Quantum fluctuations and Hawking radiation around black hole horizons in Bose–Einstein condensates

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Acoustic propagation in curved spacetime

The dynamics of a nonrelativistic, perfect and barotropic fluid is governed by the equations:

\[ \partial_t \rho + \nabla \cdot (\rho \mathbf{v}) = 0 \quad \text{(Continuity eq.)}, \]
\[ \partial_t \mathbf{v} + (\mathbf{v} \cdot \nabla) \mathbf{v} = -\frac{1}{\rho} \nabla p \quad \text{(Euler’s eq.)}, \]
\[ p = p(\rho) \quad \text{(Eq. of state)}. \]

(\[ \Rightarrow \exists \psi, \mathbf{v} = \nabla \psi \])

Acoustic approximation:

\[
\begin{bmatrix}
\psi(t, \mathbf{r}) \\
\rho(t, \mathbf{r}) \\
p(t, \mathbf{r})
\end{bmatrix}
\simeq
\begin{bmatrix}
\psi_0(t, \mathbf{r}) \\
\rho_0(t, \mathbf{r}) \\
p_0(t, \mathbf{r})
\end{bmatrix}
+ \begin{bmatrix}
\psi_1(t, \mathbf{r}) \\
\rho_1(t, \mathbf{r}) \\
p_1(t, \mathbf{r})
\end{bmatrix} \varepsilon.
\]

\[ c^2 = \frac{dp}{d\rho} (\rho_0) \quad x^\mu = (t, \mathbf{r}) \]

\[
\left\{ \begin{array}{l}
\frac{1}{\sqrt{-g}} \partial_{\mu} \left( \sqrt{-g} g^{\mu,\nu} \partial_{\nu} \right) \psi_1 = 0 \\
g_{\mu,\nu} = \frac{\rho_0}{c} \begin{bmatrix}
(\nabla \psi_0)^2 - c^2 & -\partial_j \psi_0 \\
-\partial_i \psi_0 & \delta_{i,j}
\end{bmatrix}
\end{array} \right.
\]

Painlevé–Gullstrand acoustic line-element

\[ c = C^{\text{st}} \quad \text{and} \quad \nabla \psi_0(r) = -c \sqrt{\frac{r_c}{r}} \hat{r} \]

\[ ds^2 \propto \left( \frac{r_c}{r} \right)^{\frac{3}{2}} \times \]
\[ \left[ -\left(1 - \frac{r_c}{r} \right) c^2 dt^2 + 2 \sqrt{\frac{r_c}{r}} c \, dt \, dr \\
+ dr^2 + r^2 (d\theta^2 + \sin^2 \theta \, d\varphi^2) \right] \]

- \[ \equiv ds^2_{\text{Sch.}} \] (Painlevé–Gullstrand).
- \[ r_c \equiv r_g \] = gravitational radius.
- \[ c \equiv c_0 \] = vacuum speed of light.
One-dimensional dumb holes

\[ V < c \quad V = c \quad V > c \]

Exterior of the dumb hole

Interior of the dumb hole

Horizon

\[ E_{\text{lab}}(q) = V \hbar q \quad \pm c \hbar |q| = \hbar \omega \]

Analog Hawking radiation

- **Stimulated.**— A phonon sent to the horizon gives rise to
  - two transmitted wavepackets falling down into the dumb hole and...
  - ... a reflected one propagating upstream from the horizon.
- **Spontaneous.**— Even without a source, vacuum fluctuations give rise to radiation of Hawking phonons.
Dumb holes in quasi-one-dimensional Bose–Einstein condensates

Gross–Pitaevskii field equation

\[ i\hbar \partial_t \hat{\Psi} = -\frac{\hbar^2}{2m} \partial_{xx} \hat{\Psi} + [U(x) + g(x) \hat{n} - \mu] \hat{\Psi} \]

- \( \hat{\Psi}(x, t) \): Heisenberg field operator,
- \( \hat{n}(x, t) = \hat{\Psi}^\dagger \hat{\Psi} \): density operator,
- \( U(x) \): external potential,
- \( g(x) \): four-field coupling constant,
- \( \mu \): chemical potential.

