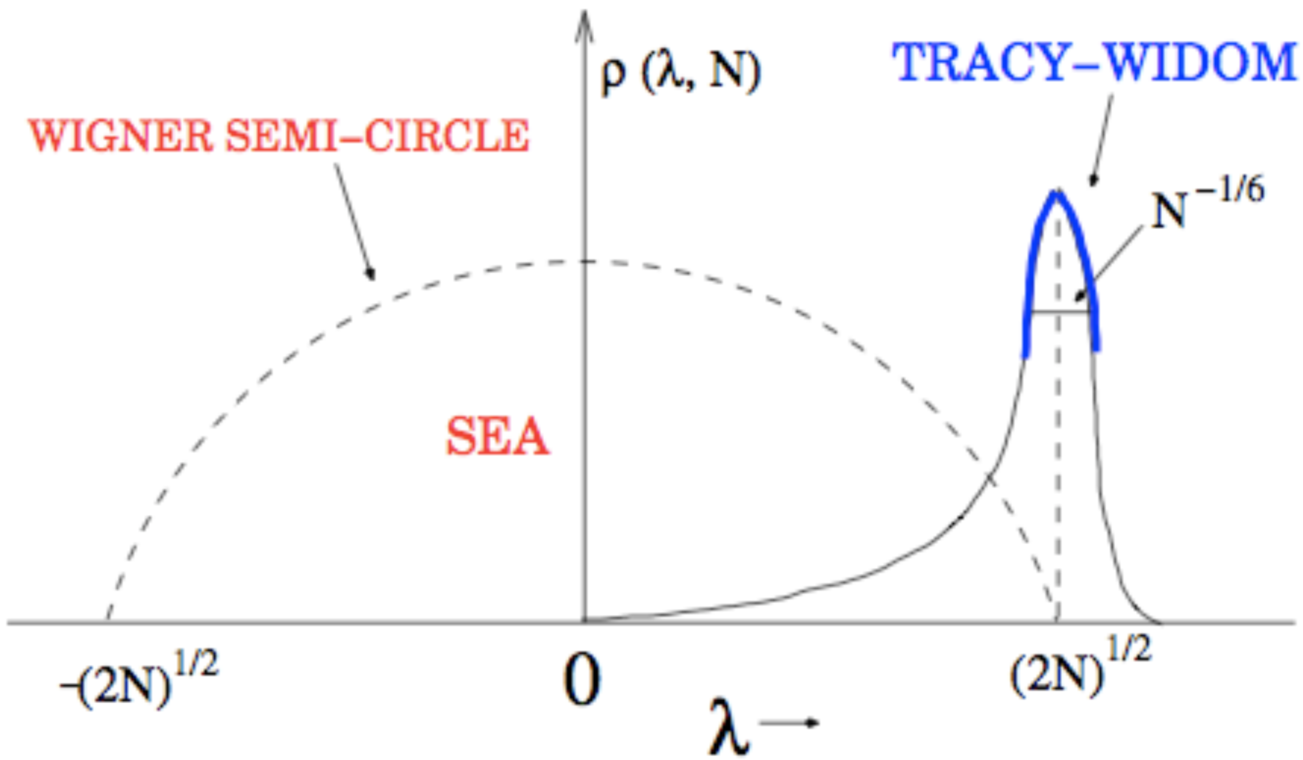
INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL
J. Phys. A: Math. Gen. **36** (2003) 3009–3032 PII: S0305-4470(03)37794-7

A new method to estimate the noise in financial correlation matrices

Thomas Guhr[1] and Bernd Kälber[2,3]

LARGEST EIGENVALUE

Tracy-Widom distribution for $\lambda_{\max}$



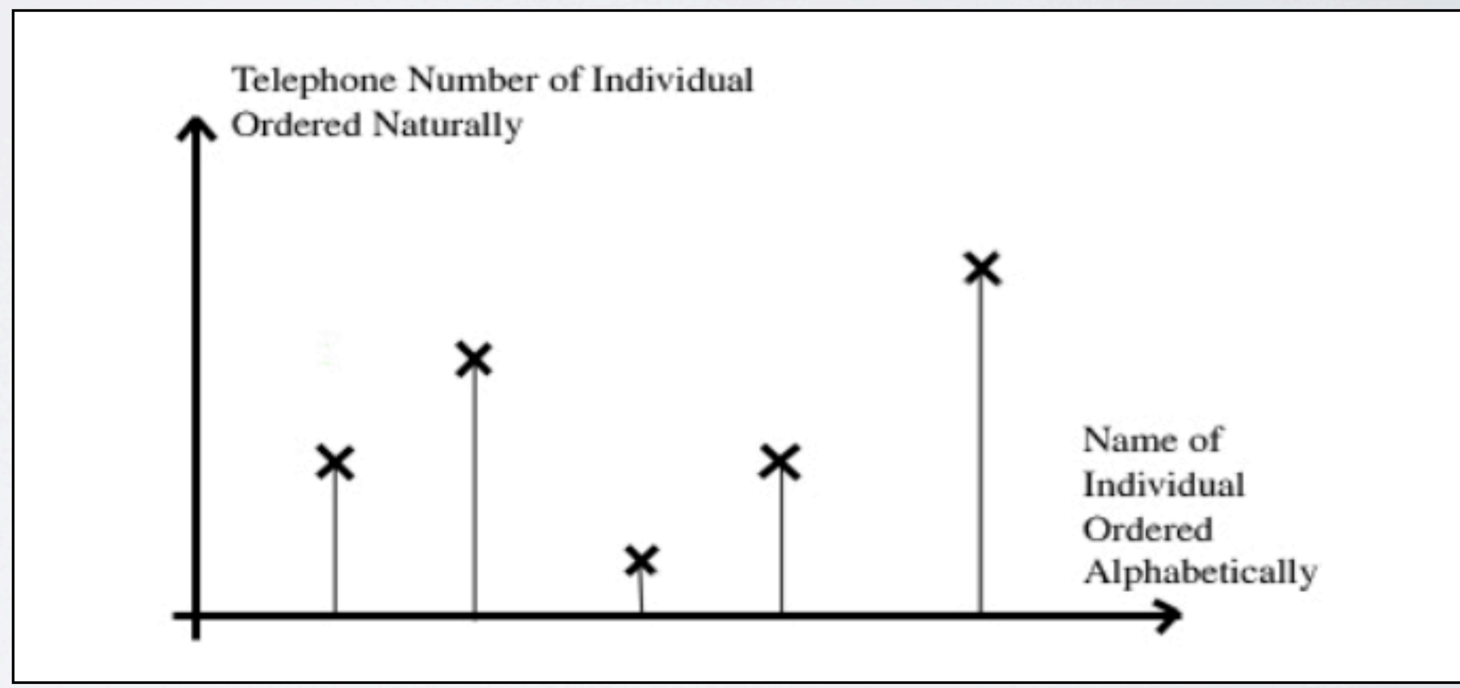
- $\langle \lambda_{\max} \rangle = \sqrt{2N}$; typical fluctuation: $|\lambda_{\max} - \sqrt{2N}| \sim N^{-1/6}$ (small)
- typical fluctuations are distributed via Tracy-Widom (1994):
- cumulative distribution:

$$\mathrm{Prob}[\lambda_{\max} \leq t, N] \to F_\beta\left(\sqrt{2}N^{1/6}(t - \sqrt{2N})\right)$$

- Prob. density (pdf): $f_\beta(z) = dF_\beta(z)/dz$
- $F_\beta(z) \to$ obtained from solution of Painlevé-II equation

“Are Tracy and Widom in Your Local Telephone Directory?”

