Conformal Geometry and Invariants of 3–strand Brownian Braids

Sergei Nechaev[1] and Raphaël Voituriez[2]

[1]*Laboratoire de Physique Théorique et Modéles Statistiques, Université Paris Sud, 91405 Orsay Cedex, France and*

[2]*Laboratoire de Physique Théorique des Liquides, Université Paris VI, 4 Place Jussieu, 75252 Paris Cedex, France*

(Dated: July 2, 2004)

We propose a simple geometrical construction of topological invariants of 3–strand Brownian braids viewed as world lines of 3 particles performing independent Brownian motions in the complex plane z. Our construction is based on the properties of conformal maps of doubly-punctured plane z to the universal covering surface. The special attention is paid to the case of indistinguishable particles. Our method of conformal maps allows us to investigate the statistical properties of the topological complexity of a bunch of 3–strand Brownian braids and to compute the expectation value of the irreducible braid length in the non-Abelian case.

I. INTRODUCTION

Our paper presents some results concerning geometry and statistics of random three-strand braids. Besides the relevance of this work at a purely algebraic level (see, for example, [12] for a review of principal topological problems), its usefulness in the understanding statistical physics of entangled lines, independent on their physical nature, is also undeniable. Statistics of ensembles of uncrossible linear objects with topological constraints has very broad application area ranging from problems of self-diffusion of directed polymer chains in flows and nematic-like textures to dynamical and topological aspects of vortex glasses in high–T_c superconductors [3, 4]. In order to have representative and physically clear image for the system of fluctuating lines with non-Abelian (i.e. noncommutative) topology we formulate the model in terms of entangled Brownian trajectories: such representation serves also as a geometrically clear image of Wilson loops in (2+1)D non-Abelian field–theoretic path integral formalism. In particular our paper focuses on the geometrical, algebraic and statistical properties of the simplest non-commutative braid group B_3.

Consider the ensemble of N particles randomly moving in the complex plane $z = x+iy$. Let us label the coordinates of these particles by $z_j(t) = x_j(t) + iy_j(t)$ where $1 \le j \le N$ and t is the current "time" (the initial configuration of the particles corresponds to $t = 0$). It is clear that in (2+1)D "space-time" (z, t) the diffusive motion of all particles is described by the statistics of N "world lines" or "directed polymers" and the time t plays the role of the length of the "world line". In what follows we shall assume the periodic boundary conditions, i.e. we shall suppose that at some time $t = T$ the configuration of the particles in the plane z is just the permutation of the initial configuration, i.e. $\{z_1(T), ..., z_N(T)\} = \mathcal{P}\{z_1(0), ..., z_N(0)\}$, $\mathcal{P}$ being a permutation of N elements. Thus, under the natural condition $z_j(t) \ne z_k(t)$ for any $1 \le \{k, j\} \le N$, the "world lines" shall be entangled.

The topology of the braid can play a crucial role in macroscopic physical properties. Let us mention only two examples. The elastic properties of polymer networks strongly depend on the initial degree of entanglement between subchains forming the sample and, hence, the elastic properties of the rubbers can be controlled by different initial conditions of samples preparation [5, 7]. In $Cu0_2$–based high–T_c superconductors in fields less than the critical magnetic field H_{c2} there exists a region where the Abrikosov flux lattice is molten, but the sample of the superconductor demonstrates the absence of the conductivity. This effect is explained by highly entangled state of flux lines due to their topological constraints [3, 8].

We restrict our study to $N = 3$, i.e. to the case of 3–strand braids produced by bunches of "word lines" of 3 particles simultaneously moving in the complex plane z (see fig.1). The simplest non-commutative group B_3 is defined on the projection of this bunch onto the plane (t, x) as it is shown in fig.1. The group B_3 is constituted by the set of generators $\{\sigma_1, \sigma_2, \sigma_1^{-1}, \sigma_2^{-1}\}$ satisfying commutation relations explicited below. Each topological configuration can be exactly encoded by an element of the braid group B_3, that is a "word" written in terms of "letters"–the braid group generators, associated to elementary moves–positive/negative crossings in the projection (see fig.1). A configuration of the braid corresponds at the time $t = T$ to an element ω_T of B_3.

We are interested in an explicit construction of topological invariants of entanglements of such world lines i.e., in the 3–strand random "braid" ω_T. We investigate and evaluate the complexity of such randomly generated braid, defined as the minimal number of generators $L(\omega_T)$ necessary to write ω_T. This quantity $L(\omega_T)$ is called the irreducible length (in the metric of words) and exactly coincides with the minimal number of crossings necessary to represent the entanglement. Therefore $L(\omega_T)$ can be chosen as an indicator of the braid complexity. In particular, if $L(\omega_t) = 0$ the braid is trivial (i.e. the strands are unentangled).