Classical stationary Gross–Pitaevskii equation

\[ \mu \Psi = -\frac{\hbar^2}{2m} \partial_{xx} \Psi + [U(x) + g(x)n] \Psi \]

- \( \Psi(x) \): wavefunction of the quasi-condensate [classical stationary version of \( \hat{\Psi}(x, t) \)],
- \( n(x) = |\Psi|^2 \): density [classical stationary version of \( \hat{n}(x, t) \)].
Quantum fluctuations around the background: Bogoliubov approach

\[ \hat{\Psi}(x, t) = \Psi(x) + \hat{\psi}(x, t) \quad \text{with} \quad \hat{\psi} \ll \Psi \]

### Bogoliubov spectrum

\[ \mathcal{E}_{\text{lab}}(q) = V \hbar q \quad (\text{Doppler shift}) \]

\[ \mathcal{E}_B(q) = c \hbar |q| \sqrt{1 + \frac{\xi^2 q^2}{4}} \]

### Scattering matrix

\[
\begin{bmatrix}
  u|\text{out} \\
  d1|\text{out} \\
  (d2|\text{in})^\dagger
\end{bmatrix} = S(\omega)
\begin{bmatrix}
  u|\text{in} \\
  d1|\text{in} \\
  (d2|\text{out})^\dagger
\end{bmatrix}
\]

\[ |S_{\ell, \ell'}(\omega)|^2: \text{transmission/reflection coefficient for a } \ell'-\text{ingoing mode oscillating at pulsation } \omega \text{ scatters into a } \ell-\text{outgoing mode.} \]
One-body Hawking signal

**Radiated power**

Energy current associated to emission of elementary excitations:

\[ \hat{\Pi}(x, t) = -\frac{\hbar^2}{2m} \partial_t \hat{\Psi}^\dagger \partial_x \hat{\Psi} + \text{H.c.} \]

Deep outside the black hole,

\[ \Pi_0 \overset{\text{def.}}{=} \langle \hat{\Pi} \rangle_{T=0} = -\int_0^{\Omega} \frac{d\omega}{2\pi} \hbar \omega |S_{u,d2}(\omega)|^2. \]

**Radiation spectrum**

\[ |S_{u,d2}(\omega)|^2 \simeq \frac{\Gamma}{\exp \left( \frac{\hbar \omega}{T_H} \right) - 1} \]

**Hawking temperature**

Low-\( \omega \) behaviour of \( S_{u,d2} \):

\[ S_{u,d2}(\omega) \simeq f_{u,d2} \left( \frac{\hbar \omega}{m c_u^2} \right)^{-\frac{1}{2}} + h_{u,d2} \left( \frac{\hbar \omega}{m c_u^2} \right)^{\frac{1}{2}}. \]

\[ \Gamma = -4 \text{Re}(f_{u,d2}^* h_{u,d2}), \quad \frac{T_H}{m c_u^2} = \frac{|f_{u,d2}|^2}{\Gamma}. \]

\[ T_H \sim 10 \text{ nK} < \mu \sim 100 \text{ nK} \]
Two-body Hawking signal

**Connected two-body density matrix**

\[ g^{(2)}(x, x') = \langle \hat{\Psi}^\dagger(x, t) \hat{\Psi}^\dagger(x', t) \hat{\Psi}(x, t) \hat{\Psi}(x', t) \rangle - n(x)n(x') = n(x)n(x')G^{(2)}(x, x') \]

**Correlated phonons**

\[ u|\text{out} \leftrightarrow x \leftrightarrow x'| \leftrightarrow d2|\text{out} \]

