On some integrals over the $U(N)$ unitary group
and their large $N$ limit

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The integral over the $U(N)$ unitary group $I = \int DU \exp \text{Tr} A U B U^\dagger$ is reexamined. Various
approaches and extensions are first reviewed. The second half of the paper deals with
more recent developments: relation with integrable Toda lattice hierarchy, diagrammatic
expansion and combinatorics, and what they teach us on the large $N$ limit of $\log I$. 

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1. Introduction and notations

1.1. Aim and plan of the paper

Techniques of integration over large matrices are important in several contexts of physics – from QCD [1] to quantum gravity [2], from disordered systems to mesoscopic physics [3] – and of mathematics – enumerative combinatorics [4], integrable systems [5, 6, 7], free probability theory [8], statistics [9], etc. In most applications, a central rôle is played by the statistics of the eigenvalues of the random matrices – spectrum and correlations – or more generally of their invariants $\text{Tr} A^p$. This is justified once “angular” variables have been integrated over, like those appearing in $\text{Tr} AB$.

It is in this general context that the integral

$$I = \int_{\text{U}(N)} DU \exp \text{Tr} A U B U^\dagger,$$

$A$ and $B$ hermitian, was studied more than twenty years ago, and an exact expression was derived [10]. Soon after, it was realised that this result had been obtained long before by Harish-Chandra [11] as a corollary of a more general problem. The purpose of this paper is to return to this integral (sometimes called the Harish-Chandra–Itzykson–Zuber integral in the physics literature), to review the known facts and to present some of its known extensions, before turning to what remains a challenge: to find a good and systematic description of the large $N$ limit of its logarithm. We shall report on some recent progress made on the latter issue.

Our paper is organised as follows. In the rest of sec. 1, we introduce notations and some basic results. The derivation of the expression of $I$ is then reviewed in sec. 2, using various methods: heat equation, character expansion and Duistermaat-Heckman theorem. Sec. 3 discusses briefly extensions in various directions, in particular the case of rectangular matrices. Connection with integrable hierarchies is then presented in sec. 4, with special attention devoted to its dispersionless limit and what can be learnt from it. The resulting expressions for the large $N$ limit of $\log I$ are then confronted in sec. 5 to those obtained by (what we believe to be) a novel diagrammatic expansion of $\log I$ and by a purely combinatorial analysis [12]. Finally sec. 6 contains a summary of results and tables.
1.2. Notations

Let $A$ and $B$ be two $N \times N$ matrices. We shall assume that they are Hermitean, even though many properties that we shall derive do not require it. The subject of study is the integral

$$I(A, B; s) = \int_{U(N)} DU \exp \frac{N}{s} \text{Tr} A U B U^\dagger$$

where $DU$ is the Haar measure on the unitary group $U(N)$, normalised to $\int DU = 1$; and the large $N$ limit (in a sense defined below) of its logarithm $\frac{1}{N^2} \log I(A, B; s)$. At the possible price of a redefinition of $U$, one may always assume that $A$ and $B$ are diagonal, $A = \text{diag} \left( a_i \right)_{1 \leq i \leq N}$, $B = \text{diag} \left( b_i \right)_{1 \leq i \leq N}$. Clearly, the real parameter $s$ could be scaled away. We find convenient to keep it as an indicator of the homogeneity in the $a_i$’s and $b_i$’s.

To any order of their $1/s$ expansions, both $I(A, B; s)$ and its logarithm are completely symmetric polynomials in the eigenvalues $a_i$ and in the $b_i$ independently, as we shall see in the next section. We want to express them in terms of the elementary symmetric functions of the $a_i$’s and of the $b_i$’s:

$$\theta_p := \frac{1}{N} \sum_{i=1}^{N} a_i^p, \quad \bar{\theta}_p := \frac{1}{N} \sum_{i=1}^{N} b_i^p.$$  \hspace{1cm} (1.2)

For finite $N$, only a finite number of these functions are independent, but this constraint disappears as $N \to \infty$. By large $N$ limit we therefore mean that we consider sequences of matrices $A$ and $B$ of size $N$ such that the moments of their spectral distributions $\theta_p$ and $\bar{\theta}_p$ converge as $N \to \infty$ (see [13] for some rigorous results on this limit). By abuse of notation we still denote by $A$ and $B$ such objects, and write

$$F(A, B; s) = \lim_{N \to \infty} \frac{1}{N^2} \log I(A, B; s).$$  \hspace{1cm} (1.3)

We shall use various combinations of the symmetric functions $\theta_p$ and $\bar{\theta}_p$. In particular, for $\alpha \vdash n$, i.e. $\alpha$ a partition of $n = \sum_p p \alpha_p$, which we also write $\alpha = [1^{\alpha_1} \cdots n^{\alpha_n}]$, we define

$$\text{tr} \alpha A := \left( \frac{1}{N} \text{Tr} A \right)^{\alpha_1} \cdots \left( \frac{1}{N} \text{Tr} A^n \right)^{\alpha_n} = \prod_{p=1}^{n} \theta_p^{\alpha_p}.$$  \hspace{1cm} (1.4)

We shall also make use of the characters of the irreducible holomorphic representations of the linear group $\text{GL}(N)$, labelled by Young diagrams $\lambda$ with rows of lengths $0 \leq \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_N$. They read

$$\chi_{\lambda} = \frac{\Delta_{\lambda}(a)}{\Delta(a)}.$$  \hspace{1cm} (1.5)
in terms of Vandermonde determinants
\[
\Delta(a) = \prod_{1 \leq j < i \leq N} (a_i - a_j) = \det(a_i^{j-1}) \tag{1.6}
\]
and their generalizations
\[
\Delta_\lambda(a) = \det(a_i^{\lambda_j+j-1}) \tag{1.7}
\]

Frobenius formula \[14\] relates these sets of symmetric functions: if \( \alpha \vdash n \)

\[
N^\Sigma \alpha; \text{tr}_\alpha A = \sum_{|\lambda| = n} \chi_\lambda(A)\check{\chi}_\lambda(\alpha) \tag{1.8}
\]
where the sum runs over all Young tableaux \( \lambda \) with \( |\lambda| = n \) boxes, and \( \check{\chi}_\lambda(\alpha) \) denotes the character of the symmetric group \( \mathfrak{S}_n \), for the representation labelled by \( \lambda \) and for the class labelled by \( \alpha \).

The expression of integrals and differential operators over hermitian matrices in terms of their eigenvalues involves a Jacobian, and problems of normalisation appear. We discuss these questions shortly.

If \( M = U A U^\dagger \), \( A \) diagonal, \( U \) unitary, we have \( dM = U dAU^\dagger + [dX, M] \), where \( dX := dUU^\dagger \) is antihermitian. Then \( \text{Tr}(dM)^2 = \sum_i d\alpha_i^2 + 2\sum_{i<j} |dX_{ij}|^2|a_i - a_j|^2 \) defines the metric tensor \( g_{\alpha\beta} \) in the coordinates \( \xi_\alpha = (a_i, X_{ij}) \). This determines first the measure \( DM = \sqrt{\det g} \prod d\xi_\alpha = 2^{N(N-1)/2} \prod_i dM_i \prod_{i<j} d\Re M_{ij} d\Im M_{ij} = 2^{N(N-1)/2} \Delta^2(a) \prod_i d\alpha_i \prod dX_{ij} = C \Delta^2(a) \prod_i d\alpha_i \prod dU \). The constant \( C \) is fixed by computing in two different ways the integral of a \( U(N) \) invariant function of \( M \), for example a Gaussian
\[
1 = \int \prod_{p=1}^N p! \quad e^{-\frac{1}{2} \text{Tr} M^2} = \frac{C}{(2\pi)^{N(N-1)/2}} \prod_{i=1}^N d\alpha_i \prod_{i<j} (a_i - a_j)^2 \tag{1.9}
\]
The latter integral equals \( \prod_{p=1}^N p! \) thus
\[
C = \frac{(2\pi)^{N(N-1)/2}}{\prod_{p=1}^N p!}, \quad DM = \frac{(2\pi)^{N(N-1)/2}}{\prod_{p=1}^N p!} \Delta^2(a) \prod_{i=1}^N d\alpha_i \prod dU. \tag{1.9}
\]

From the metric above, one also computes the Laplacian
\[
\Delta_M = \frac{1}{\sqrt{g}} \frac{\partial}{\partial \xi_\alpha} g^{\alpha\beta} \sqrt{g} \frac{\partial}{\partial \xi_\beta} = \prod_{i<j}(a_i - a_j)^2 \sum_k \frac{\partial}{\partial \alpha_k} \prod_{i<j}(a_i - a_j)^2 \frac{\partial}{\partial \alpha_k} + \Delta_X
\]
\[
= \Delta(a)^{-1} \sum_k \left( \frac{\partial}{\partial \alpha_k} \right)^2 \Delta(a) + \Delta_X, \tag{1.10}
\]
where the last equality results from the vanishing of \( \sum_k [\partial_{\alpha_k}, [\partial_{\alpha_k}, \Delta(a)]] = 0 \), as this completely antisymmetric function of the \( a \)'s is a polynomial of degree \( N(N-1)/2 - 2 \).
2. The exact expression of $I(A, B; s)$

Assume that all the eigenvalues of $A$ and $B$ are distinct. One finds \cite{11,10}

$$I(A, B; s) = \left( \prod_{p=1}^{N-1} p! \right) \left( \frac{N}{s} \right)^{-N(N-1)/2} \frac{\det \left( e^{ \frac{N}{s} a_j b_j} \right)_{1 \leq i, j \leq N}}{\Delta(a) \Delta(b)}$$

(2.1)

Note that both the numerator and the denominator of the r.h.s. are completely antisymmetric functions of the $a$’s and of the $b$’s independently, and that the limit where some eigenvalues coalesce is well defined.

This expression (2.1) may be obtained by several different routes.