Ryan Witko
Advisor: Percy Deift



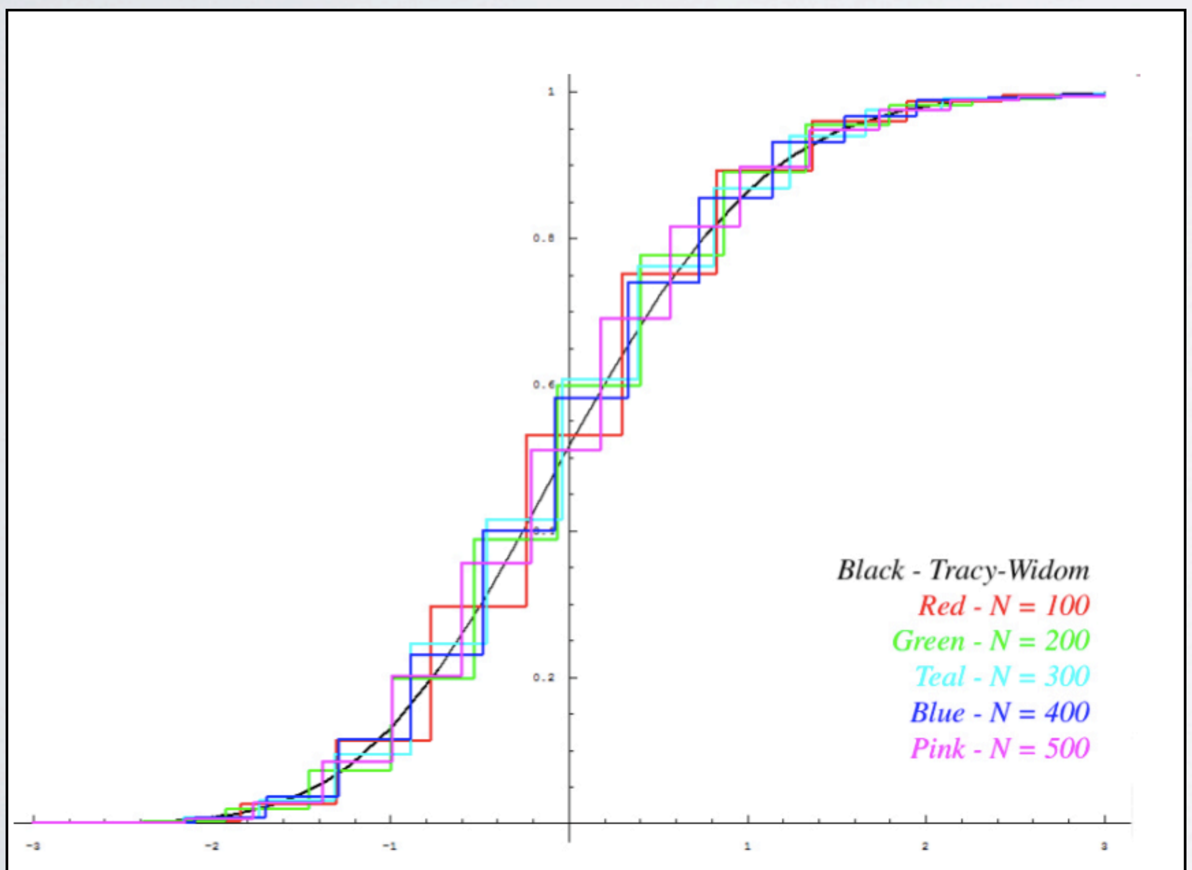
Definition:

The longest increasing (contiguous) subsequence of a given sequence is the subsequence of increasing terms containing the largest number of elements. For example, the longest increasing subsequence of the permutation {6, 3, 4, 8, 10, 5, 7, 1, 9, 2} is {3, 4, 8, 10}.

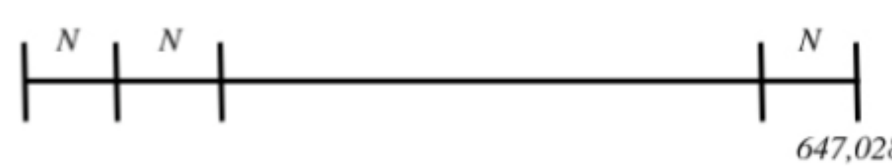
It can be coded in *Mathematica* as follows.

```
<<Combintorica`
LongestContinguousIncreasingSubsequence[p_] :=
  Last[
  Split[Sort[Runs[p]], Length[#1] >= Length[#2]&]
  ]
```

More Information »



We broke the 647,028 entries into successive samples each containing *N* entries.



Jinho Baik, Kurt Johansson and Percy Deift showed that as $N \to \infty$

(3)
$$\mathrm{Prob}\left(\frac{\ell_N - 2\sqrt{N}}{N^{1/6}} \le t\right) \to F(t)$$

The function *F(t)* was shown by Craig Tracy and Harold Widom to be the distribution of the largest eigenvalue of a random matrix in the Gaussian Unitary Ensemble (GUE). It

SUMMARY

- Eigenvalues of random matrices: strongly correlated
- Level Repulsion
- Tracy-Widom distribution: analogue of Gaussian distribution for correlated random variables
- Zeros of Riemann zeta have the same statistical properties as the eigenvalues of Gaussian matrices
- Non-intersecting Brownian bridges
- Wishart matrices: covariance matrices of random data

SECOND PART

Probability of **rare** events in **linear statistics**

"Lies, damned lies, and statistics."

A simple example of large deviation tails

- Let $M \to$ no. of heads in N tosses of an unbiased coin
- Clearly $P(M,N) = \binom{N}{M} 2^{-N}$ $(M = 0, 1, \ldots, N) \to$ binomial distribution

  with mean= $\langle M \rangle = \frac{N}{2}$ and variance=$\sigma^2 = \langle \left(M - \frac{N}{2}\right)^2 \rangle = \frac{N}{4}$
- typical fluctuations $M - \frac{N}{2} \sim O(\sqrt{N})$ are well described by the Gaussian form: $P(M,N) \sim \exp\left[-\frac{2}{N}\left(M - \frac{N}{2}\right)^2\right]$
- Atypical large fluctuations $M - \frac{N}{2} \sim O(N)$ are not described by Gaussian form
- Setting $M/N = x$ and using Stirling's formula $N! \sim N^{N+1/2} e^{-N}$ gives

$$P(M = Nx, N) \sim \exp\left[-N\Phi(x)\right] \qquad \text{where}$$

$$\boxed{\Phi(x) = x\log(x) + (1-x)\log(1-x) + \log 2} \to \text{large deviation function}$$

- $\Phi(x) \to$ symmetric with a minimum at $x = 1/2$ and for small arguments $|x - 1/2| << 1$, $\Phi(x) \approx 2(x - 1/2)^2$

  $\to$ recovers the Gaussian form near the peak

LINEAR STATISTICS

$\{x_1, \ldots, x_N\}$ Random Variables

$$A = \sum_{i=1}^{N} f(x_i)$$

Question: what is the distribution of A for large N?

Independent → Central Limit Theorems

Correlated → ?