At time \( t \) after their emission the phonons \( u|\text{out} \) and \( d2|\text{out} \) are respectively located at

\[ x = V_g(q_u|\text{out})t \quad \text{and} \quad x' = V_g(q_d2|\text{out})t, \]

inducing a correlation signal \((u - d2)\) along the line of slope

\[ \frac{x'}{x} = \frac{V_g(q_d2|\text{out})}{V_g(q_u|\text{out})} \]

in the \( \{x, x'\} \) plane.
Compressibility sum rule at zero temperature

In the absence of black hole

\[ \int_{\mathbb{R}} dx' \, g_0^{(2)}(x, x') = -n(x) \]

In the presence of black hole

The shape of the short-range anti-bunching is modified:

\[ \int_{\mathbb{R}} dx' \, g_0^{(2)}(x, x') \leftrightarrow -n(x) + \text{(terms)}_{\text{BH}}. \]

Long-range correlations allow us to recover the sum rule:

\[ \int_{\mathbb{R}} dx' \, g_0^{(2)}(x, x') \leftrightarrow -(\text{terms})_{\text{BH}}. \]

Because of the sum rule,

\[ \begin{align*}
\text{(Long-range correlations)} & \iff \\
\text{(Modifications of short-range correlations).}
\end{align*} \]
Bose–Einstein condensates of metastable helium

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\( g^{(2)}(q, q') = \langle \hat{\psi}^\dagger(q, t) \hat{\psi}(q, t) \hat{\psi}^\dagger(q', t) \hat{\psi}(q', t) \rangle - \langle \hat{\psi}^\dagger(q, t) \hat{\psi}(q, t) \rangle \langle \hat{\psi}^\dagger(q', t) \hat{\psi}(q', t) \rangle \)

\[
g^{(2)}(q, q') = g_0^{(2)}(q, q') + g_{\neq 0}^{(2)}(q, q')
\]

\[
\left\{ \begin{array}{l}
g_0^{(2)}(q, q') = \frac{1}{\xi^2 q^2 (4+\xi^2 q^2)^2} \left( \delta_{q,q'} + \delta_{q,-q'} \right) \\
g_{\neq 0}^{(2)}(q, q') = \left[ \frac{1}{4} \frac{(2+\xi^2 q^2)^2}{\xi^2 q^2 (4+\xi^2 q^2)} \delta_{q,q'} + \frac{1}{\xi^2 q^2 (4+\xi^2 q^2)} \delta_{q,-q'} \right] \sinh^{-2} \left( \frac{g n_0}{T} \frac{\xi q}{2} \sqrt{1 + \frac{\xi^2 q^2}{4}} \right)
\end{array} \right.
\]

- Infrared divergences...
- Range of correlations along the line \( \{q, q\}: \sim \xi^{-1} \)?
- Width of the correlation lines \( \{q, q\} \) and \( \{q, -q\}: \sim L^{-1}, \ell_\varphi^{-1} \)?

\[\xi q = \frac{v}{c}\]
Two-body Hawking signal in momentum space

Outside the BH

\[ q_u \rightarrow q_{u|\text{out}} \quad q \quad q_{d1|\text{out}} \quad q_{d2|\text{out}} \rightarrow q_d \]

Inside the BH

\[ q' \quad q_{d1|\text{out}} \quad q_{d2|\text{out}} \]

correlated

\[ g^{(2)}(q, q') \]

Waterfall (for example)

\[ n(x) \quad U(x) \]

\[ q_{\text{out}}^\text{min} \quad q_{\text{out}}^\text{max} \]

\[ q_{d1|\text{out}}^\text{max} \equiv q^* \]

\[ q_{d2|\text{out}}^\text{max} \equiv q^* \]

\[ \xi_d q^* = \left(-2 + \frac{m_d^2}{2} + \frac{m_d}{2} \sqrt{8 + m_d^2}\right)^{\frac{1}{2}} \]

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Hawking radiation in BECs  
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Hawking radiation in quasi-1D Bose–Einstein condensates

Conclusion

• Bose–Einstein condensates offer interesting prospects to observe a spontaneous—so fully quantum—Hawking radiation.