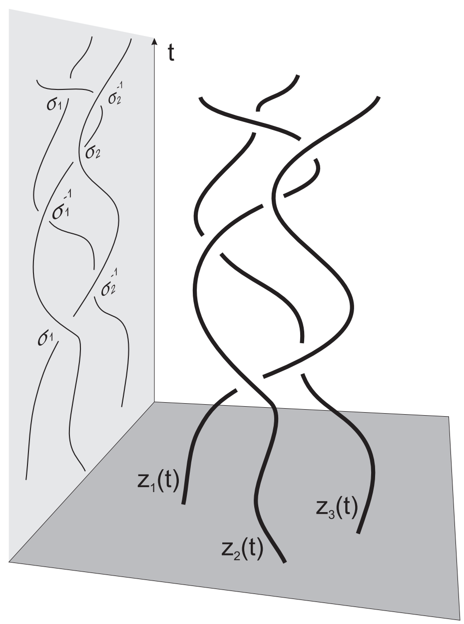

FIG. 1: Braiding of three directed lines. Elementary moves σ_i in the projection generate the group B_3.

In what follows we shall repeatedly use the so-called Burau matrix representation of the braid group. In the particular case of the group B_3 the Burau representation is given by 2×2 matrices:

$$\sigma_1 = \begin{pmatrix} -t & 1 \\ 0 & 1 \end{pmatrix}; \qquad \sigma_2 = \begin{pmatrix} 1 & 0 \\ t & -t \end{pmatrix} \tag{1}$$

where t is a free parameter. It is known that for B_3 this representation is faithful [12]. When $t = -1$ the group B_3 coincides with the modular group $PSL(2, \mathbb{Z})$.

The paper is structured as follows. The first part introduces the basic notions of homotopy groups and homotopic length, necessary to self-contained description of topological invariants. The method proposed in this part, demonstrates on the simplest examples how the monodromy representations of groups allows one to build the universal coverings, giving rise to topological invariants, associated at a physical level to fluxes of non-Abelian extension of "solenoidal magnetic fields". Using the same ideas, the second part deals with the case of B_3 for which we explicitly derive a topological invariant, directly linked to the irreducible braid length, and construct its non-Abelian flat connection. For the model of ideal Brownian braids, the stochastic evolution of this topological invariant is given and its asymptotic distribution is derived.

II. TOPOLOGICAL INVARIANTS AND MONODROMY REPRESENTATIONS

A. Basic concepts and definitions

Let us consider the double–punctured complex plane $\mathbb{C}^{\star\star} = \mathbb{C} - \{z_1, z_2\}$, of variable $z = x + iy$ and suppose the coordinates of the punctures M_1 and M_2 to be $z_1 = (0,0)$ and $z_2 = (c,0)$ correspondingly. We set $c \equiv 1$ by appropriate rescaling of the plane z.

Take now two closed elementary paths on z such that the first one (γ_1) encloses only the point M_1 and the second one (γ_2) surrounds only the point M_2. With the usual composition law of paths, γ_1 and γ_2, generate a fundamental group F_2, which is the first homotopy group, π_1, of the double–punctured complex plane z:

$$\pi_1(\mathbb{C}^{\star\star}) = F_2$$

The trivial path (the unit element of the group F_2) is the composition of an arbitrary loop with its inverse: $e = \gamma_i\gamma_i^{-1} = \gamma_i^{-1}\gamma_i$, $i = \{1,2\}$. The loops γ_i and $\tilde{\gamma}_i$ are equivalent if γ_i can be continuously deformed into $\tilde{\gamma}_i$ (the equivalent loops represent the same element of F_2). We denote the class of equivalent paths by γ_i.

Any element of the group F_2 being finitely generated, corresponds to a closed path on z and can be represented by a "word" consisting of a sequence of letters–generators of the group F_2: $\{\gamma_1, \gamma_2, \gamma_1^{-1}, \gamma_2^{-1}\}$. Each word can be reduced to a minimal (or "irreducible") representation. For example, the word $W = \gamma_1\gamma_2^{-1}\gamma_1\gamma_1\gamma_1^{-1}\gamma_2^{-1}\gamma_2\gamma_1^{-1}\gamma_2^{-1} \equiv \gamma_1\gamma_2^{-1}\left[\gamma_1\left[\gamma_1\gamma_1^{-1}\right]\left[\gamma_2^{-1}\gamma_2\right]\gamma_1^{-1}\right]\gamma_2^{-1}$ can be reduced to $W = \gamma_1\gamma_2^{-1}\gamma_2^{-1}$. Consider now a finitely generated group $G = \langle g_1, g_2\rangle$ (to be more precise, the group admitting only one– or two–generators presentation). We are interested in the monodromy representation of F_2 into G defined by the following group homomorphism Ψ:

$$\Psi : \begin{cases} F_2 \longrightarrow G \\ \gamma_i \longrightarrow g_i \quad (i = 1,2) \end{cases} \tag{2}$$

Note that this straightforwardly can be generalized to the case of multi–punctured surfaces or other topological spaces. We do not discuss in details the existence of such homomorphism, assuming that it results mainly from the fact that F_2 is a free group.

Our main goal in the present paper consists in defining an explicit geometrical construction of a non-Abelian generalization of a Gauss topological invariant (i.e. a "linking number") for different groups G using the monodromy representations of G. In particular, we construct a complex flat connection $A(z)$ on $\mathbb{C}^{\star\star}$, i.e. a "Bohm-Aharonov–like vector potential" $A(z)$, whose holonomy gives rise to a representation of G. A special attention is paid to the case when $G = B_3$. Moreover, we use the developed approach to estimate the averaged complexity of a 3–strand braid represented by independent motion of 3 Brownian particles in (2+1) dimensions.