The first rigorous results concerning large deviations are due to the Swedish mathematician Harald Cramér, who applied them to model the insurance business. From the point of view of an insurance company, the earning is at a constant rate per month (the monthly premium) but the claims X_i come randomly. For the company to be successful over a certain period of time (preferably many months), the total earning should exceed the total claim. Thus to estimate the premium you have to ask the following question : "What should we choose as the premium q such that over N months the total claim C = \Sum_i X_i should be less than Nq ? " Cramér gave a solution to this question for i.i.d. random variables

What if the random variables are strongly correlated?

A Trivial Problem

DIAGONAL MATRIX

$$\begin{bmatrix} x_{11} & & & & \\ & x_{22} & & 0 & \\ & & x_{33} & & \\ & 0 & & \ddots & \\ & & & & x_{NN} \end{bmatrix}$$

$$\Pr[x_{ii} = x] = (2\pi)^{-1/2} \exp[-x^2/2]$$

↓

GAUSSIAN

N Eigenvalues: $\lambda_i = x_{ii} \longrightarrow$ Independent

- $P_N = \mathrm{Prob}[\lambda_1 \leq 0,\ \lambda_2 \leq 0,\ \ldots,\ \lambda_N \leq 0] = 2^{-N} = \exp[-(\ln 2)\, N]$

A Nontrivial Problem

REAL SYMMETRIC MATRIX (NxN)

$$X = \begin{vmatrix} x_{11} & x_{12} & \cdots & x_{1N} \\ x_{21} & x_{22} & \cdots & x_{2N} \\ \vdots & \vdots & & \vdots \\ x_{N1} & \cdots & \cdots & x_{NN} \end{vmatrix}$$

GAUSSIAN

↓

$$\Pr[X] \propto \exp[-\tfrac{1}{2}\mathrm{Tr}(X^2)]$$

N eigenvalues : $\lambda_1, \lambda_2, \ldots, \lambda_N$

→ strongly correlated

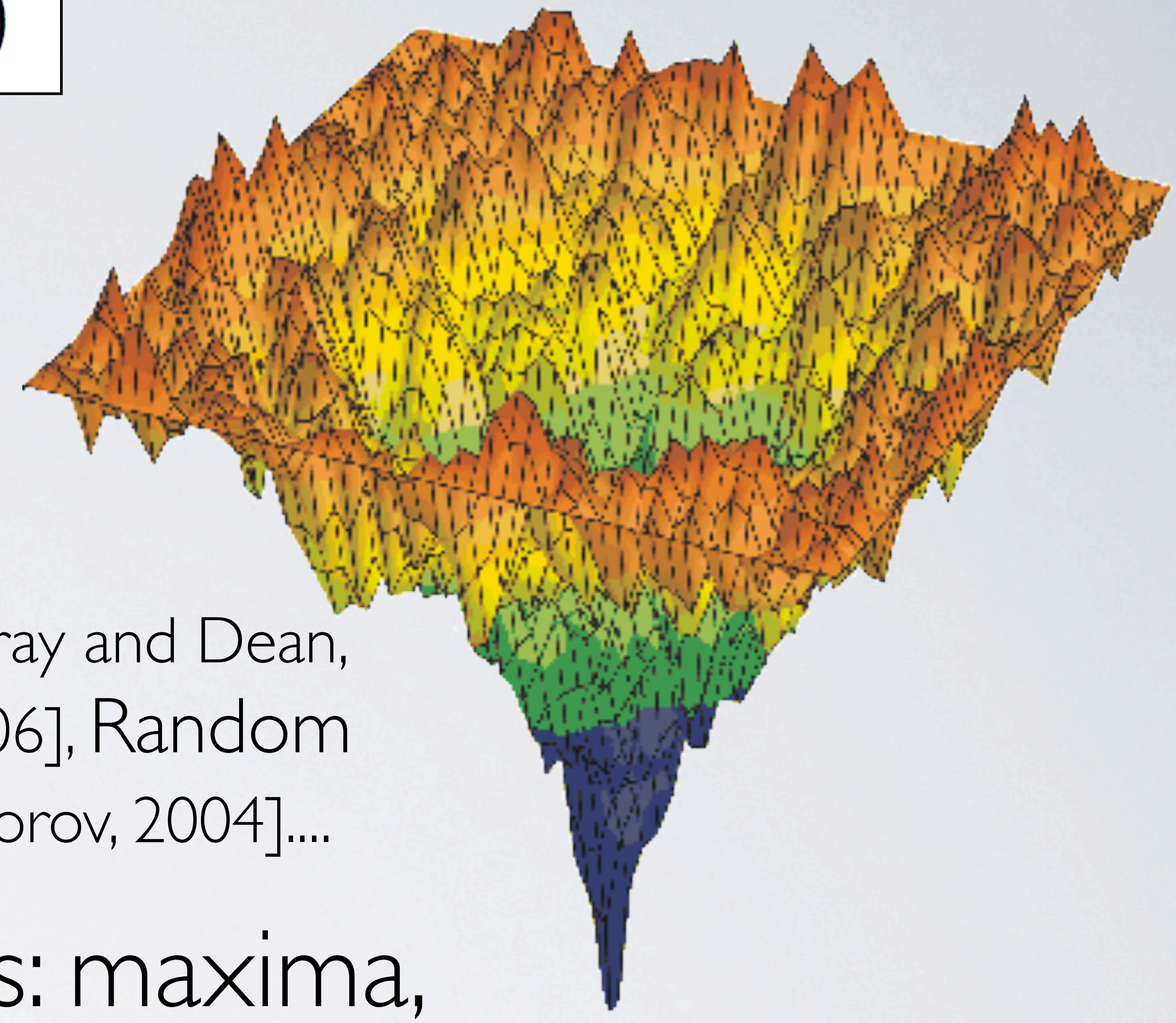
- $P_N = \mathrm{Prob}[\lambda_1 \le 0,\ \lambda_2 \le 0,\ \ldots,\ \lambda_N \le 0] = \mathrm{Prob}[\lambda_{\max} \le 0] = ?$

[R.M. May, Nature, 238, 413 (1972)—Ecosystems]

[Cavagna et. al. 2000, Fyodorov 2004, —— Glassy systems]

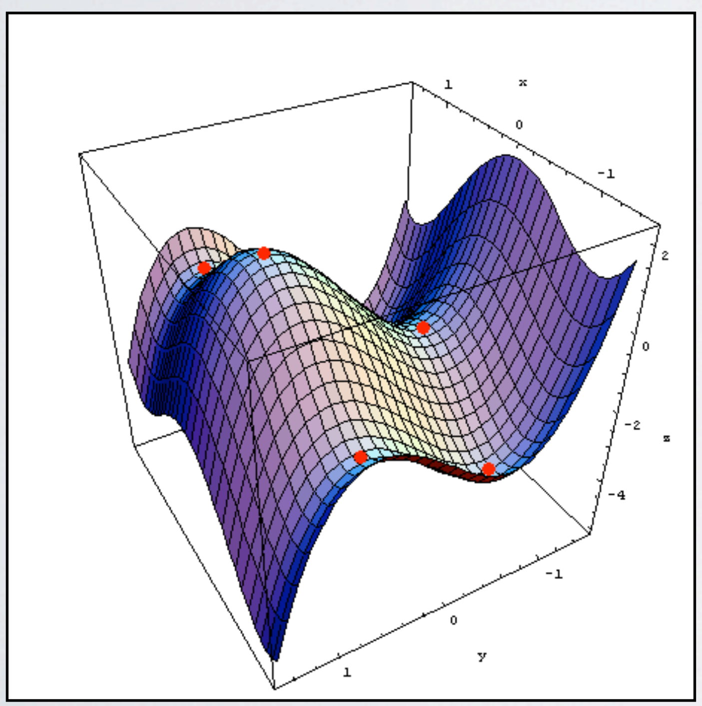
[Susskind 2003, Douglas et. al. 2004, Aazami & Easther 2006— String theory].....