• New dumb-hole configurations of experimental interest.

• Analytical formula for the Hawking temperature $T_H$. ($T_H \sim 10\text{ nK} < (\mu \sim 100\text{ nK})$: the one-body Hawking signal is lost in the thermal noise, but...

• ...nonlocal density correlations (in the real space and also in the momentum space) provide a clear qualitative signature of the two-body Hawking signal, even at finite temperature.

• The usual normalization sum rule at zero temperature is also verified in the presence of an acoustic horizon: long-range density correlations have to be associated to short-range modifications of the connected two-body density matrix.

• Prospects: Acoustic horizon in the flow of a polariton condensate: emission of a “Hawking polarization-flux” from the horizon with $T_H|\text{Polariton BEC} > \mu|\text{Polariton BEC}$.

Supplementary 1: Quasi-1D BECs in the mean-field regime

\[ \Psi_{\perp}(r_{\perp}, \Psi_{\parallel}(x, t)) \text{ (“frozen”)} \]

\[ V_{\perp}(r_{\perp}) = \frac{1}{2} m \omega_{\perp}^2 r_{\perp}^2 \]

**Harmonic radial confinement**

\[ \Psi(r, t) = \Psi_{\parallel}(x, t) \times \Psi_{\perp}(r_{\perp}, \Psi_{\parallel}(x, t)) \]

**Born–Oppenheimer approximation**

\[ i \hbar \partial_t \Psi_{\parallel} = -\frac{\hbar^2}{2m} \partial_{xx} \Psi_{\parallel} + \left[ U_{\text{ext}}(x, t) + g_{1D} n_{1D} - \mu \right] \Psi_{\parallel} \]

- \( n_{1D} = |\Psi_{\parallel}(x, t)|^2 \): longitudinal density of the condensate.
- \( g_{1D} = 2 \frac{\hbar \omega_{\perp}}{a_s} \Rightarrow a_s > 0 \): 3D s-wave scattering length.

**1D Gross–Pitaevskii equation**

\[ \frac{\hbar \omega_{\perp}}{\left( \frac{\hbar^2 a_s^{-2}}{m} \right)} \ll n_{1D} a_s \sim \frac{\mu}{\hbar \omega_{\perp}} \ll 1 \]

- (1) allows to avoid the Tonks–Girardeau regime and implies \( \mathcal{E}_{\text{int}} \ll \mathcal{E}_{\text{kin}} \) and \( \ell_{\varphi} \gg \xi \), where \( \ell_{\varphi} = \xi \exp(\pi \sqrt{\frac{\hbar^2 n_{1D}}{g_{1D} m}}) \).
- (2) allows to avoid the 3D-like transverse Thomas–Fermi regime and implies that the transverse motion is frozen.
Supplementary 2: Location of the acoustic horizon

\[ \Psi(x < 0) = \sqrt{n_u} \left[ \sqrt{1 - m_u^2} \tanh \left( \frac{x - x_0}{\xi_u} \sqrt{1 - m_u^2} \right) - im_u \right] e^{i q_u x} \]

\[ n(x) = |\Psi(x)|^2 \]

\[ \frac{V(x_0)}{c(x_0)} = \sqrt{\frac{2}{m_u^2} - 1} > 1 \]

\[ \Rightarrow x \bigg|_{\text{Horizon}} < x_0 \]

\[ \Rightarrow x \bigg|_{\text{Horizon}} = x \bigg|_{\text{Horizon}}(\omega) \]

Waterfall configuration \( (x_0 = 0) \)
Supplementary 3: Quantum reflection

Real space

Quantum particle incoming from the left with an energy \( \varepsilon > U_{\text{max}} \).

\[
\varepsilon = \frac{V(X)}{c} Q \pm |Q| \sqrt{1 + \frac{Q^2}{4}} = f(X, Q)
\]

Phase space

\[
\varepsilon = \frac{Q^2}{2} + U(X) = f(X, Q)
\]