B. Topological invariants from conformal maps

In order to explain the basic notions, we begin with the simplest possible case—the Abelian group $\mathbb{Z}$ and construct the corresponding topological invariant by means of conformal transforms.

1. Abelian case: commutative group $G = \mathbb{Z}$

The central point of the approach deals with the reconstruction of a linear differential equation on the manifold $\mathbb{C}^{\star\star}$ by its monodromy group G. In other words, defining the action of G on the space of solutions of some 2^{nd}–order liner differential equation with two branching points, we are attempting to recover the form of this differential equation and its solutions. In the case of $G=\mathbb{Z}$ it is natural to consider a usual 1-dimensional additive action

$$n[w(z)] = w(z) + 2i\pi n$$

for w defined on $\mathbb{C}^{\star\star}$ and $n \in \mathbb{Z}$. It turns out that the simplest linear differential equation with two branching points, satisfying those conditions is:

$$(z-z_1)(z-z_2)\frac{dw}{dz} - 2z + z_1 + z_2 = 0 \tag{3}$$

whose solution reads

$$w(z) = \ln(z-z_1) + \ln(z-z_2) \tag{4}$$

One can check that the free group Γ_2 acts on the space of solutions as follows

$$w(z) \stackrel{\{\gamma_1,\gamma_2\}}{\longrightarrow} w(z) + 2i\pi \tag{5}$$

which means that the homomorphism Ψ is trivial in for this example: $\Psi(\gamma_i) = 1$ $(i = 1,2)$. The function $w(z)$ conformally maps the doubly punctured plane to the universal covering space $w = u+iv$ free of any branching points. In the complex plane w we have

$$\begin{cases} u = \ln|(z-z_1)(z-z_2)| \\ v = \arg(z-z_1) + \arg(z-z_2) \end{cases}$$

The function $w(z|z_1,z_2)$ is a topological invariant because of equality (5). In particular, the function $v(z|z_1,z_2)$ defines the total number of turns in the plane z around the branching points z_1 and z_2 and hence it is nothing else as the Gauss linking number.

Thus, knowing the conformal transform $w(z)$ of the multiply punctured plane $\mathbb{C}^{\star\star}$ to the universal (i.e. uniformizing) covering surface, we can easily extract a topological invariant $\mathrm{Inv}(\gamma)$ of a closed path γ (starting and ending at some arbitrary point $z_0 \neq \{0,1\}$ in the plane z) from the difference $w_{\rm in}(z_0) - w_{\rm fin}(z_0)$. The path γ connects the images of the point z_0 on different Riemann sheets—the "copies" of the fundamental domain of G. Recall that by definition the fundamental domain of a group G is a minimal connected domain tessellating the whole covering space under the action of G. Representing the topological invariant $\mathrm{Inv}_{(z)}(\gamma)$ as a full derivative along the contour γ, we get:

$$\mathrm{Inv}_{(z)}(\gamma) \stackrel{\rm def}{=} w_{\rm fin} - w_{\rm in} \equiv \oint\limits_{\gamma} \frac{dw(z)}{dz} dz = \oint A(z) dz \tag{6}$$

The physical interpretation of the derivative $A(z) = \frac{dw(z)}{dz}$ is very straightforward. The conformal transform $w(z)$ plays the role of a complex potential of a field $A(z)$, which defines a *flat connection* of a multiple–punctured plane $\mathbb{C}^{\star\star}$.

For the commutative group $G=\mathbb{Z}$ we obtain $A(z)$ by taking the derivative of (4):

$$A(z) = \frac{1}{z-z_1} + \frac{1}{z-z_2} \tag{7}$$

This expression can be easily identified with the standard (Abelian) Bohm–Aharonov vector potential of two solenoidal magnetic fields orthogonal to the plane z and crossing it in the points z_1 and z_2.

In the next section the same construction of the flat connection associated to a specific group G will be generalized to the non-Abelian case.

2. Non-Abelian case: non-commutative groups F_2, H_q

We consider a special class of hyperbolic and hyperbolic–like groups: the free (F_2) and the Hecke (H_q) groups, as well as the braid group B_3. By hyperbolic–like groups we mean a class broader than hyperbolic groups in the classification of M.Gromov [13]. (According to the Gromov's definition, the group B_3 does not belong to the class of hyperbolic groups). The important feature for us would be just the exponential growth of the group. From this point of view the group B_3 fits our scheme.

1. The free group F_2 (in general, F_n $(n = 2, 3, 4, ...)$) by definition is the free product of two (in general of n) copies of cyclic groups of second order, $\mathbb{Z}_2$. The matrix representation of the generators of the free groups $F_2\{f_1, f_2\}$ and $F_3\{f_1, S\}$ are:

$$\begin{aligned} F_2: \quad & f_1 = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix};\ f_2 = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \\ F_3: \quad & f_1 = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix};\ S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \end{aligned} \tag{8}$$

2. The Hecke group H_q is the free product of two cyclic groups $\mathbb{Z}_2$, and $\mathbb{Z}_q$ of orders 2 and q respectively. The Hecke group is defined by the relations

$$\begin{aligned} (ST_q)^q = b_q^q = 1 \\ S^2 = a_2^2 = 1 \end{aligned} \tag{9}$$

where the generators T_q and S have the following matrix representation

$$H_q: \quad T_q = \begin{pmatrix} 1 & 2\cos\frac{\pi}{q} \\ 0 & 1 \end{pmatrix};\ S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{10}$$

The parameter q takes discrete values $q = 3, 4, 5, 6, \ldots$. The Hecke group H_q "interpolates" between the modular group $PSL(2, \mathbb{Z})$ (for $q = 3$) and the free group F_3 with 3 generators ($q \to \infty$).