2.1. Heat equation

For two hermitian $N \times N$ matrices $M_A$ and $M_B$, let us consider

$$K(M_A, M_B; s) = \left( \frac{N}{2\pi s} \right)^{N^2/2} \exp \left( -\frac{N}{2s} \text{Tr}(M_A - M_B)^2 \right).$$

(2.2)

$K(M_A, M_B; s)$ satisfies the heat equation

$$\left( \frac{\partial}{\partial s} - \frac{1}{2} \Delta_M \right) K(M_A, M_B; s) = 0,$$

(2.3)

where $\Delta_M$ is the Laplacian over $M$, together with the boundary condition that for $t \to 0$, $K(M_A, M_B; s) \to \delta(M_A - M_B)$.

The heat kernel $K(M_A, M_B; s)$ is invariant under the simultaneous adjoint action on $M_A$ and $M_B$ by the same unitary matrix $U$. If we diagonalise $M_A = U_A A U_A^\dagger$ and $M_B = U_B B U_B^\dagger$, $K(M_A, M_B; s) = K(A, U B U^\dagger; s)$ where $U = U_A^\dagger U_B$. Upon integration

$$\tilde{K}(A, B; s) := \int DU K(M_A, U M_B U^\dagger; s) = \int DU K(A, U B U^\dagger; s)$$

$$= \left( \frac{N}{2\pi s} \right)^{N^2/2} e^{-\frac{N}{2s} \text{Tr}(A^2 + B^2)} I(A, B; s)$$

(2.4)

is again a solution of the heat equation (2.3), but depends only on the eigenvalues $a_i$ of $A$ (and $b_i$ of $B$). Using the explicit form of the Laplacian in terms of the eigenvalues $a_i$, (see (1.10)), $\tilde{K}$ satisfies

$$\left( \frac{\partial}{\partial s} - \frac{1}{2} \sum_k \left( \frac{\partial}{\partial a_k} \right)^2 \right) \Delta(a) \tilde{K}(M_A, M_B; s) = 0.$$

(2.5)
The product $\Delta(a)\Delta(b)\tilde{K}(A, B; s)$ is an antisymmetric function of the $a$’s and of the $b$’s, is a solution of the heat equation with the flat Laplacian, and satisfies the boundary conditions that for $s \to 0$, $C\Delta(a)\Delta(b)\tilde{K}(A, B; s) \to \frac{1}{N!} \sum_{P \in S_N} \epsilon_P \prod_i \delta(a_i - b_{P_i})$, with $C$ the constant computed in (1.9).

In physical terms it is the Green function of $N$ independent free fermions, and it is thus given by the Slater determinant

$$C\Delta(a)\Delta(b)\tilde{K}(A, B; s) = \frac{1}{N!} \left( \frac{N}{2\pi s} \right)^{N/2} \det \left[ \exp -\frac{N}{2s} (a_i - b_j)^2 \right],$$

which is consistent with (2.1).

2.2. Character expansion

We may expand the exponential in (1.1) to get

$$I(A, B; s) = \sum_{n=0}^{\infty} \frac{(N/s)^n}{n!} \int DU \langle \text{Tr} A Ub \rangle^n.$$  

(2.7)

Frobenius formula (1.8) evaluated for the partition $[1^n]$ gives for any matrix $X$ of $\text{GL}(N)$

$$\text{Tr}^n X = \sum_{|\lambda|=n} \hat{d}_\lambda \chi_\lambda(X),$$

(2.8)

where $\hat{d}_\lambda = \hat{\chi}_\lambda([1^n])$ is the dimension of the $\lambda$-representation of $S_n$. Integration over the unitary group then yields

$$\int DU \chi_\lambda(A Ub) = \frac{\chi_\lambda(A)\chi_\lambda(B)}{\chi_\lambda(I)} = \frac{\chi_\lambda(A)\chi_\lambda(B)}{d_\lambda},$$

(2.9)

and a well known formula [14] gives

$$\frac{\hat{d}_\lambda}{d_\lambda} = n! \prod_{p=1}^{N} \frac{(p-1)!}{(\lambda_p + p - 1)!}.$$  

(2.10)

Using (1.5) and putting everything together we find

$$\Delta(a) \Delta(b) I(A, B; s) = \left( \prod_{p=1}^{N-1} p! \right) \sum_{n=0}^{\infty} \frac{(N/s)^n}{n!} \sum_{|\lambda|=n} \frac{1}{\prod_{p}(\lambda_p + p - 1)!} \Delta_\lambda(a) \Delta_\lambda(b)$$

(2.11a)

$$= \left( \prod_{p=1}^{N-1} p! \right) \sum_{0 \leq \ell_1 < \ldots < \ell_N} \prod_{q=1}^{N} \frac{(N/s)^{\ell_q - q + 1}}{(\ell_q)!} \det(a_{i,j}^{\ell_i}) \det(b_{i,j}^{\ell_j})$$

(2.11b)
Eq. (2.11d) is interesting in its own sake, while an extension of Binet–Cauchy theorem enables one to resum (2.11b) into

\[ \Delta(a) \Delta(b) I(A, B; s) = \left( \prod_{p=1}^{N-1} p! \right) (N/s)^{N(N-1)/2} \det \left( e^{\frac{N}{s} a_{ij}} \right) \] (2.12)

which is precisely (2.1).

2.3. Duistermaat-Heckman theorem

Let us compute the integral (1.1) by the stationary phase method. The stationary points \( U_0 \) of the “action” \( \text{Tr} A U B U^\dagger \) satisfy \( \text{Tr} \delta U U_0^\dagger [U_0 B U_0^\dagger, A] = 0 \) for arbitrary antihermitian \( \delta U U_0^\dagger \), hence \( [U_0 B U_0^\dagger, A] = 0 \). For diagonal matrices \( A \) and \( B \) with distinct eigenvalues, this implies that \( U_0 B U_0^\dagger \) is diagonal and therefore that the saddle point \( U_0 \) are permutation matrices.

Gaussian fluctuations around the stationary point \( U_0 = P \) may be computed by writing \( U = e^X P \), \( X \) antihermitian, and by integrating over \( X \) after expanding the action to second order. Summing over all stationary points thus gives the “one-loop approximation” to the integral (1.1):

\[ I(A, B; s) = C' \sum_{P \in \mathcal{S}_N} e^{s \pi \sum_{i=1}^{N} a_{Pi} b_{Pi}} \int \prod_{i<j} d^2 X_{ij} e^{-\frac{s^2}{N} \sum_{i<j} |X_{ij}|^2} \prod_{i<j} (a_i - a_j)(b_{Pi} - b_{Pj}) \]

\[ = C' \sum_{P \in \mathcal{S}_N} e^{s \pi \sum_{i=1}^{N} a_{Pi} b_{Pi}} \left( \frac{\pi s}{N} \right)^{N(N-1)/2} \prod_{i<j} (a_i - a_j)(b_{Pi} - b_{Pj}) \]

\[ = C' \left( \frac{s \pi}{N} \right)^{N(N-1)/2} \frac{1}{\Delta(a) \Delta(b)} \sum_{P \in \mathcal{S}_N} \epsilon_P e^{s \pi \sum_{i=1}^{N} a_{Pi} b_{Pi}} \]

\[ = C' \left( \frac{s \pi}{N} \right)^{N(N-1)/2} \frac{\det \left( e^{s \pi a_{ij}} b_{ij} \right)}{\Delta(a) \Delta(b)} \] (2.13)

1 Recall [13] that if \( f(x) = \sum_{\ell \geq 0} f_\ell x^\ell \),

\[ \sum_{0 \leq \ell_1 < \ell_2 < \cdots < \ell_N} f_{\ell_1} \cdots f_{\ell_N} \det a_{ij} \det b_{ij} = \frac{1}{N!} \sum_{\ell_i \geq 0} \epsilon_P \epsilon_{P'} f_{\ell_1} \cdots f_{\ell_N} \prod_{i} a_{iP_i} b_{iP'_i} \]

\[ = \frac{1}{N!} \sum_{P, P'} \epsilon_P \epsilon_{P'} \prod_{i} \sum_{\ell_i} f_{\ell_i} (a_{P_i} b_{P'_i}) \]

\[ = \det f(a_i b_j) \]
which reproduces the previous result up to a constant $C'$. The latter may be determined,
for example by considering the $s \to 0$ limit, and the result reproduces (1.1). Thus the
stationary phase approximation of the original integral (1.1) or (2.4) turns out to give the
effect result! This well known empirical fact turned out to be a particular case of a general
situation analysed later by Duistermaat and Heckman [16]: if a classical system has only
periodic trajectories with the same period, the stationary phase (or saddle point) method
is exact.

In more precise mathematical terms, let $\mathcal{M}$ be a $2n$-dimensional symplectic manifold
with symplectic form $\omega$, and suppose that it is invariant under a $U(1)$ action. Let $H$ be the
Hamiltonian corresponding to this action (i.e. $dH = i_V \omega$, $V$ the vector field of infinitesimal
$U(1)$ action). Assume also that the fixed point set (the critical points) is discrete. Then
the theorem of Duistermaat-Heckman asserts that the stationary phase method is exact,
\[ \int \frac{\omega^n}{n!} e^{itH} = \left( \frac{2\pi}{t} \right)^n \sum_{P_c} e^{i\frac{\pi}{4} \text{sign}(\text{Hess}(P_c))} e^{itH(P_c)} \frac{\sqrt{\det \omega(P_c)}}{\sqrt{\det \text{Hess}(P_c)}} \quad (2.14) \]
where the sum is over (isolated) critical points $P_c$; the phase involves the signature
\[ \text{sign}(\text{Hess}(P_c)), \]
not the number of positive minus the number of negative eigenvalues,
of the Hessian matrix $\text{Hess}_{ij} = \frac{\partial^2 H}{\partial \xi_i \partial \xi_j} \bigg|_{P_c}$.