A particle moving in a N-dim. landscape $V(y_1, \ldots, y_N)$



$$\frac{dy_i}{dt} = -\nabla_{y_i} V$$

- Spin and structural glasses, Gaussian fields [Bray and Dean, 2006], String landscapes [Aazami and Easther, 2006], Random Energy Landscapes and Glass Transition [Fyodorov, 2004]....



Stationary points: maxima, minima and saddles

$$H_{i,j} = \left[\frac{\partial^2 V}{\partial y_i \partial y_j}\right]$$

Hessian matrix

Eigenvalues of Hessian matrix determine the nature of the stationary point

RANDOM HESSIAN MODEL

Draw the elements of the Hessian matrix independently at random

$$H_{i,j} = \left[\frac{\partial^2 V}{\partial y_i \partial y_j} \right]$$

It belongs to the **GOE** of random matrices

The index distribution (number of positive eigenvalues) provides information about the ***typical*** stability pattern of a Random Hessian model

Most of the stationary points are saddles!

Journal of Cosmology and Astroparticle Physics
An IOP and SISSA journal

Cosmology from random multifield potentials

Amir Aazami and Richard Easther

Department of Physics, Yale University, New Haven, CT 06520, USA
E-mail: amir.aazami@yale.edu and richard.easther@yale.edu

Received 23 January 2006

Despite the approximation used to obtain equation (8), we have confirmed that the likelihood that all the eigenvalues of an $N \times N$ symmetric matrix have the same sign scales as e^{-cN^2}. The measured constant differs slightly from -0.25, although given the simplicity of our approximation the agreement is perhaps surprisingly good.

- Based on numerics, Aazami & Easther (2006) predicted for large N:

$$\boxed{P_N \sim \exp[-\theta N^2]} \text{ with } \theta_{\rm num} \approx 0.27$$

$\rightarrow$ very small probability $\rightarrow$ RARE EVENT

- Exact result: $\boxed{\theta = \frac{1}{4}\ln(3) = 0.274653..}$ (Dean and S.M., 2006)

More generally, for $\beta = 1$ (GOE), $\beta = 2$ (GUE) and $\beta = 4$ (GSE)

$$\boxed{P_N \sim \exp[-\beta\theta N^2]} \text{ for large } N$$

GAUSSIAN MATRIX NxN

$N = 5$

$$\begin{pmatrix} 0.5377 & 0.2631 & -1.8044 & 0.3286 & 0.4951 \\ 0.2631 & -0.4336 & 1.6888 & 1.7271 & 0.7810 \\ -1.8044 & 1.6888 & 0.7254 & 0.7133 & 0.7160 \\ 0.3286 & 1.7271 & 0.7133 & 1.4090 & 1.5237 \\ 0.4951 & 0.7810 & 0.7160 & 1.5237 & 0.4889 \end{pmatrix}$$

Real Symmetric or Complex Hermitian or Quaternion self-dual :
eigenvalues are real

$$\vec{\lambda} = [-2.4341 \ -0.8386 \ -0.5203 \ 2.2594 \ 4.2610]$$

$$\mathcal{N}_+ = \text{number of positive eigenvalues}$$

The index

$$P_\beta(\lambda_1,\ldots,\lambda_N)$$

Joint probability density of eigenvalues

$$\mathcal{P}(\mathcal{N}_+, N) = \int_{-\infty}^{\infty} d\lambda_1 \cdots d\lambda_N P_\beta(\lambda_1,\ldots,\lambda_N)\delta\left(\mathcal{N}_+ - \sum_{i=1}^N \theta(\lambda_i)\right)$$

Probability distribution of linear statistics

[Chen '94 - '98
Forrester '98
Beenakker '93
..........]

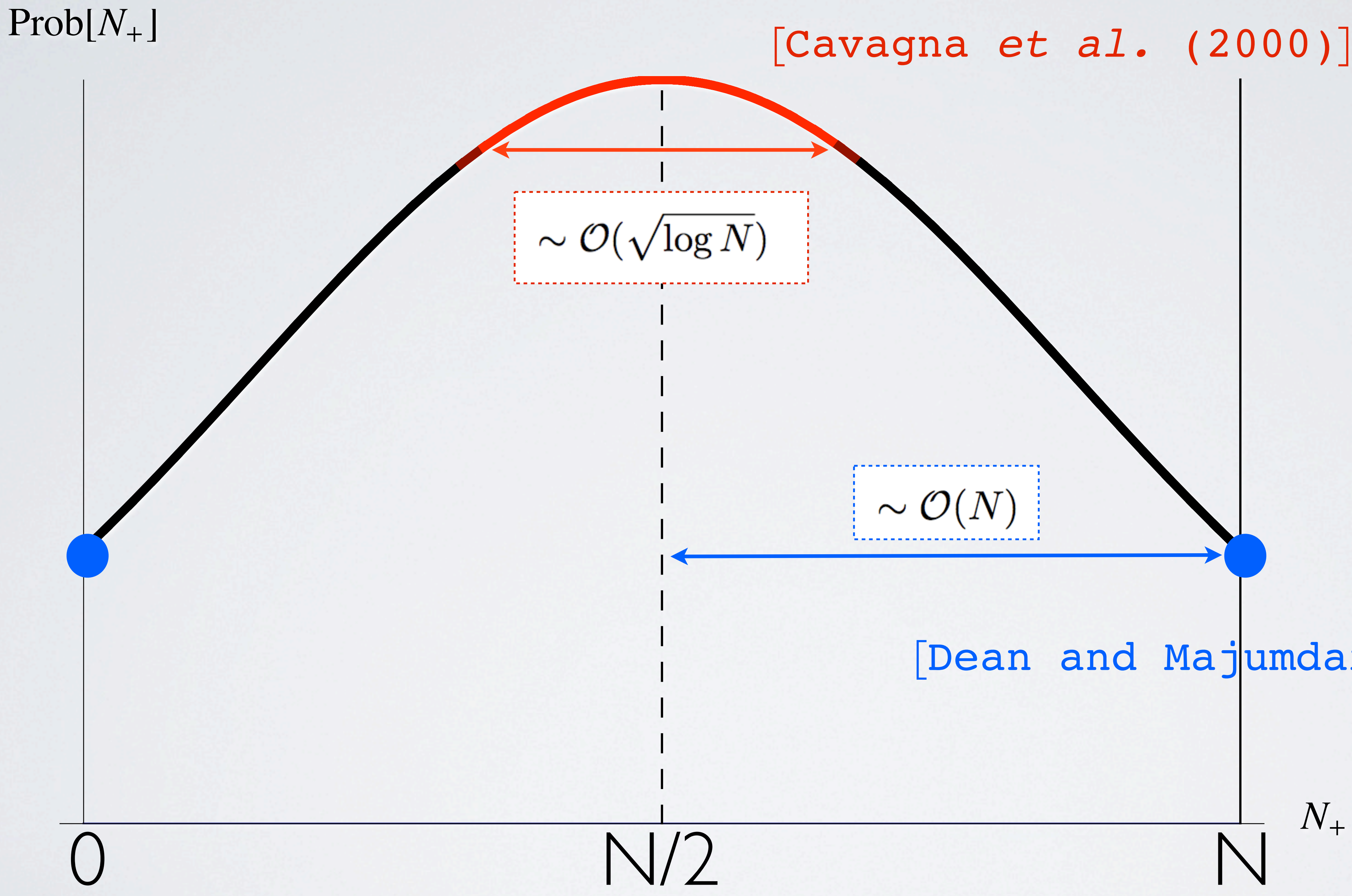
$$P_\beta(\lambda_1,\ldots,\lambda_N) = \frac{1}{Z_N} \mathrm{e}^{-\frac{\beta}{2}\sum_{i=1}^N \lambda_i^2} \prod_{j<k} |\lambda_j - \lambda_k|^\beta = \frac{1}{Z_N} \mathrm{e}^{-\beta \mathcal{H}(\vec{\lambda})}$$