We have stressed in the previous section that the topological invariant can be constructed on the basis of the conformal map $w(z)$ of multiple–punctured plane to the universal covering space of a group. We now extend the described method to more interesting cases than considered at length of the section II B 1.

We derive the conformal mapping of the half–plane $\mathrm{Im}\, z > 0$ onto the fundamental domain of the triangular group G—a curvilinear triangle lying in the upper half–plane $\mathrm{Im}\, w > 0$. The action of G on this fundamental domain generates the whole covering space. Each copy of the fundamental domain represents a Riemann sheet corresponding to the fibre bundle above z and the whole covering space w is the unification of all such Riemann sheets—see fig.2.

The coordinates of initial and final points of a trajectory on the universal covering w determine:

- The coordinates of the corresponding points on z;
- The element of G corresponding to the homotopy class of the path on z.

Following the construction described in the previous section let us define the action of $G\{g_1, g_2\}$ in the complex plane $w = u + iv$. The group G admits the faithful 2–dimensional representations and acts in the covering space w by fractional–linear transforms. Consider two basic contours γ_1 and γ_2, associated to the action of G on z ($z \neq \{z_1, z_2, \infty\}$. The contours γ_1 and γ_2 enclose the branching points located at z_1 and z_2 correspondingly ($z \neq \{z_1, z_2, \infty\}$). The function $w(z)$ ($z \neq \{z_1, z_2, \infty\}$) obeys the following transformations:

$$w\left[z \xrightarrow{\gamma_1} z\right] \to \tilde{w}_1(z) = \frac{a_1 w(z) + b_1}{c_1 w(z) + d_1}; \quad w\left[z \xrightarrow{\gamma_2} z\right] \to \tilde{w}_2(z) = \frac{a_2 w(z) + b_2}{c_2 w(z) + d_2} \tag{11}$$

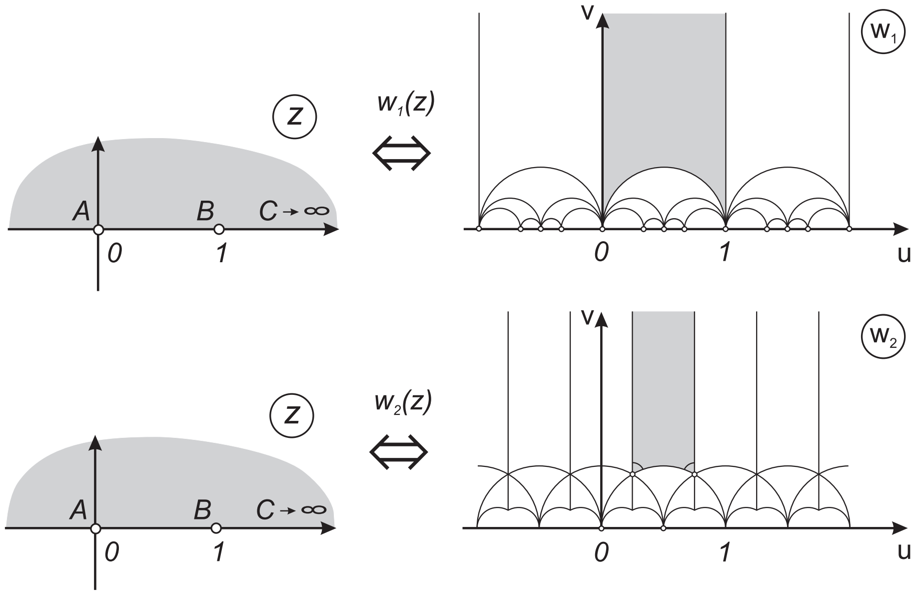

FIG. 2: Conformal transforms of the upper half-plane $\mathrm{Im}\, z > 0$ to the fundamental domain of the group F_2 (up) and of the group $PSL(2, \mathbb{Z})$ (down).

The matrices g_1 and g_2

$$g_1 = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}; \qquad g_2 = \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} \tag{12}$$

are the matrices of basic substitutions of the group $G\{g_1, g_2\}$, i.e. g_1 and g_2 are the generators of G.