The integral (1.1) satisfies the conditions of the above theorem. The integration runs
over the orbit $\mathcal{O} = \{ M = UBU^\dagger \}$ of $B$ under the coadjoint action of $U$: this orbit, home-
omorphic to the manifold $U(N)/U(1)^N$, has the even dimension $N(N-1)$ and is in fact
a symplectic manifold. On two tangent vectors $V_i = [X_i, M], i = 1, 2$, ($X_i$ antihermitian),
tangent to $\mathcal{O}$ on $M$, the symplectic form reads $\omega(V_1, V_2) = \text{Tr} M [X_1, X_2]$. The Hamilto-
nian $H = \text{Tr} AM$ defines a periodic flow $M(t) = e^{itA} M(0)e^{-itA}$ if all eigenvalues of $A$ are
relatively rational. As the latter configurations form a dense set among diagonal matrices
$A$, this constraint may in fact be removed and this justifies a posteriori the stationary
phase calculation above. For further details on the Duistermaat-Heckman theorem in the
present context, the reader may consult also [17],[18].

3. Generalizations and extensions

3.1. Other groups

As already mentioned, the integral (2.1) appeared first in the work of Harish-Chandra [11],
as a simple application to compact groups of a general discussion of invariant differential
operators on Lie algebras. Following Harish-Chandra, let $G$ be a compact Lie group. Denote by $\text{Ad}$ the adjoint action of $G$ on itself and on its Lie algebra $\mathfrak{g}$, by $\langle . , . \rangle$ the invariant bilinear form on $\mathfrak{g}$, and by $\mathfrak{h} \subset \mathfrak{g}$ the Cartan subalgebra. If $h_1$ and $h_2 \in \mathfrak{h}$

$$
\Delta(h_1)\Delta(h_2) \int_G \text{d}g \exp(\text{Ad}(g)h_1,h_2) = \text{const} \sum_{w \in W} \epsilon(w) \exp\langle w(h_1),h_2 \rangle ,
$$

where $w$ is summed over the Weyl group $W$, $\epsilon(w) = (-1)^{\ell(w)}$, $\ell(w)$ is the number of reflections generating $w$, and $\Delta(h) = \prod_{\alpha > 0} \langle \alpha, h \rangle$, a product over the positive roots of $\mathfrak{g}$. In the case of $U(N)$, if we take $h_1 = iA$, $h_2 = iB$, (3.1) reduces to (2.1).

This extension to general compact $G$ may also be derived by constructing the heat kernel $K(g_1, \text{Ad}(g)g_2; s)$ on $G$ in terms of characters, by using Weyl’s formulae for characters and by averaging $K$ over $G$ [19]. Let’s sketch the derivation. One may always assume that $g_j = e^{h_j}$, $h_j \in \mathfrak{h}$, $j = 1, 2$ and one writes

$$
K(g_1, g_2; s) = \sum_{\lambda} d_{\lambda} \chi_{\lambda}(g_1 g_2^{-1}) e^{-\frac{1}{2} s C_{\lambda}}
$$

where $C_{\lambda}$ is the value of quadratic Casimir for the representation of weight $\lambda$, $d_{\lambda}$ the dimension of the latter. Integration over the adjoint action gives

$$
\int \text{d}g K(g_1, \text{Ad}(g)g_2; s) = \sum_{\lambda} \chi_{\lambda}(g_1) \chi_{\lambda}(g_2^{-1}) e^{-\frac{1}{4} s C_{\lambda}} .
$$

Weyl’s formula for the characters reads

$$
\chi_{\lambda}(e^{ih}) = \sum_{w \in W} \epsilon_{w} e^{i(\lambda + \rho, w(h))},
$$

same for $\lambda = 0$}

where $\rho$ is the Weyl vector, sum of all fundamental weights. As $C_{\lambda} = |\lambda + \rho|^2 - |\rho|^2$, the r.h.s of (3.3) is the exponential of a quadratic form in $\lambda$. The summation over $\lambda$ may be extended from the Weyl chamber to the full weight lattice, Poisson summation formula is then used and after taking a limit of infinitesimal $s$, $h_1$ and $h_2$, one is led to (3.1).

For completeness we also mention the generalisation of (1.1) involving Grassmannian coordinates and the integration over the supergroup $U(N_1 | N_2)$, see [20,21]. The case of integration over a pseudounitary group $U(N_1, N_2)$ has also been discussed, see [22] and further references therein.
3.2. Rectangular matrices

Consider the integral
\[ I^{(2)}(A, B; s) = \int_{U(N_2)} DU \int_{U(N_1)} DV \exp \frac{N}{s} \text{Tr}(AUBV^\dagger + \text{h.c.}) \quad (3.5) \]
where \( A \) is a complex \( N_1 \times N_2 \) matrix, \( B \) a complex \( N_2 \times N_1 \) matrix, and \( N = \min(N_1, N_2) \). Without loss of generality, one may assume that \( N_1 \geq N_2 \) and that the \( N_2 \times N_2 \) matrices \( A^\dagger A \) and \( BB^\dagger \) are diagonal, with real non negative eigenvalues \( a_i, b_i \) respectively. We assume again that the \( a \)'s on the one hand, the \( b \)'s on the other, are all distinct, so that neither of the Vandermonde \( N_2 \times N_2 \) determinants \( \Delta(a), \Delta(b) \), vanishes. Then the methods of heat equation or of character expansion presented in sec. 2 yield the following expression
\[ \Delta(a)\Delta(b)I^{(2)}(A, B; s) = \frac{\prod_{p=1}^{N_2-1} p! \prod_{q=1}^{N_1-1} q!}{\prod_{r=1}^{N_1-N_2-1} r!} \frac{1}{(s/N)^N(N_1-1)} \frac{\det I_{N_1-N_2}(2N \sqrt{a_ib_j}/s)}{\prod_{i=1}^{N_2}(a_ib_i)^{\frac{1}{2}(N_1-N_2)}}, \]
with \( I_\nu(z) \) the Bessel function \( I_\nu(z) = \sum_{n=0}^\infty \frac{1}{n!(n+\nu)!} \left( \frac{z}{2} \right)^{2n+\nu} \). This expression has been obtained for \( N_1 = N_2 \) in [21,15] and in the general case in [23].

Integrals (1.1) and (3.5) are the cases \( K = 1, 2 \) of an infinite set of unitary integrals which are exactly calculable:
\[ I^{(K)}(A_k, B_k; s) = \int_{U(N_k)} \prod_{k=1}^K DU_k \exp \frac{N}{s} \sum_{k=1}^K \text{Tr}A_kU_{k+1}B_kU_k^\dagger \]
\[ = \text{const} \frac{\det \phi(a_ib_j(N/s)^K)}{\Delta(a)\Delta(b)} \quad (3.7) \]
(index \( k \) is cyclic modulo \( K \)) where \( N = \min_k(N_k) \), \( A_k \) and \( B_k^\dagger \) are \( N_k \times N_{k+1} \) matrices, \( 1 \leq k \leq K \); the generalized hypergeometric series \( \phi \) is given by \( \phi(x) = \sum_{n=0}^\infty x^n / \prod_{k=1}^K(n + N_k - N)! \); and, assuming \( N_1 = N \), the \( a \) (resp. \( b \)) are the \( N \) (distinct) eigenvalues of \( A_1A_2\ldots A_K \) (resp. \( B_K\ldots B_1 \)).

In what follows, we shall concentrate on integrals (1.1) and (3.5); however the analysis applies equally well to the more general integral (3.7), see in particular [7] for a discussion of hypergeometric tau-functions of Toda lattice.

3.3. Correlation functions

Returning to the unitary group and the integral (1.1), it is also natural to consider the correlation functions associated with it, i.e. to compute the integrals
\[ \int_{U(N)} DU_{i_1j_1} \cdots U_{i_mj_m} U_{k_1\ell_1}^\dagger \cdots U_{k_m\ell_m}^\dagger \exp \frac{N}{s} \text{Tr}AUU^\dagger \quad (3.8) \]
Partial results have been obtained in [24] and in [25]. We still lack explicit and general expressions for these correlation functions.
4. Connection with integrable hierarchies

The integral \( (1.1) \) turns out to provide a non-trivial solution of the two-dimensional Toda lattice hierarchy [26]. This stems from the following observation: define

\[
\tau_N = \det \left( e^{\frac{1}{\hbar} a_i b_j} \right)_{1 \leq i, j \leq N}
\]

where \( \hbar = s/N \). Comparing with Eq. (2.1), we see that \( \tau_N = \hbar^{-N(N-1)/2} \prod_{p=0}^{N-1} (p!)^{-1} \). Then the following formula holds: (see also [26,27] for a two-matrix-model formulation)

\[
\tau_N = \det \left( \oint \oint \frac{du}{2\pi i u} \frac{dv}{2\pi i v} u^i v^j e^{\frac{1}{\hbar} \left( \sum_{q \geq 1} t_q u^q + \sum_{q \geq 1} \bar{t}_q v^q + u^{-1}v^{-1} \right)} \right)_{0 \leq i, j \leq N-1}
\]

where the integration contours are small enough circles around the origin, and with the traditional notations:

\[
t_q = \hbar \frac{1}{q} \sum_{i=1}^{N} a_i^q = \frac{s}{q} \theta_q \quad \bar{t}_q = \hbar \frac{1}{q} \sum_{i=1}^{N} b_i^q = \frac{s}{q} \bar{\theta}_q \quad q \geq 1 \]

Formula (4.2) can be easily proved by noting that \( e^{\frac{1}{\hbar} \sum_{q \geq 1} t_q u^q} = \prod_{i=1}^{N} (1 - ua_i)^{-1} \), \( e^{\frac{1}{\hbar} \sum_{q \geq 1} \bar{t}_q v^q} = \prod_{i=1}^{N} (1 - vb_i)^{-1} \), and expanding the contours to catch the poles at \( u = a_i^{-1} \), \( v = b_i^{-1} \). This makes \( \tau_N \) a tau function of the 2D Toda lattice hierarchy, as we shall discuss now.