$$\mathcal{H}(\vec{\lambda}) = \frac{1}{2}\sum_{i=1}^N \lambda_i^2 - \frac{1}{2}\sum_{j\neq k} \log|\lambda_j - \lambda_k|$$

Canonical weight of an auxiliary thermodynamical system

WHAT IS KNOWN? 2 SCALES IN THIS PROBLEM

$\mathcal{N}_+ =$ number of positive eigenvalues



TYPICAL VS. ATYPICAL FLUCTUATIONS: A PUZZLE?

Peak

PHYSICAL REVIEW B VOLUME 61, NUMBER 6 1 FEBRUARY 2000-II

Index distribution of random matrices with an application to disordered systems

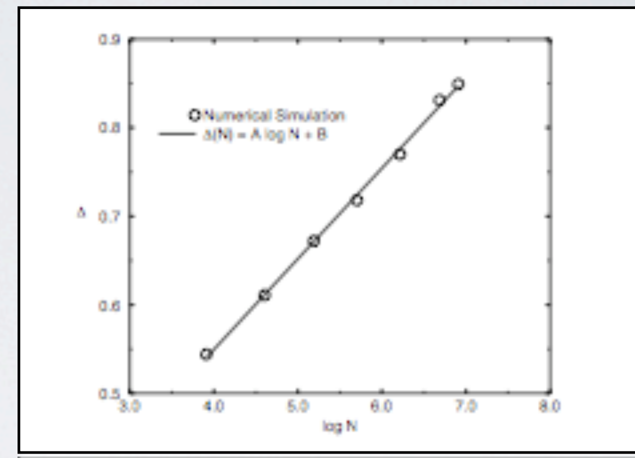
Andrea Cavagna,* Juan P. Garrahan,† and Irene Giardina‡
Theoretical Physics, University of Oxford, 1 Keble Road, Oxford, OX1 3NP, United Kingdom

(Received 21 July 1999; revised manuscript received 15 October 1999)

$$c = \frac{\mathcal{N}_+}{N}$$

$$\mathcal{P}(\mathcal{N}_+ = cN, N) \simeq \exp\left(-\frac{\pi^2 N^2}{2\ln N}(c-1/2)^2\right)$$

$$\Delta(N) = \left\langle \left(\mathcal{N}_+ - \frac{N}{2}\right)^2 \right\rangle \approx \frac{\ln N}{\pi^2}$$



Tails

PRL 97, 160201 (2006) PHYSICAL REVIEW LETTERS 20 OCTOBER 2006

Large Deviations of Extreme Eigenvalues of Random Matrices

David S. Dean[1] and Satya N. Majumdar[2]
[1]*Laboratoire de Physique Théorique (UMR 5152 du CNRS), Université Paul Sabatier, 118, route de Narbonne, 31062 Toulouse Cedex 4, France*
[2]*Laboratoire de Physique Théorique et Modèles Statistiques (UMR 8626 du CNRS), Université Paris-Sud, Bâtiment 100, 91405 Orsay Cedex, France*

$$\mathcal{P}(\mathcal{N}_+ = N, N) \simeq \exp\left(-\beta N^2 \theta\right), \qquad \theta = (\ln 3)/4$$

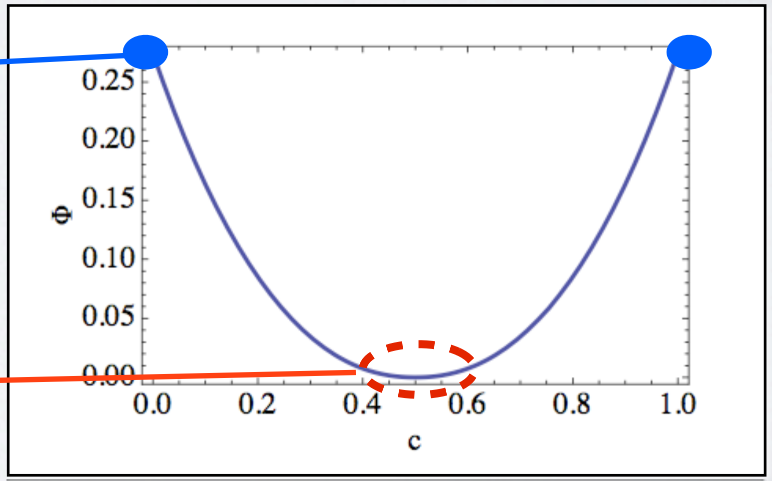
MAIN RESULT FOR LARGE N

$$\mathcal{P}(\mathcal{N}_+ = cN, N) \approx \exp\left(-\beta N^2 \Phi(c)\right)$$

$$\lim_{N\to\infty}\left[\frac{-\log \mathcal{P}(\mathcal{N}_+ = cN, N)}{\beta N^2}\right] = \Phi(c)$$

in agreement with
Dean & Majumdar

$$\Phi(0) = \Phi(1) = \theta = \log 3/4 \approx 0.27..$$



$$\Phi(c = 1/2 + \delta) = -\frac{\pi^2}{2}\frac{\delta^2}{\log \delta}$$

$$\mathcal{P}(\mathcal{N}_+, N) \approx \exp\left[-\frac{\beta\pi^2}{2\ln(N)}\left(\mathcal{N}_+ - N/2\right)^2\right]$$