It is well known that the function $w(z)$ can be defined as the quotient of two fundamental solutions $u_1(z)$ and $u_2(z)$ of a second order differential equation with branching points $\{z_1 = (0,0), z_2 = (0,1), z_3 = \infty\}$. As it follows from the analytic theory of differential equations, the solutions $u_1(z)$ and $u_2(z)$ undergo the linear substitutions when the variable z moves along the contours γ_1 and γ_2:

$$\gamma_1 : \begin{pmatrix} \tilde{u}_1(z) \\ \tilde{u}_2(z) \end{pmatrix} = g_1 \begin{pmatrix} u_1(z) \\ u_2(z) \end{pmatrix}; \quad \gamma_2 : \begin{pmatrix} \tilde{u}_1(z) \\ \tilde{u}_2(z) \end{pmatrix} = g_2 \begin{pmatrix} u_1(z) \\ u_2(z) \end{pmatrix} \tag{13}$$

where g_1 and g_2 are the generators of the monodromy group of this equation. The problem of restoring the form of a differential equation by the monodromy matrices g_1 and g_2 of the group G of the differential equation, is known as the Riemann-Hilbert problem. Yet we restrict ourselves with the groups F_2 and H_q, the group B_3 shall be considered separately.

The free group F_2. For the free group F_2 the solution of the Riemann-Hilbert problem gives rise to the following second–order differential equation [14]:

$$z(z-1)\frac{d^2u(z)}{dz^2} + (2z-1)\frac{du(z)}{dz} + \frac{1}{4}u(z) = 0 \tag{14}$$

Indeed, a possible basis of solutions of this equation is as follows:

$$\begin{cases} u_1(z) = F(1/2, 1/2, 1, z) \\ u_2(z) = iF(1/2, 1/2, 1, 1-z) \end{cases} \tag{15}$$

Using the well–known properties of hypergeometric functions, one can restore the monodromy matrices defined in (13) for this basis:

$$g_1 = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}; \qquad g_2 = \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix} \tag{16}$$

which coincides with the generating set of the group F_2. The function $w(z)$ performing the conformal map of the upper half–plane Im$z > 0$ onto the fundamental domain (the curvilinear triangle ABC) of the universal covering w satisfies eq.(11) and can be written as:

$$w(z) = \frac{u_1(z)}{u_2(z)} \tag{17}$$

The function $w(z)$ is well known in the literature (see, for example, [14]) and its inverse $z(w)$ is the elliptic modular function

$$z(w) \equiv \lambda(w) = \frac{\theta_2^4(0,w)}{\theta_3^4(0,w)} \tag{18}$$

Now we can give an explicit expression of the flat connection $\mathbf{A}(z)$ for the doubly punctured plane corresponding to the monodromy of the free group F_2. Taking the derivatives and using the properties of the hypergeometric functions, we get:

$$A(z) = \frac{dw(z)}{dz} = -i\frac{E(z)K(1-z) + (E(1-z) - K(1-z))K(z)}{2z(z-1)K^2(1-z)} \tag{19}$$

where

$$K(z) = \int_0^{\pi/2} \frac{d\theta}{\sqrt{1 - z\sin^2\theta}}; \qquad E(z) = \int_0^{\pi/2} d\theta\sqrt{1 - z\sin^2\theta} \tag{20}$$

are correspondingly the complete elliptic integrals $K(z)$ and $E(z)$ (see, for example, [15]).

In the Fig.3 we have plotted the absolute values $|A(z)|$ of the Abelian (Eq.(7)) and the non-Abelian (Eq.(19)) flat connections.

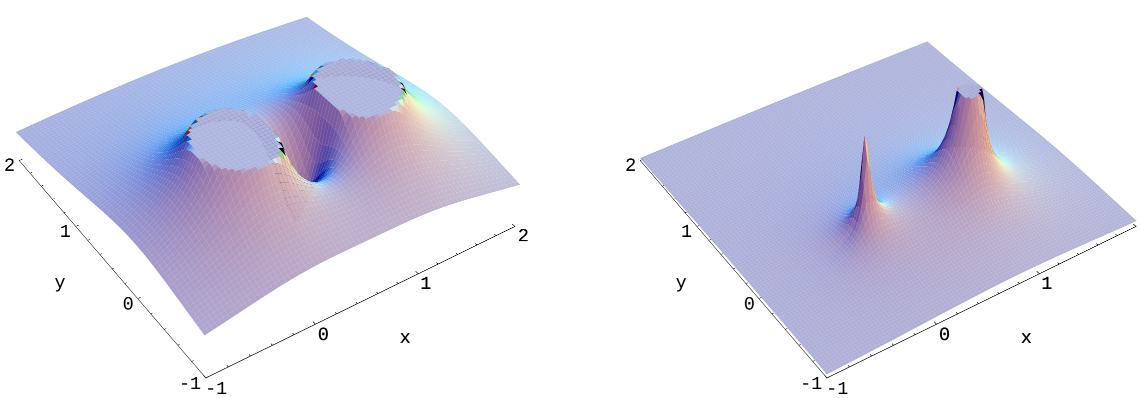

FIG. 3: The cuts of the function $|A(z)|$ in the Abelian (left) and non-Abelian (right) cases.

The leading asymptotics of (19) are as follows:

$$A(z) \sim \begin{cases} \dfrac{-i\pi}{z\ln^2 z} & \text{as } z \to 0 \\ \dfrac{-i}{\pi(z-1)} & \text{as } z \to 1 \end{cases} \tag{21}$$

Hence even in the vicinity of the branching point $z = 0$ the function $A(z)$ defined in (19) does not coincide asymptotically with the flat connection for the Abelian case (7).