4.1. Biorthogonal polynomials and 2D Toda lattice hierarchy

Noting that the parameter \( \hbar = s/N \) can always be scaled away we set \( \hbar = 1 \) throughout this section. We take Eq. (1.2) to be the definition of \( \tau_N \) as a function of the two infinite sets of times \( (t_q, \bar{t}_q), q \geq 1 \). We also set \( \tau_0 = 1 \).

Formula (4.2) suggests to introduce a non-degenerate bilinear form on the space of polynomials by

\[
\langle q | p \rangle = \oint \oint \frac{du}{2\pi i u} \frac{dv}{2\pi i v} p(u) q(v) e^{\frac{1}{\hbar} \left( \sum_{q \geq 1} t_q u^q + \sum_{q \geq 1} \bar{t}_q v^q + u^{-1}v^{-1} \right)}
\]

and normalized biorthogonal polynomials \( q_n(v) \) and \( p_n(u) \) with respect to the bilinear form above, that is polynomials of the form \( p_n(u) = \hbar_n^{-1} u^n + \cdots \) and \( q_n(v) = \hbar_n^{-1} v^n + \cdots \), such
that \( \langle q_m \| p_n \rangle = \delta_{m,n} \) for all \( m, n \geq 1 \). One can now replace monomials in Eq. (4.2) with biorthogonal polynomials and obtain immediately

\[
\tau_N = \prod_{i=0}^{N-1} h_i^2 .
\] (4.5)

Next we introduce the matrices of multiplication by \( u \) and \( v \) in the basis of biorthogonal polynomials:

\[
L_{mn} = \langle q_m \| u \| p_n \rangle \quad \bar{L}_{mn} = \langle q_m \| v \| p_n \rangle \quad m, n \geq 1 .
\] (4.6)

Note that

\[
L_{mn} = 0 \quad m > n + 1 , \quad \bar{L}_{mn} = 0 \quad n > m + 1 .
\] (4.7)

A standard calculation \[26\] leads to the following evolution equations for \( L \) and \( \bar{L} \) with respect to variations of the \( t_q, \bar{t}_q \):

\[
\frac{\partial L}{\partial t_q} = -[(L^q)_+, L] \quad \frac{\partial \bar{L}}{\partial \bar{t}_q} = -[(\bar{L}^q)_-, \bar{L}]
\] (4.8a)

\[
\frac{\partial \bar{L}}{\partial t_q} = -[(L^q)_+, \bar{L}] \quad \frac{\partial L}{\partial \bar{t}_q} = -[(\bar{L}^q)_-, \bar{L}] .
\] (4.8b)

Here \((\cdot)_\pm\) denotes the lower / upper triangular part plus one half of the diagonal part. Eqs. (4.8) are the standard form of the two-dimensional Toda lattice hierarchy \[28\] (up to a choice of sign of the \( t_q \)).

It is often more convenient to write these equations as quadratic equations in the set of \( \tau_N \). These are the “bilinear” Hirota equations. They are obtained by picking two sets of times \( (x_q, \bar{x}_q) \) and \( (y_q, \bar{y}_q) \) and writing in two different ways

\[
\oint \oint \frac{du}{2\pi iu} \frac{dv}{2\pi iv} p_{n,x,\bar{x}}(u)q_{m,y,\bar{y}}(v)e^{\sum_{q \geq 1} x_q u^q + \sum_{q \geq 1} \bar{x}_q u^{-q} + u^{-1} v^{-1}}
\]

where \( p_{n,x,\bar{x}} \) is the right biorthogonal polynomial associated to times \( (x_q, \bar{x}_q) \), whereas \( q_{m,y,\bar{y}} \) is the left biorthogonal polynomial associated to times \( (y_q, \bar{y}_q) \). One finds:

\[
\oint \frac{du}{2\pi iu} u^{-n} \tau_n(x_q - \frac{1}{q} u^{-q}, \bar{x}_q) \tau_{m+1}(y_q + \frac{1}{q} u^{-q}, \bar{y}_q) e^{\sum_{q \geq 1} (x_q - y_q) u^{-q}}
\]

\[
= \oint \frac{dv}{2\pi iv} v^{-m} \tau_m(y_q, \bar{y}_q - \frac{1}{q} v^{-q}) \tau_{n+1}(x_q, \bar{x}_q + \frac{1}{q} v^{-q}) e^{\sum_{q \geq 1} (y_q - \bar{x}_q) v^{-q}} .
\] (4.9)
Equivalently, equation (4.9) can be derived directly from the original expression (4.1) using
the set of determinant identities
\[
\sum_{i=1}^{n+1} \det(A - a_i, B) \det(A' + a_i, B') = \sum_{j=1}^{m+1} \det(A', B' - b'_j) \det(A + b'_j) \tag{4.10}
\]
where \(A = \{a_1, \ldots, a_{n+1}\}, B = \{b_1, \ldots, b_n\}, A' = \{a'_1, \ldots, a'_m\}, B' = \{b'_1, \ldots, b'_{m+1}\}\), and
a symbolic notation \(\det(\cdot, \cdot)\) is used for determinants of the form \(\det(\exp(\frac{1}{\hbar}a_ib_j))\).

Expanding Eq. (4.9) in powers of \(x - y\) and \(\bar{x} - \bar{y}\), expanding in power series in \(u\), \(v\) and performing the integration results in an infinite set of partial differential equations satisfied by the \((\tau_N)\).

Example: Expand to first order in \(x_1 - y_1\), and set \(m = n + 1\). The result is:
\[
\tau_{n+1} \tau_{n-1} = \tau_n \partial \bar{\partial} \tau_n - \partial \tau_n \bar{\partial} \tau_n \quad \forall n \geq 1 \tag{4.11}
\]
(with \(\partial = \partial / \partial t_1\), \(\bar{\partial} = \partial / \partial \bar{t}_1\)) which is a form of the Toda lattice equation. It is of course also the Desnanot–Jacobi determinant identity applied to Eq. (4.2).

Finally, the matrices \(L\) and \(\bar{L}\) satisfy an additional relation, the so-called string equation, which takes the form \(\tag{4.12}
\)
\[
[L^{-1}, \bar{L}^{-1}] = 1 .
\]

4.2. Large \(N\) limit as dispersionless limit

We now restore the parameter \(\hbar\), which is required for the large \(N\) limit. Indeed, as \(N \to \infty\), \(\hbar\) must be sent to zero, keeping \(s = \hbar N\) fixed. We define
\[
f(t_q, \bar{t}_q; s) = \lim_{N \to \infty} \hbar^2 \log \tau_N(t_q, \bar{t}_q) \tag{4.13}
\]
where \(\tau_N(t_q, \bar{t}_q)\) is defined by Eq. (4.2). In a region of the space of parameters \((t_q, \bar{t}_q)\) which includes a neighborhood of the origin \(t_q = \bar{t}_q = 0\), \(f(t_q, \bar{t}_q; s)\) is governed by “dispersionless” equations we shall describe now. \(f(t_q, \bar{t}_q; s)\) is called in this context the dispersionless tau-function.

First, comparing with the expression (1.3) of the free energy \(F(\theta_q s^{-q/2}, \bar{\theta}_q s^{-q/2})\) (where we have explicitly stated the dependence of \(F\) on the \(\theta_q\) and the \(\bar{\theta}_q\) and scaled away the parameter \(s\)), using Eq. (2.1) and definitions (1.3), we see that
\[
f(t_q, \bar{t}_q; s) = -\frac{1}{2} s^2 \log s + \frac{3}{4} s^2 + s^2 F(q t_q s^{-q/2-1}, q \bar{t}_q s^{-q/2-1}) . \tag{4.14}
\]
This is the scaling form of the dispersionless tau function.\footnote{2}

The dispersionless Toda hierarchy is a classical limit of the Toda hierarchy: it is obtained by replacing commutators with Poisson brackets defined by

\[ \{g(z,s), h(z,s)\} = z \frac{\partial g}{\partial z} \frac{\partial h}{\partial s} - z \frac{\partial h}{\partial z} \frac{\partial g}{\partial s} \]  

Here \( z \) is the classical analogue of the shift operator \( Z_{ij} = \delta_{i,j+1} \), which justifies that \( \{\log z, s\} = 1 \).

The Lax Eqs. (4.8) become

\[ \frac{\partial \ell}{\partial t_q} = -\{(\ell^q)_+, \ell\} \quad \frac{\partial \ell}{\partial t_q} = \{(\ell^q)_-, \ell\} \]  

\[ \frac{\partial \bar{\ell}}{\partial t_q} = -\{(\bar{\ell}^q)_+, \bar{\ell}\} \quad \frac{\partial \bar{\ell}}{\partial t_q} = \{(\bar{\ell}^q)_-, \bar{\ell}\} \]  

(4.16a)

(4.16b)

where \((\cdot)_\pm\) now refers to positive and negative parts of Laurent expansion in \( z \), and \( \ell(z,s,t) \) and \( \bar{\ell}(z,s,t) \) have the following \( z \) dependence:

\[ \ell = rz + \sum_{k=0}^{\infty} \lambda_k z^{-k} \]  

\[ \bar{\ell} = rz^{-1} + \sum_{k=0}^{\infty} \bar{\lambda}_k z^k \]  

(4.17a)

(4.17b)

related to the structure (4.7) of \( L \) and \( \bar{L} \).