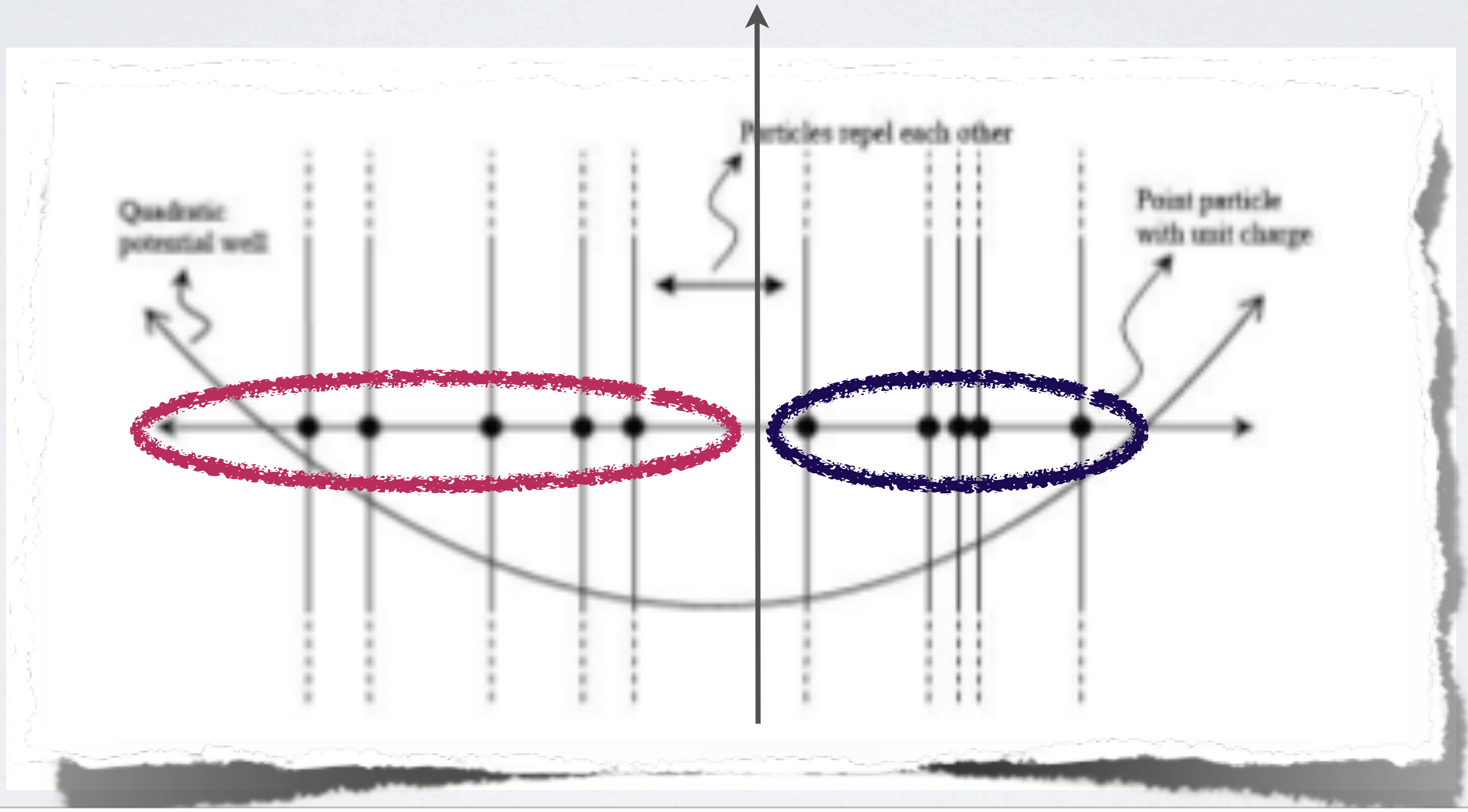
in agreement with
Cavagna *et al.*

$$\mathcal{P}(\mathcal{N}_+, N) = \int_{-\infty}^{\infty} d\lambda_1 \cdots d\lambda_N P_\beta(\lambda_1, \ldots, \lambda_N)\delta\left(\mathcal{N}_+ - \sum_{i=1}^{N}\theta(\lambda_i)\right)$$

$$e^{-\beta\mathcal{H}(\vec{\lambda})}$$

$$\mathcal{H}(\vec{\lambda}) = \frac{1}{2}\sum_{i=1}^{N}\lambda_i^2 - \frac{1}{2}\sum_{j\neq k}\log|\lambda_j - \lambda_k|$$

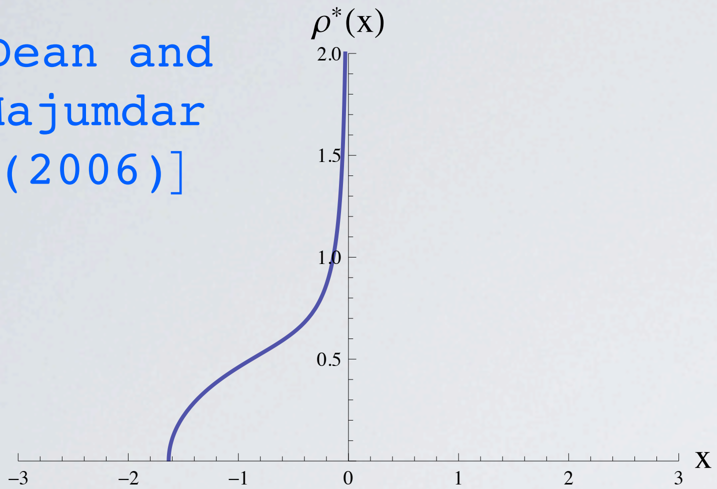
Task: evaluate this integral for large N by mapping it to a Coulomb gas problem

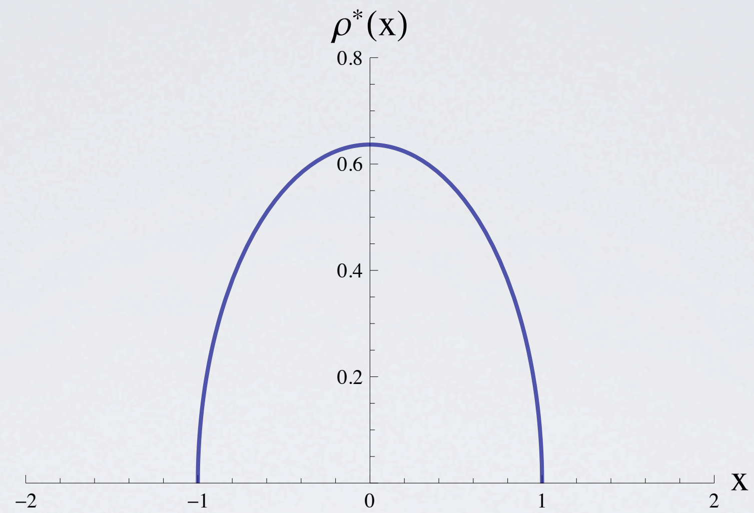


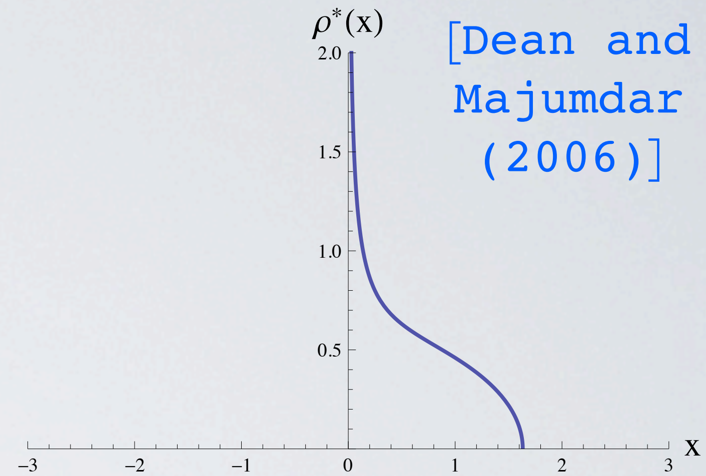
$\mathcal{P}(\mathcal{N}_+, N)$ is the canonical partition function of an auxiliary Coulomb gas with an extra hard constraint