THE HECKE GROUP H_q. We now pass to the case of H_q, keeping in mind that for $q = 3$ we recover the group $PSL(2,\mathbb{Z})$, directly linked to B_3 (B_3 is a central extension of $PSL(2,\mathbb{Z})$, see [12]). The solution of the corresponding Riemann-Hilbert problem leads to the following differential equation: [16]

$$z(z-1)\frac{d^2u(z,q)}{dz^2} + \left[\left(\frac{3}{2}-\frac{1}{q}\right)z - \left(1-\frac{1}{q}\right)\right]\frac{du(z,q)}{dz} + \frac{1}{4}\left(\frac{1}{2}-\frac{1}{q}\right)^2 u(z,q) = 0 \tag{22}$$

whose possible basis of solution is:

$$\begin{cases} u_1(z) = F\left(\frac{q-2}{4q}, \frac{q-2}{4q}, 1-\frac{1}{q}, z\right) - \lambda(q)e^{i\pi/q}z^{1/q}F\left(\frac{q+2}{4q}, \frac{q+2}{4q}, 1+\frac{1}{q}, z\right) \\ u_2(z) = -e^{i\pi/q}F\left(\frac{q-2}{4q}, \frac{q-2}{4q}, 1-\frac{1}{q}, z\right) + \lambda(q)z^{1/q}F\left(\frac{q+2}{4q}, \frac{q+2}{4q}, 1+\frac{1}{q}, z\right) \end{cases} \tag{23}$$

where

$$\lambda(q) = e^{2i\pi/q}\frac{\Gamma(1-\frac{1}{q})\Gamma(\frac{q+2}{4q})\Gamma(\frac{3q+2}{4q})}{\Gamma(1+\frac{1}{q})\Gamma(\frac{q-2}{4q})\Gamma(\frac{3q-2}{4q})} \tag{24}$$

Taking into account that the series defining the hypergeometric functions $F(z)$ converges for $|z| < 1$ and corresponds to the so-called logarithmic case (see [14]) one obtains from (23) the following monodromy matrices after proper analytic continuation:

$$g_1 = e^{i\pi/q}\begin{pmatrix} 2\cos\frac{\pi}{q} & 1 \\ -1 & 0 \end{pmatrix}; \qquad g_2 = i\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tag{25}$$

One can check that after normalization by the determinant, the matrices (25) become the generators of H_q (correspondingly of orders q and 2). Let us point out that for $q = 3$, one can identify the inverse function $z(w)$ with the Klein's "absolute modular invariant"

$$z(w) \equiv J(w) = \frac{(\theta_2^8(0,w) + \theta_3^8(0,w) + \theta_4^8(0,w))^3}{54\theta_2^8(0,w)\theta_3^8(0,w)\theta_4^8(0,w)}$$

where $\theta_2, \theta_3, \theta_4$ are the Jacobi elliptic θ–functions. As in the previous section, the corresponding complex flat connection $A_{H_q}(z)$ can be obtained as the full derivative

$$A_{H_q}(z) = \frac{dw_{H_q}(z)}{dz}$$

from the conformal map

$$w_{H_q}(z) = \frac{u_1(z)}{u_2(z)}$$

with $u_1(z)$ and $u_2(z)$ defined in (23).

III. INVARIANT OF 3-STRAND BRAIDS

Let us return to the model of random braids discussed in the very beginning of the paper. The system of three independently moving particles is described by 3 complex variables (i.e. 6 degrees of freedom) and hence our system can not be directly viewed as a monodromy problem. In order to describe the whole system by one complex variable ζ

and to be able to use the tools elaborated in the previous section, with the minimal loss of information, we introduce the anharmonic quotient:

$$\zeta(t) = \frac{z_{23}(t)}{z_{13}(t)} \equiv \frac{z_2(t) - z_3(t)}{z_1(t) - z_3(t)} \tag{26}$$

One can easily check that the variable ζ contains the complete topological information of a mutual configuration of three entangled lines, except the global phase, or global twist of the braid. It means that we do not take into account the center of B_3 considering the factor group $PSL(2,\mathbb{Z}) = B_3/\mathbb{Z}$. We can always take into account the global twist afterwards by passing to the rotating coordinate system with the origin located in the center of mass of the system of three particles. From the statistical point of view the possibility of neglecting the center of B_3 (which is equivalent to considering the factor group $PSL(2,\mathbb{Z})$ only), has been discussed in the papers [18, 19]. In fact, in these paper it has been shown that the escape rates for random walks on B_3 and $PSL(2,\mathbb{Z})$ are the same in the limit of infinitely long trajectories.

Thus, the parametrization of the three-particle system $\{z_1(t),\ z_2(t),\ z_3(t)\}$ by the function $\zeta(t)$ living in the doubly punctured complex plane enables us to preserve all topological characteristics of the braid of mutually entangled world lines of these particles. However the precise derivation of the expression for the flat connection, as it is shown below, explicitly depends on the fact whether the particles are identical or not.