These equations imply that the differential forms \( d\varphi(\ell,s,t) \), \( d\bar{\varphi}(\bar{\ell},s,t) \) are closed:

\[ d\varphi = m d\ell/\ell + \log z ds + \sum_{q \geq 1} (\ell^q)_+ dt_q + \sum_{q \geq 1} (\bar{\ell}^q)_- d\bar{\ell}_q \]  

\[ d\bar{\varphi} = \bar{m} d\bar{\ell}/\bar{\ell} + \log z ds + \sum_{q \geq 1} (\ell^q)_+ dt_q + \sum_{q \geq 1} (\bar{\ell}^q)_- d\bar{\ell}_q \]  

(4.18a)

(4.18b)

where \( m \) and \( \bar{m} \) are Orlov–Shulman functions, which can be characterized by

\[ m = \sum_{q \geq 1} q t_q \ell^q + s + \sum_{q \geq 1} \frac{\partial f}{\partial t_q} \ell^{-q} \]  

\[ \bar{m} = \sum_{q \geq 1} q \bar{t}_q \bar{\ell}^q + s + \sum_{q \geq 1} \frac{\partial f}{\partial \bar{t}_q} \bar{\ell}^{-q} \]  

(4.19a)

(4.19b)

In fact, one can get rid of one more parameter since \( A \) and \( B \) can be scaled independently, but we choose not to do so for symmetry reasons.
and satisfy the “dressed” Poisson bracket relations

\[ \{ \log \ell, m \} = 1 \quad \{ \log \bar{\ell}, \bar{m} \} = 1. \tag{4.20} \]

In the present case, we have the following constraints, which determine uniquely the solution of the dispersionless Toda hierarchy:

\[ m = \bar{m} = (\ell \bar{\ell})^{-1}. \tag{4.21} \]

Eqs. (4.21) are directly related (via Eqs. (4.20)) to the classical limit of the string equation (4.12), i.e. \( \{ \ell^{-1}, \bar{\ell}^{-1} \} = 1 \).

Let us call \( a = \ell^{-1} \) and \( b = \bar{\ell}^{-1} \). The fact that \( m = \bar{m} \) implies that \( \varphi(a) \) and \( \bar{\varphi}(b) \) are related by Legendre transform, or that their derivatives \( b(a) = \frac{\text{d}}{\text{d}a} \varphi(a) \) and \( a(b) = \frac{\text{d}}{\text{d}b} \bar{\varphi}(b) \) are functional inverses of each other (the latter fact was derived by \textit{ad hoc} methods in [30] and [31]).

### 4.3. Application of the dispersionless formalism

In what follows we set \( s = 1 \), so that \( \theta_q = qt_q \). In order to explore the structure of the function \( f(t_q, \tilde{t}_q; s) \), we now assume that only a finite number of \( t_q \) and \( \tilde{t}_q \) is non-zero. Note that this cannot happen if the eigenvalues are real; however, here we are interested in properties of the integral (1.1) as a formal power series in the \( t_q \) and therefore we do not worry about the actual support of the eigenvalues. Then, according to Eqs. (4.17) and (4.19), the Laurent expansion of \( a(z) \) and \( b(z) \) is finite, of the form:

\[ a = \sum_{q=1}^{\bar{n}+1} \alpha_q z^{-q} \quad b = \sum_{q=1}^{n+1} \beta_q z^q. \tag{4.22} \]

where \( n = \max\{ q \mid t_q \neq 0 \} \), \( \bar{n} = \max\{ q \mid \tilde{t}_q \neq 0 \} \), \( \alpha_1 = \beta_1 = 1/r \), \( \alpha_{n+1} = \bar{n}t_n r^{n+1} \), \( \beta_{n+1} = nt_n r^{n+1} \) (\( r \) is defined by Eqs. (4.17)). Then it is easy to show that \( a \) and \( b \) satisfy an algebraic equation, some coefficients of which can be worked out explicitly:

\[ b^{n+1} a^{n+1} - b^n a^n b^n(a^{-1} \theta_1 + \cdots + \theta_n) - a^n(b^{-1} \tilde{\theta}_1 + \cdots + \bar{\theta}_n) + \text{lower order terms} = 0. \tag{4.23} \]

Plugging Eq. (4.22) into Eq. (4.23) leads to algebraic equations for the coefficients \( \alpha_q, \beta_q \) as a function of the \( \theta_q, \tilde{\theta}_q \).

\(^3\) The constraint \( m = \bar{m} \) is true for a more general class of solutions of Toda, see for example sec. 4.4 and [23].
Examples:

- If $\theta_q = \delta_{q1}\bar{\theta}_1$, one can set $\bar{\theta}_1 = 1$ by homogeneity. In this case Eq. (4.23) is quadratic in $b$. In terms of $b$ and $\ell = 1/a$ it reads

$$b^2 - b(\ell + \theta_1\ell^2 + \cdots + \theta_n\ell^{n+1}) - \ell P(\ell) = 0$$

(4.24)

where $P$ is a polynomial with $P(0) = 1$. One way to determine $P$ is to note that Eq. (4.22) provides a rational parameterization of $a$ and $b$, so that the resulting curve $(a, b)$ has genus zero, which forces the discriminant of Eq. (4.24) to be of the form $\ell(1 + \ell/(4\psi^2))Q(\ell)^2$ where $Q$ is a polynomial of degree $n$ with $Q(0) = 2$, and $\psi$ is a constant. This yields the expression

$$b(\ell) = \frac{1}{2} \left( \ell + \theta_1\ell^2 + \cdots + \theta_n\ell^{n+1} + \sqrt{\ell(1 + \frac{\ell}{4\psi^2})Q(\ell)} \right).$$

(4.25)

Considering Eq. (4.19a) as an asymptotic expansion of $m = b/\ell$ as $\ell \to \infty$ fixes $Q$: $Q$ is the polynomial part of $2\psi(1 + \theta_1\ell + \cdots + \theta_n\ell^n)/\sqrt{1 + 4\psi^2/\ell}$. Imposing $Q(0) = 2$ leads to

$$\psi = 1 + \sum_{q=1}^{n} (-1)^{q+1}(\frac{2q)!}{(q!)^2}\theta_q\psi^{2q+1}$$

(4.26)

Note that $\psi = r^2 = \partial^2 F/\partial\theta_1\partial\bar{\theta}_1 = \exp(\partial^2 F/\partial s^2)$ (the latter equality being the dispersionless limit of Toda Eq. (1.11)).

By Lagrange inversion, Eq. (4.26) allows to get the exact expansion of $\psi$ as well as of $F$ in powers of the $\theta_q$. One finds, restoring the $\bar{\theta}_1$ dependence:

$$\psi = \sum_{\alpha_1,\ldots,\alpha_n=0}^{\infty} \frac{(\sum_{q\geq 1}(2q+1)\alpha_q)!}{(\sum_{q\geq 1}2q\alpha_q+1)!} \Pi_{q\geq 1}^{\alpha_q} \frac{1}{\alpha_q!} \left( (-1)^{q+1}(\frac{2q)!}{(q!)^2}\theta_q \right)^\alpha_q \sum_{q=0}^{\infty} q^{\alpha_q}$$

$$F = \sum_{\alpha_1,\ldots,\alpha_n=0}^{\infty} \frac{(\sum_{q\geq 1}(2q+1)\alpha_q-3)!}{(\sum_{q\geq 1}2q\alpha_q)!} \Pi_{q\geq 1}^{\alpha_q} \frac{1}{\alpha_q!} \left( (-1)^{q+1}(\frac{2q)!}{(q!)^2}\theta_q \right)^\alpha_q \sum_{q=0}^{\infty} q^{\alpha_q}.$$  

(4.27)

- If $\theta_q = \delta_{qn}\bar{\theta}_n$, $\bar{\theta}_q = \delta_{qn}\bar{\bar{\theta}}_n$ (with the same $n$), one can show using the formalism above that $\psi = \partial^2 F/\partial\theta_1\partial\bar{\theta}_1$ satisfies the equation:

$$\psi = 1 + (n+1)\theta_n\bar{\theta}_n\psi^{n+2}$$

(4.28)

so that

$$\psi = \sum_{k\geq 1} (n+1)^k \frac{(n+2)k!}{((n+1)k+1)!k!} (\theta_n\bar{\theta}_n)^k \quad F = \sum_{k\geq 1} (n+1)^k \frac{(n+2)k-3)!}{((n+1)k)!k!} (\theta_n\bar{\theta}_n)^k.$$  

(4.29)

For example, for $n = 2$, $F = \frac{1}{2}(\theta_2\bar{\theta}_2) + \frac{3}{4}(\theta_2\bar{\theta}_2)^2 + \frac{1}{2}(\theta_2\bar{\theta}_2)^3 + \cdots$
4.4. Case of rectangular matrices

All of the formalism above applies equally well to the integral (3.5) over rectangular matrices. The latter has a “diagonal” character expansion i.e.

\[ \tau_N^{(2)} = \text{const} I^{(2)}(A, B; s) = \sum_{\lambda} \hbar^{-|\lambda|} \frac{1}{\prod_p (\lambda_p + p + 1)!} \frac{\Delta_\lambda(a) \Delta_\lambda(b)}{\Delta(a) \Delta(b)} \]

where \( \hbar = s/N, N = \min(N_1, N_2), \nu = |N_1 - N_2| \).

The form of Eq. (4.30) alone proves that the \( \tau_N^{(2)} \) form a tau function of 2D Toda lattice (for fixed \( \nu \)). There is also a determinant formula similar to Eq. (4.2) which we shall not write.

The large \( N \) limit is taken keeping \( \hbar N = s \) and \( \hbar \nu \equiv \xi \) fixed. The times are \( t_q = \hbar q \text{Tr}(AA^\dagger)^q, \bar{t}_q = \hbar q \text{Tr}(BB^\dagger)^q \). Biorthogonal polynomials and the dispersionless equations can be obtained similarly as above.