PHASE TRANSITIONS IN THE CONSTRAINED GAS

[Dean and Majumdar (2006)]







[Dean and Majumdar (2006)]

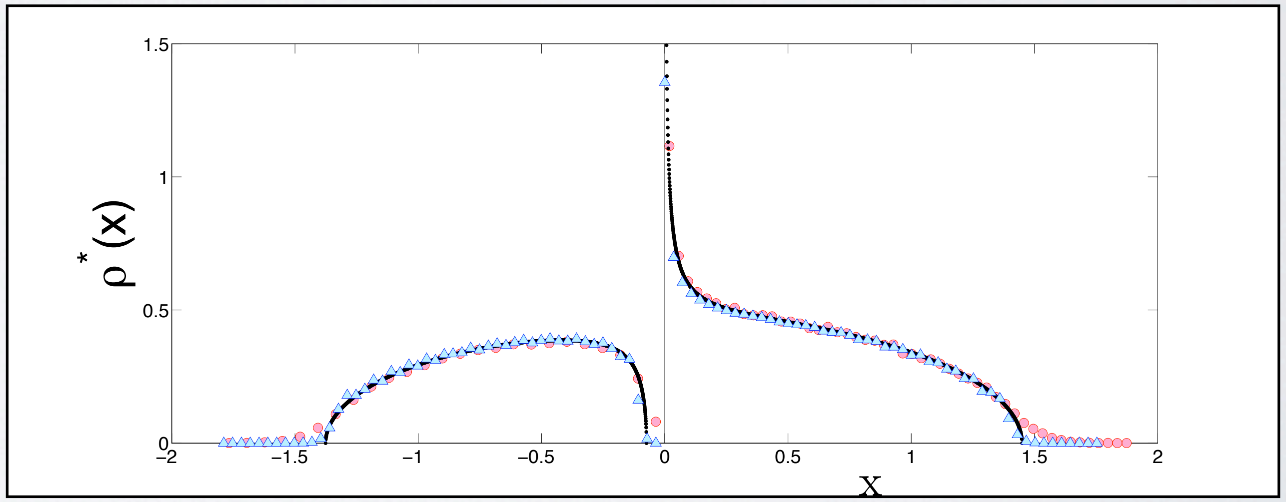
$c = 0$

$c = \frac{1}{2}$

$c = 1$

?

$$\rho^\star(x) = \frac{1}{\pi}\sqrt{\frac{L-x}{x}(x+L/a)(x+(1-1/a)L)}$$



$\rho^\star(x)$ → Partition Function → $\mathcal{P}(\mathcal{N}_+, N)$

$$\mathcal{P}(\mathcal{N}_+, N) = \int_{-\infty}^{\infty} d\lambda_1 \cdots d\lambda_N P_\beta(\lambda_1, \ldots, \lambda_N) \delta\left(\mathcal{N}_+ - \sum_{i=1}^{N} \theta(\lambda_i)\right)$$

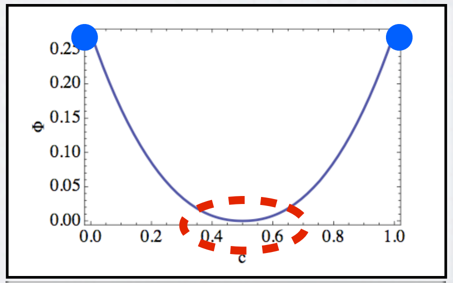
$$\mathcal{P}(\mathcal{N}_+, N) \propto \int \mathcal{D}[\rho] \mathrm{e}^{-\beta N^2 \mathcal{F}_c[\rho]}$$

where

$$\mathcal{F}_c[\rho] = \frac{1}{2}\int_{-\infty}^{\infty} dx\ x^2 \rho(x) - \frac{1}{2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dx dx' \rho(x)\rho(x') \ln|x - x'| + \\ + A_1\left(\int_{-\infty}^{\infty} dx \theta(x)\rho(x) - c\right) + A_2\left(\int_{-\infty}^{\infty} dx \rho(x) - 1\right)$$

Saddle point of the free energy: equilibrium density of the fluid $\rho^\star(x)$

$$\mathcal{P}(\mathcal{N}_+ = cN, N) \simeq \exp\left[-\beta N^2 \left(\underbrace{\mathcal{F}_c[\rho^\star] - \mathcal{F}_{1/2}[\rho^\star]}_{\Phi(c)}\right)\right]$$



SOLVING THE SADDLE-POINT EQUATION

$$\frac{\delta \mathcal{F}_c[\rho]}{\delta \rho} = 0 \qquad \Rightarrow \qquad \rho^\star(x)$$

Inverse Electrostatic Problem

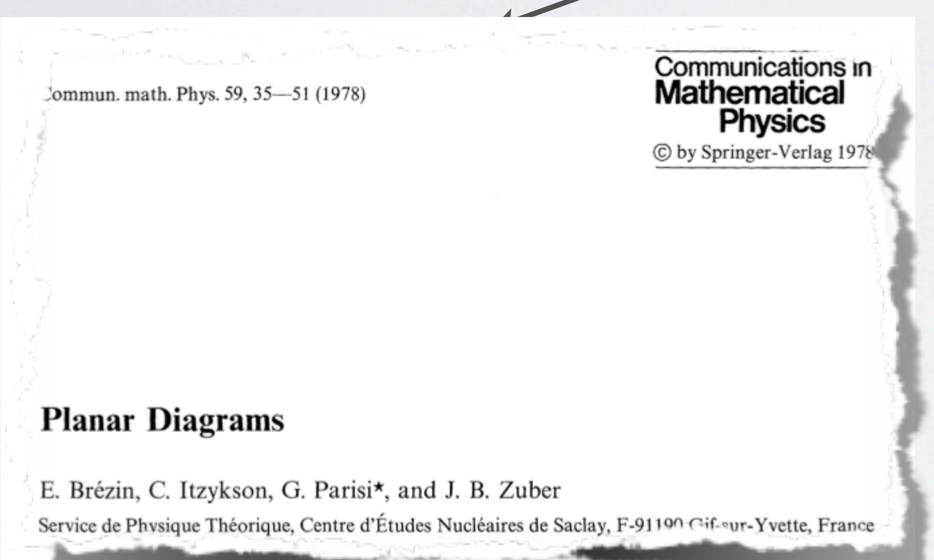
$$x^2 + A_1\theta(x) + A_2 = 2\int_{-\infty}^{\infty} \rho^\star(y) \ln|x-y| dy$$

$$x = \mathrm{Pr}\int dy \frac{\rho^\star(y)}{x-y}.$$

Equazioni integrali singolari del tipo di Carleman.

FRANCESCO G. TRICOMI (a Torino).

A Mauro Picone nel suo 70mo compleanno.



Commun. math. Phys. 59, 35—51 (1978)

Communications in Mathematical Physics

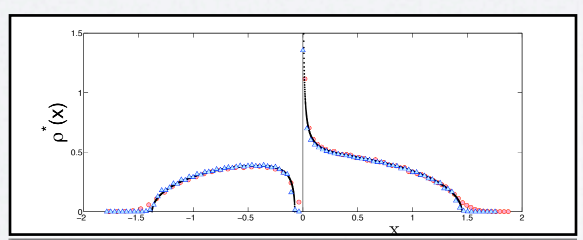
© by Springer-Verlag 1978

Planar Diagrams

E. Brézin, C. Itzykson, G. Parisi*, and J. B. Zuber

Service de Physique Théorique, Centre d'Études Nucléaires de Saclay, F-91190 Gif-sur-Yvette, France

[Brezin *et al.* 1978]



New!