Looking at the elementary moves associated with the generators $\sigma_1,\ \sigma_2$ of B_3, we obtain the transformations for the variable ζ. For example, let us pick up some path in the homotopy class of σ_1 (see fig.1). By definition of σ_1, we have at some time t:

$$\sigma_1[z_1 \leftrightarrow z_2, z_3 \leftrightarrow z_3] : \begin{cases} z_1(t) = \dfrac{z_1 + z_2}{2} + \dfrac{1}{2} z_{12} e^{-i\pi t} \\ z_2(t) = \dfrac{z_1 + z_2}{2} - \dfrac{1}{2} z_{12} e^{-i\pi t} \\ z_3(t) = z_3 \end{cases} \tag{27}$$

In the same way we can get the transformation of ζ along a path corresponding to the homotopy class $\sigma_2 \equiv \sigma_2[z_1 \leftrightarrow z_1, z_2 \leftrightarrow z_3]$. Finally we arrive at the following set of transformations of ζ:

$$\begin{aligned} \sigma_1 : \zeta \ &\rightarrow \ \frac{1}{\zeta} \\ \sigma_2 : \zeta \ &\rightarrow \ \frac{\zeta}{\zeta - 1} \end{aligned} \tag{28}$$

which precisely define the representation of the symmetric group S_3.

Indistinguishable particles. Our main goal consists in constructing the flat connection for the system of three *identical* particles moving in the plane. The condition of the indistinguishability of particles requires to factor the action of the group B_3 by S_3, and hence to consider ζ as living in the factor space $\mathbb{C}^{\star\star}/S_3$. The case of *distinguishable* particles shall be discussed at the end of this section.

All the trajectories $\zeta(t)$ (parameterized by t) are obviously closed in the space $\mathbb{C}^{\star\star}/S_3$. Keeping in mind our strategy of the previous section, we want to define a monodromy representation of B_3 (more precisely of $PSL(2,\mathbb{Z})$) acting on $\mathbb{C}^{\star\star}/S_3$. To do that we construct a conformal map $w_s(\zeta)$ of the doubly punctured plane factorized over the action of the symmetric group, $\mathbb{C}^{\star\star}/S_3$ onto the fundamental domain of the modular group, $PSL(2,\mathbb{Z})$:

$$w_s(\zeta) : \mathbb{C}^{\star\star}/S_3 \longrightarrow PSL(2,\mathbb{Z}) \tag{29}$$

To our knowledge, the explicit method of solving the uniformization problem (29) has not been yet considered in the context of our problem. So, our own way to tackle it deals with the following observation.

Consider the so-called pure braid group P_3 defined as follows:

$$P_3 = \langle \sigma_1^2, \sigma_2^2, (\sigma_1\sigma_2)^2 \rangle \tag{30}$$

It is obvious that the group P_3 is the subgroup of B_3. In particular, the group P_3 can be identified with the factor group $F_2 \times \mathbb{Z}$ through the obvious correspondence

$$\sigma_1^2 = g_1, \ \sigma_1^2 = g_1 \tag{31}$$

where g_1 and g_2 are the generators of the free group F_2. The relations (14) follow from the Burau representation of braid group generators (1) and the obvious geometric construction shown in the fig.4.

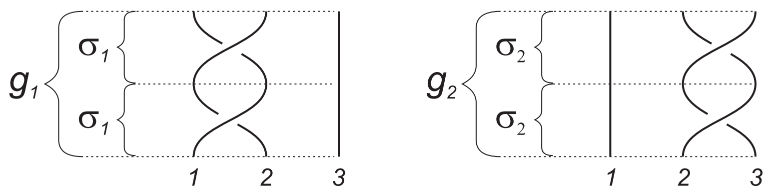

FIG. 4: The squares of the braid group generators σ_i act as the free group generators g_i: $g_i = \sigma_i^2$ $(i = 1, 2)$.

In the case of the pure braid group P_3 there is no difference between distinguishable and indistinguishable particles moving in the plane z because the generators g_1 and g_2 correspond to "full turns" of one world line of the particle with respect to another one (and not to "half turns" as in case of the braid group B_3). That is, we deal with the monodromy representation of F_2. Let us remind that we factor out the global turns (i.e. the center of the group) in the space $\mathbb{C}^{\star\star}$. The uniformization problem in this case is solved by the conformal map $w_s(\zeta)$ of the doubly punctured plane to the covering space

$$w_s(\zeta) : \mathbb{C}^{\star\star} \longrightarrow F_2$$

Thus, we return to the topic of the previous section, where an explicit form of such conformal map $w_s(\zeta) \equiv w(\zeta)$ is given in (17).