We have thus constructed a whole family \( g_\xi(t_q, \bar{t}_q; s) = \lim_{N \to \infty} \hbar^2 \log \tau_N^{(2)} \) of dispersionless tau functions. The diagonal character expansion (4.30) implies that they belong to the class of solutions that satisfy \( m = \bar{m} \). After some calculations, one finds that the constraint analogous to Eq. (4.21) is

\[ m(m + \xi) = \bar{m}(\bar{m} + \xi) = (\ell \bar{\ell})^{-1} \]

Note that if \( \xi = 0 \) i.e. \( N_1 = N_2 \), the constraint (4.31) becomes \( m = \bar{m} = (\ell \bar{\ell})^{-1/2} \), which suggests to redefine \( \ell = \ell_2, \bar{\ell} = \bar{\ell}_2 \). The new functions \( \ell_2, \bar{\ell}_2 \) satisfy the string equation \( \{\ell_2^{-1}, \bar{\ell}_2^{-1}\} = 1 \) of the integral (1.1), and of course the same Toda equations (4.16) for even times, with the replacement \( t_{2q} \to q, \bar{t}_{2q} \to \bar{q} \). In fact, one can check that we recover in this case the dispersionless tau function of the usual integral (1.1) with odd moments equal to zero: \( g_0(u_q, \bar{u}_q; s) = 2f(t_{2q} = u_q, t_{2q+1} = 0, \bar{t}_{2q} = \bar{u}_q, \bar{t}_{2q+1} = 0; s) \).

5. Diagrammatic expansion and combinatorics

5.1. A Feynman diagram expansion of \( F \)

To develop a Feynman diagram expansion of (1.1), we first trade the integration over the unitary group for an integration over \( N \times N \) complex matrices \( X \) by writing

\[ e^{\frac{N}{2} \text{Tr} AUBU^\dagger} = \left( \frac{\pi}{N} \right)^{-N^2} \int DXDX^\dagger e^{-N \text{Tr} XX^\dagger} \exp \frac{N}{\sqrt{s}} \text{Tr}(AX^\dagger + XBU^\dagger) \]

(5.1)
Thus,
\[
I(A, B; s) = \text{const} \int DX DX^\dagger e^{-N \text{Tr}(XX^\dagger) + N^2 W(AXBX^\dagger; s)}
\]  
(5.2)
where
\[
e^{N^2 W(X_1, X_2; s)} = \int DU \exp \frac{N}{\sqrt{s}} \text{Tr}(UX_1 + U^\dagger X_2).
\]  
(5.3)

This integral is known exactly in this large \(N\) limit [32]: with the same abuse of notations as mentioned in sec. 1,
\[
W(Y; s) = \sum_{\alpha \vdash n} \frac{\text{tr}_\alpha Y}{\prod_p (\alpha_p! p^{\alpha_p})},
\]  
(5.4a)
\[
W_\alpha = (-1)^n \frac{(2n + \sum \alpha_p - 3)!}{(2n)!} \prod_{p=1}^n \left( \frac{-(2p)!}{p!(p-1)!} \right)^{\alpha_p},
\]  
(5.4b)
and \(W_\alpha\) is an integer, as follows from the recursion formulae discussed in [32].

The form (5.2) is adequate to develop a diagrammatic expansion à la Feynman of \(F(A, B; s)\) (see [33,34] for reviews). Recall that double lines are conveniently introduced to encode the conservation of indices [35]. The inverse of the first term in the exponential of (5.2) yields the propagator, \(\langle X_{ij}X_{kl}^\dagger \rangle = \frac{1}{N} \delta_{il} \delta_{jk}\), while each term in \(W\), i.e. each monomial \(\text{tr}_\alpha(AXBX^\dagger) / \prod_p (\alpha_p! p^{\alpha_p})\) gives rise to a multi-vertex of type \(\alpha\), (see Fig. 1), which comes with a weight \(N^2 - \sum \alpha_p W_\alpha\) times products of \(a\)'s and \(b\)'s and Kronecker symbols expressing the conservation of indices. When these Kronecker symbols are “contracted”, they leave sums over closed circuits (or faces) of powers of \(a\)'s or \(b\)'s. A face of side \(p\) thus contributes a factor \(N\theta_p\), resp. \(N\bar{\theta}_q\) with the notation of (1.2). To put it another way, the Feynman diagrams may be regarded as bicolourable, with faces carrying alternatingly \(a\) or \(b\) “colour”.

As is well known, only connected diagrams contribute to the free energy \(\log I\). In the discussion of this connectivity, each multi-vertex just introduced must be regarded as a connected object, and it is useful to keep track of this fact by drawing a tree of dotted lines which connect the various traces which compose it, see Fig. 1.b.

\[\text{Fig. 1: (a) a vertex of type [2]; (b) a multi-vertex of type [1^2 2^1 3^1].}\]
If a diagram contributing to $\log I$ has $P$ propagators, $V_\alpha$ vertices of type $\alpha$ and $L$ loops of indices, and builds a surface of total genus $g$ with $c$ connected components, it carries a power of $N$ equal to

$$
\#(N) = -P + \sum_\alpha V_\alpha (2 - \sum_p \alpha_p) + L
= (\sum_\alpha V_\alpha \sum_p \alpha_p - P + L) + 2 \sum_\alpha V_\alpha (1 - \sum_p \alpha_p)
= 2 - 2g + 2 \left ( c - 1 - \sum_\alpha V_\alpha (\sum_p \alpha_p - 1) \right ) \leq 2 - 2g ,
$$

where the last inequality expresses that to separate the diagram into $c$ connected components, one must cut at most its $\sum_\alpha V_\alpha (\sum_p \alpha_p - 1)$ dotted lines.

This simple counting (see the second ref [32]) shows that the leading $O(N^2)$ terms contributing to $F$ are obtained as the sum of planar (i.e. genus $g = 0$) and minimally connected Feynman diagrams, such that cutting any dotted line makes them disconnected. They are thus trees in these dotted lines.

These Feynman rules, supplemented by the usual prescriptions for the symmetry factor of each diagram (equal to the inverse of the order of its automorphism group), are what is needed to get a systematic expansion of $F$ in powers of the $\theta$ and $\bar{\theta}$.

$$
F = \sum_{\text{minimally connected diagrams } \mathcal{G}} \frac{s^{-P}}{|\text{Aut } \mathcal{G}|} \prod_{\text{vertices}} W_\alpha \prod_{\text{a-faces}} \theta_p \prod_{\text{b-faces}} \bar{\theta}_q . \tag{5.5}
$$

Note that these rules imply that the sign attached to each diagram (coming from the signs of the $W_\alpha$) is also $(-1)^L = (-1)^\text{power of } \theta \text{+ power of } \bar{\theta}$.

**Examples:**

$W(XX^\dagger; s)$ contains a term linear in $XX^\dagger$ (with coefficient $1/s$), which may be inserted in arbitrary number on any propagator without changing the topology of the diagram: each individual insertion (which raises the corresponding power of $a$ and $b$ by one unit) is depicted by a cross on the propagator.

Terms in $F$ linear in $\theta$ and $\bar{\theta}$ come solely from these insertions, which yields

$$
\sum_{p=1}^{\infty} \frac{1}{p} \frac{\theta_p \bar{\theta}_p}{s^p} \quad \text{(a single loop with } p \text{ insertions has a symmetry factor equal to } p).$$

Terms in $F$ of the form $(\theta_1 \bar{\theta}_1/s)^p$ come entirely from a multivertex of type $tr_{[1^n]}XX^\dagger$, whence a contribution $2^p \frac{(2p-3)!}{p!(2p)!}$. More generally, the term in $F$ of the form $\prod_p (\theta_p)^{\alpha_p} \bar{\theta}_p^{\alpha_p}$ comes solely from the multivertex of type $\alpha$, $\alpha \vdash n$, in which the contraction of $b$ lines into “petals” is unique and determines that of $a$ lines. The weight $W_\alpha$ together with the symmetry factor $\prod_p \alpha_p^p \alpha_p!$ reproduces the result of (4.27).

Likewise, it is easy to find the diagrams contributing to the $(\theta_2 \bar{\theta}_2/s^2)^2$ and $(\theta_2 \bar{\theta}_2/s^2)^3$ terms in $F$ which give $(\frac{1}{2} + (\frac{1}{2})^2)(\theta_2 \bar{\theta}_2/s^2)^2 = \frac{9}{4}(\theta_2 \bar{\theta}_2/s^2)^2$ and

\[ \left(\frac{1}{2} + \frac{1}{3!} \times 8 + \frac{1}{2} \times 4 + \frac{1}{3!} \times 2^2\right) \left(\frac{\theta_2 \bar{\theta}_2}{s^2}\right)^3 = \frac{9}{2} \left(\frac{\theta_2 \bar{\theta}_2}{s^2}\right)^3, \]

in agreement with the expansion of the end of sec. 4. Conversely, one sees how effective the methods of sec. 4 are to resum classes of Feynman diagrams. Indeed it is not obvious how to derive directly from the diagrammatic expansion that $F(\theta_2 \bar{\theta}_2)$, or more generally $F(\theta_n \bar{\theta}_n)$ and $\psi(\theta_n \bar{\theta}_n)$, have the simple form given by Eq. (4.29). In particular it would be interesting to find a direct combinatorial proof of the equation (4.28) satisfied by $\psi$. Its form suggests a possible connection with decorated rooted trees, perhaps in the spirit of (36).

Remark: Actions of the form of Eq. (5.2) and their diagrammatic expansion are a generalization of the “dually weighted graphs” of (37): the extra ingredient is that graphs are required to be bicolored and the black/white faces are weighted separately. They are also a generalization of the bicolored diagrams of (38) – the faces of the latter being weighted only according to their color and not to their number of edges (this corresponds to the case where $A$ and $B$ are projectors).

5.2. Combinatorics

We now show that this expansion is in fact equivalent to the one recently obtained in (12). What is remarkable is that the methods are orthogonal: (12) is based on manipulations in the symmetric group, and in particular uses some results on the number of solutions of equations for permutations (which are however themselves related to planar constellations...
We now explain the results of [12] that are relevant here, and show their equivalence to our Feynman diagram expansion.

First we explain how to associate to a (not necessarily connected) bicolored map $g$, a pair of permutations $\sigma(g)$ and $\tau(g)$ of the set of edges. $\sigma(g)$ (resp. $\tau(g)$) is the permutation which to an edge associates the next edge obtained by clockwise rotation around the white (resp. black) vertex to which it is connected, see the example of Fig. 2a). This is a one-to-one correspondence.