Iterated single-support solution by Tricomi (1957)

Phys. Rev. E **83**, 041105 (2011)

IN SUMMARY...

$$\mathcal{P}(\mathcal{N}_+ = cN, N) \simeq \exp\left[-\beta N^2 \left(\underbrace{\mathcal{F}_c[\rho^\star] - \mathcal{F}_{1/2}[\rho^\star]}_{\Phi(c)}\right)\right]$$

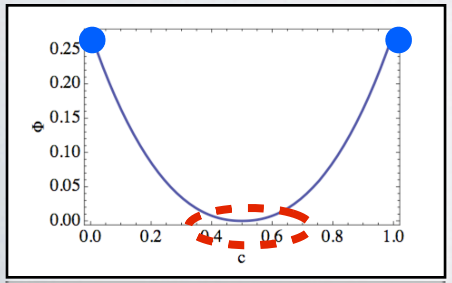
where

$$\rho^\star(x) = \frac{1}{\pi}\sqrt{\frac{L-x}{x}(x+L/a)(x+(1-1/a)L)}$$

and

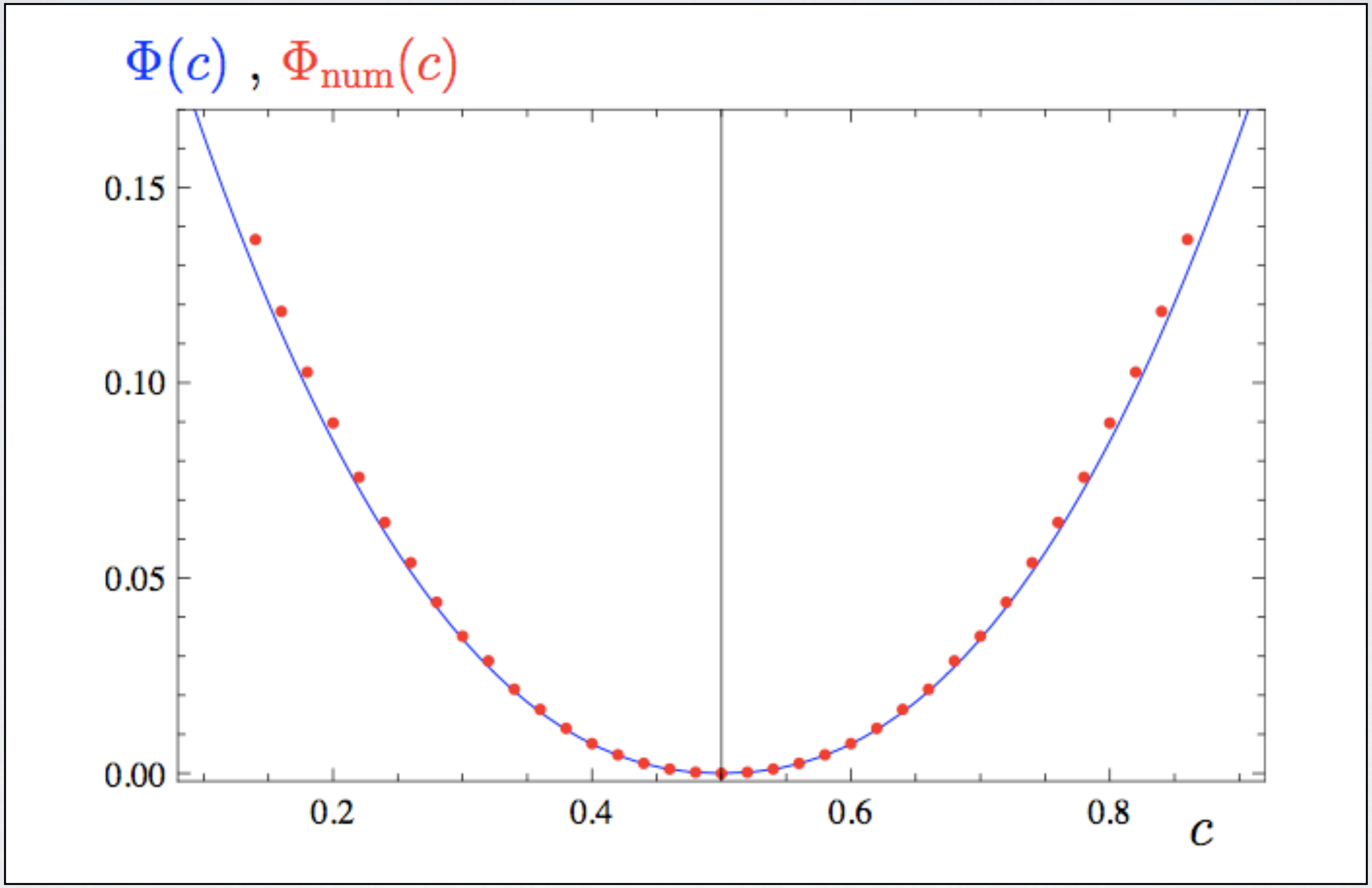
$$\mathcal{F}_c[\rho] = \frac{1}{2}\int_{-\infty}^{\infty} dx\ x^2\rho(x) - \frac{1}{2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dx dx'\rho(x)\rho(x')\ln|x-x'| + + A_1\left(\int_{-\infty}^{\infty} dx\theta(x)\rho(x) - c\right) + A_2\left(\int_{-\infty}^{\infty} dx\rho(x) - 1\right)$$

$$\Phi(c) = \frac{1}{4}[L^2 - 1 - \log(2L^2)] + \frac{(1-c)}{2}\log(a) - \frac{(1-c)(a^2-1)}{4a^2}L^2 + \frac{c}{2}\int_L^\infty W_1(x)dx + \frac{(1-c)}{2}\int_{L/a}^\infty W_2(x)dx$$



$$W_1(x) = F(x) - \frac{1}{x} = x - \frac{1}{x} - \sqrt{\frac{(x-L)}{x}\left(x+\frac{L}{a}\right)\left(x+\left(1-\frac{1}{a}\right)L\right)}$$

Some numerics...



SUMMARY

- Any matrix coming up in your research? Randomize! (and come to my office later...)

- Strongly correlated random variables
- Ubiquity - Universality of local statistics
- Rare events for strongly correlated random variables ---> exactly solvable cases!

REFERENCES

- S.N. Majumdar, C. Nadal, A. Scardicchio, and PV, Phys. Rev. Lett. **103**, 220603 (2009); Phys. Rev. E **83**, 041105 (2011)
- S.N. Majumdar and PV, [arXiv: 1112.5391] (2011)
- A. Cavagna, J.P. Garrahan, and I. Giardina, Phys. Rev. B **61**, 3960 (2000)
- D. Dean and S.N. Majumdar, Phys. Rev. Lett. **97**, 160201 (2006); Phys. Rev. E **77**, 041108 (2008)
- N.S. Witte and P.J. Forrester, [arXiv: 1110.0119] (2011)

Thank you.