An important remark is that the permutation $\sigma(g)\tau(g)$ encodes the faces of the map. Indeed, cycles of $\sigma(g)\tau(g)$ are obtained by turning clockwise around each face and keeping only the edges going from a white to a black vertex. It is implied that each connected component has its own face at infinity (for which rotation must be made counterclockwise). Consequently, the lengths of the cycles of $\sigma(g)\tau(g)$ are precisely one half of the sizes of the faces.

![Fig. 2: a) A bicolored planar map and b) its dual. $\sigma = (1\ 2\ 11\ 3\ 4\ 5\ 6\ 7\ 10\ 8\ 9)$, $\tau = (1\ 2\ 3\ 6\ 4\ 5\ 7\ 8\ 9\ 11\ 10)$, $\sigma\tau = (1\ 3\ 5\ 6\ 8\ 11\ 2\ 10\ 4\ 7\ 9)$.](image)

Also, for $\rho$ a permutation of the set of edges, define $\Pi_\rho$ to be the partition of the set of edges into the orbits (cycles) of $\rho$. For two such partitions $\Pi$ and $\Pi'$, we say that $\Pi \leq \Pi'$ iff every block of $\Pi$ is included in a block of $\Pi'$; and define $\Pi \vee \Pi'$ to be the smallest common majorant. Also, call $\#\Pi$ the number of blocks of $\Pi$. Finally, if $n$ is the number of edges, define $C_\rho \vdash n$ to be the partition of the integer $n$ corresponding to $\Pi_\rho$, i.e. the lengths of the cycles of $\rho$.

Then the results of [12] (Theorems 2.12, 4.2) can be reformulated as follows:

$$F(A,B;s) = \sum_{n=1}^{\infty} \frac{s^{-n}}{n!} \sum_{\sigma,\tau \in S_n} \sum_{\Pi_\sigma \leq \Pi_\tau} \gamma(\sigma\tau, \Pi_\sigma \vee \Pi_\tau) \text{tr}_{C_\sigma} A \text{tr}_{C_\tau} B.$$ (5.6)

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The coefficient $\gamma(\rho, \Pi)$ is only defined explicitly if $\Pi = \Pi_{\rho}$, in which case $\gamma(\rho, \Pi_{\rho}) = W_{C_{\rho}}$, where $W_{\alpha}$ is given by Eq. (5.4). In general, we have the following expression for $\gamma(\rho, \Pi)$:

$$
\gamma(\rho, \Pi) = \sum_{\Pi' \geq \Pi_{\rho}, \#(\Pi \cup \Pi') = 1} \prod_{\text{blocks } i} W_{C_{\rho_i}}
$$

(5.7)

where the $\rho_i$ are the restrictions of $\rho$ to each block $i$ of the partition $\Pi'$. This expression looks somewhat complicated but it is very easy to interpret once we have related this formalism to the diagrammatic expansion of the previous section.

In order to proceed with the equivalence, we first associate to the pair of permutations $\sigma$ and $\tau$ a map according to the construction above. Since $\sigma$ and $\tau$ are permutations of $\{1, \ldots, n\}$, this produces a map with labelled edges from 1 to $n$. The quantity $\#\Pi_{\sigma} + \#\Pi_{\tau} + \#\Pi_{\sigma\tau} - n$ is simply the Euler–Poincaré characteristic of the map and the condition on the summation simply imposes planarity of the map (or more precisely, of each of its connected components).

Secondly, we unlabel the resulting map. That is, since the summand of Eq. (5.6) does not actually depend on the labelling, one can sum together maps which are only distinguished by the labelling of edges; by definition of the symmetry factor of a map (inverse of the order of the automorphism group, i.e. here number of permutations that commute with both $\sigma$ and $\tau$), the only modification this produces is to replace the $1/n!$ with the symmetry factor of the unlabelled map. Thus, we rewrite Eq. (5.6):

$$
F(A, B, s) = \sum_g \frac{s^{-n}}{|\text{Aut } g|} \gamma(\sigma(g)\tau(g), \Pi_{\sigma(g)} \lor \Pi_{\tau(g)}) \ tr_{C_{\sigma(g)}} A \ tr_{C_{\tau(g)}} B
$$

(5.8)

where the summation is over inequivalent (possibly disconnected) planar maps $g$, and $n$ is the number of edges.

Thirdly, we replace these maps $g$ with their dual maps $\hat{g}$. Note that this must be performed separately for each connected component. The correspondence is again one-to-one. An example is given on Fig. 2. The new maps have bicolored faces; equivalently, one can orient their edges so that white (resp. black) vertices correspond to clockwise (resp. counterclockwise) faces.

The resulting maps $\hat{g}$ are very similar to the Feynman diagrams of sec. 5.1; however, they still lack the “dotted lines” which link together the various connected components. This is where Eq. (5.7) comes in. Indeed to each term in the sum (5.7) we associate one
particular set of dotted lines as follows: since $\Pi' \geq \Pi_{\sigma(g)\tau(g)}$, and the cycles of $\sigma(g)\tau(g)$ are associated to faces of $g$, one can think of $\Pi'$ as ways of grouping together faces of $g$, that is vertices of $\hat{g}$: these are precisely the dotted lines. Finally, the other conditions in the summation of Eq. (5.7) can be interpreted as follows: $\#(\Pi' \lor \Pi_{\sigma(g)} \lor \Pi_{\tau(g)}) = 1$ ensures that the diagram including dotted lines is connected; and $\#\Pi_{\tau(g)\sigma(g)} - \#\Pi' = \#(\Pi_{\sigma(g)} \lor \Pi_{\tau(g)}) - 1$ ensures that it is “minimally connected”, i.e. that it has a tree structure.

Finally, we compare the weights: the $WC_{\rho_i}$ are associated to groups of vertices of $\hat{g}$ linked together by dotted lines, just as in the previous section; as to the $trC_{\sigma(g)}A$ (resp. $trC_{\tau(g)}B$), they associate to each white (resp. black) vertex of $g$, that is each clockwise (resp. counterclockwise) face of $\hat{g}$, a weight of $\theta_p$ (resp. $\bar{\theta}_p$) where $p$ is the number of edges surrounding it, which is again what is required.

Remark: the special case $\Pi_{\sigma(g)} \lor \Pi_{\tau(g)} = \Pi_{\sigma(g)\tau(g)}$, for which the associated weight is a single $WC_{\sigma(g)\tau(g)}$, occurs when $g$ is a disjoint union of bicolored trees, or equivalently when all vertices of $\hat{g}$ are linked together by dotted lines.

5.3. Case of rectangular matrices

We sketch here how the diagrammatic expansion of sec. 5.1 can be generalized to the case of the integral (3.5) over rectangular matrices. Introduce complex rectangular $N_1 \times N_2$ matrices $X$ and $Y$; then, applying the same trick as above, we find

$$\int_{U(N_2)} DU \int_{U(N_1)} DV \exp \frac{N}{s} \text{Tr}(AUBV^\dagger + VB^\dagger U^\dagger A^\dagger)$$

$$= \text{const} \int DX DX^\dagger DY DY^\dagger e^{-N' \text{Tr}(XX^\dagger + YY^\dagger)} + N_1^2 W(X^\dagger AA^\dagger Y; s) + N_2^2 W(B^\dagger Y^\dagger XB; s) \quad (5.9)$$

where $N = \min(N_1, N_2)$, $N' = \max(N_1, N_2)$. The diagrammatic expansion is identical to that of sec. 5.1, except that now there are two types of vertices, weighted by extra factors $N_1/N'$ and $N_2/N'$ respectively. Due to the presence of $X$ and $Y^\dagger$, resp. $Y$ and $X^\dagger$, in these vertices, we see that they must alternate and therefore these diagrams have both faces and vertices bicolored. Up to a factor of 2 corresponding to the two possible colorings of the vertices, this is the same as requiring diagrams to have bicolored even-sized faces. Each face of size $2p$ receives a weight $\text{Tr}(AA^\dagger)^p$ or $\text{Tr}(BB^\dagger)^p$ depending on its color (orientation).

Note that if $N_1 = N_2 = N$, the diagrams are weighted identically as in sec. 5.1; this means that the integral (3.5) for square matrices can be considered in the large $N$ limit as the particular case of the integral (1.1) for which all odd moments vanish (cf a similar observation in sec. 4.4).
6. Summary of results, Tables

One may compute the first terms of the expansion \( F(A, B; s) = \sum_{n=1}^{\infty} \frac{1}{n!} F_n(\theta, \tilde{\theta}) \). It is sufficient to tabulate them for \( \theta_1 = \tilde{\theta}_1 = 0 \) since if we write \( A = A' + \theta_1 I, B = B' + \tilde{\theta}_1 I \), with \( A' \) and \( B' \) traceless, \( F(A, B; s) = F(A', B'; s) + \theta_1 \tilde{\theta}_1 / s \). Or alternatively

\[
F_n(\theta_1, \tilde{\theta}_1, \theta_2, \tilde{\theta}_2, \ldots) = \theta_1 \tilde{\theta}_1 \delta_{n1} + F_n(0, 0, \theta_2, \tilde{\theta}_2, \ldots)
\]

with \( \theta_p = \sum_{q=0}^{p} \binom{p}{q} \theta_q (-\theta_1)^{p-q} \) and likewise for \( \tilde{\theta}_p \). Up to order 8, we find:

\[
F_1 = 0, \quad F_2 = \frac{1}{8} \theta_2 \tilde{\theta}_2, \quad F_3 = \frac{1}{8} \theta_3 \tilde{\theta}_3,
\]

\[
F_4 = \frac{1}{8} \left[ \theta_2^2 \left( 3 \theta_4^2 - 2 \theta_4 + \theta_4 \right) - \theta_4 \left( -2 \theta_2^2 + \theta_4 \right) \right],
\]

\[
F_5 = \frac{1}{8} \left[ \theta_2 \theta_3 \left( 20 \theta_2 \theta_3 - 6 \theta_2 \theta_3 + 7 \theta_2 \theta_3 - 5 \theta_2 + 3 \theta_3 \right) \right],
\]

\[
F_6 = \frac{1}{8} \left[ \theta_2^3 \left( 27 \theta_2^2 - 16 \theta_2^2 - 30 \theta_2 \theta_3 + 7 \theta_2 \theta_3 + 15 \theta_2 \theta_3 - 3 \theta_3 \right) \right] + 3 \theta_2 \theta_4 \left( -10 \theta_2^2 + 5 \theta_2^2 + 10 \theta_2 \theta_4 - 2 \theta_6 \right) + \theta_6 \left( 7 \theta_2^2 - 3 \theta_2^2 - 6 \theta_2 \theta_4 + \theta_6 \right),
\]

\[
F_7 = \theta_2^2 \theta_3 \left( 66 \theta_2^2 \theta_3 - 21 \theta_2 \theta_3 - 20 \theta_3 \theta_4 + 4 \theta_7 \right) - \theta_3 \theta_4 \left( 20 \theta_2^2 \theta_3 - 5 \theta_2 \theta_4 + 6 \theta_2 \theta_5 + \theta_7 \right) - \theta_2 \theta_5 \left( 21 \theta_2^2 \theta_3 - 6 \theta_2 \theta_4 - 6 \theta_2 \theta_5 + \theta_7 \right) + \frac{1}{7} \theta_7 \left( 28 \theta_2^2 \theta_3 - 7 \theta_2 \theta_4 + 2 \theta_7 \right),
\]

\[
F_8 = \frac{1}{8} \left[ 3 \theta_2^4 \left( 117 \theta_2^4 - 192 \theta_2^4 - 180 \theta_2^2 \theta_4 - 25 \theta_2^2 \theta_4 + 56 \theta_2 \theta_5 + 56 \theta_2 \theta_5 - 10 \theta_8 \right) - 4 \theta_2 \theta_4^2 \left( 144 \theta_2^2 - 176 \theta_2^2 - 200 \theta_2^2 \theta_4 + 25 \theta_2^2 + 28 \theta_2 \theta_5 + 56 \theta_2 \theta_6 - 9 \theta_8 \right) - 4 \theta_2^2 \theta_4 \left( 135 \theta_2^4 - 195 \theta_2^4 - 25 \theta_2^4 + 54 \theta_2 \theta_5 + \theta_2 \left( -200 \theta_2^2 + 56 \theta_2 \theta_6 \right) - 9 \theta_8 \right) + \theta_4^2 \left( 75 \theta_2^4 - 100 \theta_2^4 - 100 \theta_2^4 \theta_4 + 10 \theta_2^4 + 24 \theta_2 \theta_5 + 28 \theta_2 \theta_6 - 4 \theta_8 \right) + \theta_2 \theta_5 \left( 168 \theta_2^4 - 192 \theta_2^4 - 216 \theta_2^4 \theta_4 + 24 \theta_2^4 + 48 \theta_2 \theta_5 + 56 \theta_2 \theta_6 - 8 \theta_8 \right) + \theta_2 \theta_6 \left( 168 \theta_2^4 - 224 \theta_2^4 \theta_4 - 224 \theta_2^4 \theta_4 + 28 \theta_2^4 + 56 \theta_2 \theta_5 + 56 \theta_2 \theta_6 - 8 \theta_8 \right) + \theta_8 \left( -30 \theta_2^4 + 36 \theta_2^4 \theta_4 + 36 \theta_2^4 \theta_4 - 4 \theta_2^4 - 8 \theta_3 \theta_5 - 8 \theta_2 \theta_6 + \theta_8 \right) \right]
\]

One may slightly simplify this expansion by making the following asymmetric change of variables: we introduce instead of the moments \( \tilde{\theta}_q \), the free cumulants \( \tilde{\phi}_q \) which are defined by [33,10,40]

\[
\tilde{\phi}_q = -\sum_{\alpha_1 \ldots \alpha_q \geq 0 \atop \sum_i i \alpha_i = q} \frac{(q + \sum_i \alpha_i - 2)!}{(q - 1)!} \prod_i \frac{(-\tilde{\theta}_i)^{\alpha_i}}{\alpha_i!}.
\]

Indeed, if one now expands \( F \) as

\[
F = \sum_{q \geq 1} \frac{\partial F}{\partial \theta_q} \bigg|_{\theta = 0} \theta_q + \frac{1}{2!} \sum_{q, r \geq 1} \frac{\partial^2 F}{\partial \theta_q \partial \theta_r} \bigg|_{\theta = 0} \theta_q \theta_r + \cdots
\]
then the first 3 derivatives were computed exactly in \cite{26} using the formalism of dispersionless Toda hierarchy, and turn out to be simply expressible in terms of the $\bar{\phi}_q$. In particular, $\frac{\partial F}{\partial \bar{\phi}_q}|_{\theta=0} = \frac{1}{q} \bar{\phi}_q$. The same expansion to order 8 now takes the form:

\[
F_2 = \frac{1}{4} \theta_2 \bar{\phi}_2, \quad F_3 = \frac{1}{4} \theta_3 \bar{\phi}_3, \quad F_4 = \frac{1}{4} \theta_4 \bar{\phi}_4 - \frac{1}{4} \theta_2^2 \left( \bar{\phi}_4 + \frac{1}{4} \bar{\phi}_2^2 \right)
\]

\[
F_5 = \frac{1}{4} \theta_5 \bar{\phi}_5 - \theta_3 \bar{\phi}_2 (\bar{\phi}_5 + \bar{\phi}_3 \bar{\phi}_2)
\]

\[
F_6 = \frac{1}{4} \theta_6 \bar{\phi}_6 - \theta_4 \theta_2 (\bar{\phi}_6 + \bar{\phi}_4 \bar{\phi}_2 + \frac{1}{4} \bar{\phi}_2^3) - \frac{1}{4} \theta_2^3 \left( \bar{\phi}_6 + \bar{\phi}_4 \bar{\phi}_2 + \frac{1}{4} \bar{\phi}_2^3 \right) + \frac{1}{4} \theta_2 \theta_4 \left( 7 \bar{\phi}_6 + 12 \bar{\phi}_4 \bar{\phi}_2 + 5 \bar{\phi}_2^3 + 2 \bar{\phi}_2^2 \right)
\]

\[
F_7 = \frac{1}{4} \theta_7 \bar{\phi}_7 - \theta_5 \theta_2 \left( \bar{\phi}_7 + \bar{\phi}_5 \bar{\phi}_2 + \bar{\phi}_4 \bar{\phi}_3 \right) - \theta_5 \theta_3 \left( \bar{\phi}_7 + \bar{\phi}_5 \bar{\phi}_2 + 2 \bar{\phi}_4 \bar{\phi}_3 + \bar{\phi}_3 \bar{\phi}_2^2 \right) + \theta_3 \theta_2^2 \left( 4 \bar{\phi}_7 + 7 \bar{\phi}_5 \bar{\phi}_2 + 8 \bar{\phi}_4 \bar{\phi}_3 + 5 \bar{\phi}_3 \bar{\phi}_2^2 \right)
\]

\[
F_8 = \frac{1}{4} \theta_8 \bar{\phi}_8 - \theta_6 \theta_2 \left( \bar{\phi}_8 + \bar{\phi}_6 \bar{\phi}_2 + \bar{\phi}_5 \bar{\phi}_3 + \frac{1}{4} \bar{\phi}_2^3 \right) - \theta_5 \theta_3 \left( \bar{\phi}_8 + \bar{\phi}_6 \bar{\phi}_2 + 2 \bar{\phi}_5 \bar{\phi}_3 + \bar{\phi}_4 \bar{\phi}_2^2 + \bar{\phi}_3 \bar{\phi}_2^2 \right) + \theta_3 \theta_2^2 \left( 9 \bar{\phi}_8 + 16 \bar{\phi}_6 \bar{\phi}_2 + 18 \bar{\phi}_5 \bar{\phi}_3 + 11 \bar{\phi}_2^4 + 11 \bar{\phi}_4 \bar{\phi}_2^2 + 14 \bar{\phi}_3 \bar{\phi}_2^2 + \bar{\phi}_2^3 \right) + \theta_2^4 \left( 9 \bar{\phi}_8 + 16 \bar{\phi}_6 \bar{\phi}_2 + 24 \bar{\phi}_5 \bar{\phi}_3 + 11 \bar{\phi}_2^4 + 16 \bar{\phi}_4 \bar{\phi}_2^2 + 20 \bar{\phi}_3 \bar{\phi}_2^2 + 2 \bar{\phi}_2^3 \right) - \frac{1}{4} \theta_4 \theta_2^2 \left( 10 \bar{\phi}_8 + 24 \bar{\phi}_6 \bar{\phi}_2 + 24 \bar{\phi}_5 \bar{\phi}_3 + 15 \bar{\phi}_2^4 + 24 \bar{\phi}_4 \bar{\phi}_2^2 + 24 \bar{\phi}_3 \bar{\phi}_2^2 + 3 \bar{\phi}_2^3 \right)
\]

One notes the intriguing feature that in either basis, for all known terms (up to $p = 10$), $pF_p$ has only integer coefficients. In other words, the quantity $s \partial F/\partial s$ has only integer coefficients. The rationale behind this fact and its reinterpretation in terms of the counting of “rooted” objects has remained elusive to us. The integral (1.1) still has a few mysteries... .

In this article, we have presented the standard lore on the integral (1.1) as well as recent developments. The integral itself is well understood by a variety of methods (sect. 2) and admits interesting generalizations (sect. 3). However the explicit expression, in terms of symmetric functions of the eigenvalues, of the integral itself (not to mention the more complicated problem of the associated correlation functions) is more subtle; sect. 4 was devoted to the discussion of this problem in the large $N$ limit. These considerations, based on integrable hierarchies, are complemented by other recent work based on combinatorial arguments, as discussed in sect. 5. It is our feeling that the exact interrelations between these various approaches require further study.

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