Consider now the whole group $B_3/\mathbb{Z}$, take the same function $w(\zeta)$ and check the effect of the basic substitutions of the symmetric group (28) onto the transformations of $w(\zeta)$. Using the well known properties of the modular functions (see, for example, [20, 21, 22]), we get:

$$\begin{cases} w_s\left(\zeta \to \dfrac{1}{\zeta}\right) & \to \ -w_s + 1 \\[2ex] w_s\left(\zeta \to \dfrac{\zeta}{\zeta - 1}\right) & \to \ \dfrac{w_s}{w_s - 1} \end{cases} \tag{32}$$

The transformations of w_s precisely coincide with the basic transformations of the modular group $PSL(2,\mathbb{Z})$. In fact, technically it is much easier to check (instead of (32)) the transformations for the inverse function $\zeta(w_s) = \frac{\theta_2^4(0,w_s)}{\theta(0,w_s)}$:

$$\begin{cases} \zeta\left(w_s \to -w_s + 1\right) & \to \ \dfrac{1}{\zeta} \\[2ex] \zeta\left(w_s \to \dfrac{w_s}{w_s - 1}\right) & \to \ \dfrac{\zeta}{\zeta - 1} \end{cases} \tag{33}$$

Hence the function $w_s(\zeta)$ explicitly solves the desired uniformization problem (29). Thus, we can use this function $w_s(\zeta)$ to construct the non-Abelian flat connection $A_s(\zeta)$ for the trajectories on $\mathbb{C}^{\star\star}/S_3$ parametrized by $\zeta(t)$:

$$A_s(\zeta) = \frac{dw_s(\zeta)}{d\zeta} \tag{34}$$

where

$$w_s(\zeta) \equiv w(\zeta) = \frac{u_1(\zeta)}{u_2(\zeta)} = -i\,\frac{F(1/2,1/2,1,z)}{F(1/2,1/2,1,1-z)}$$

(see eq.(15)).

The expression (34) is the source for the explicit construction of the topological invariant via equation (6) of three entangled world lines $z_1(t)$, $z_2(t)$ and $z_3(t)$ shown in fig.1.

Distinguishable particles. For *distinguishable* particles the action of the symmetric group S_3 should be neglected and hence ζ is living in the doubly punctured plane, $\mathbb{C}^{\star\star}$. The problem of uniformization in this case is solved by the conformal map $w_d(\zeta)$ mapping the doubly punctured plane, $\mathbb{C}^{\star\star}$, onto the fundamental domain of the modular group, $PSL(2,\mathbb{Z})$:

$$w_d(\zeta) : \mathbb{C}^{\star\star} \longrightarrow PSL(2,\mathbb{Z}) \tag{35}$$

(compare this expression to (29)). The corresponding function $w_d(\zeta)$ coincides with $w_2(\zeta) = \frac{u_1(\zeta)}{u_2(\zeta)}$ found in the privious section (use the equations (23)–(24) for $q=3$).

IV. STOCHASTIC BEHAVIOR OF THE INVARIANT FOR BROWNIAN 3-STRAND BRAIDS

We are interested in this section in the *stochastic* behavior of the invariant $|w(\eta)|$ when each of the three particles perform a 2D Brownian motion (BM) represented by its complex coordinate z_i. This model naturally appears as the generalization of the Edwards' problem of determining the distribution of the winding angle of two independent 2D BM [5]. The intrinsic noncommutative structure of the 3–particles model, contrary to the commutative 2–particles problem, induces a completely different behavior of the invariant, which characterizes quantitatively the tangle complexity. The method employed here is inspired by a work of Gruet [26].

A. The Ideal Model

We begin with an ideal case of point–like particles diffusing in the infinite plane with the diffusion constants D set to 1. Following previous sections, one has to study the motion of the reduced variable η in the space $\mathbb{C}^{\star\star}/S_3$. Our strategy relies on the following theorem on conformal transforms of complex BM often referred to as a generalized Paul Lévy's theorem [28]. Denote by B_t the generic BM in the complex plane z. For any conformal map f, the image $Y_t = f(B_t)$ of a BM B_t is a *time changed* BM $B_u(t)$ such that

$$du = \frac{1}{4} dY d\bar{Y} = |f'(B_t)|^2 dt \tag{36}$$

In terms of the diffusion equation (which corresponds to the Langevin equation (36), the time change can be interpreted as a space dependent diffusion coefficient [25]:

$$\partial_t P(Y_t) = \frac{du}{dt} \Delta P(Y_t) \tag{37}$$

A slightly modified version of the generalized Lévy theorem shows that $\zeta(t)$, being the ratio of two complex BM (see (26)), is also a time changed BM, denoted $\zeta(t) = B_{u(t)}$. The time change then reads, following the rules of Itô calculus:

$$du = \frac{1}{4} d\zeta d\bar{\zeta} = \frac{1+|\zeta(t)|^2}{|z_{32}(t)|^2} dt \tag{38}$$

Now we should pass to the covering space w and describe the evolution of the invariant $w(\zeta) = w(B_u)$. Using once more the generalized Lévy's theorem, we obtain that there exists a *hyperbolic* BM $\tilde{B}$ such that $w(\zeta)$ is again a time–changed $\tilde{B}$:

$$w(\zeta(t)) = \tilde{B}_{\tau(u(t))} \tag{39}$$

with

$$d\tau = \frac{|w'(B_u)|^2}{\mathrm{Im}^2(w(B_u))} du \tag{40}$$

The choice of the hyperbolic metric is not artificial and anticipates the geometrical properties of the transform $w(\zeta)$. Note the importance of obtained result: we know that the topological invariant $w(\zeta)$, giving the braid complexity, performs a hyperbolic diffusion in a new time, or equivalently a hyperbolic diffusion in a space with the metric–dependent diffusion coefficient. We now have to study the full time change $\tau(t)$. Combining (38) and (40) and using the well known inverse function $w^{-1} \equiv z(w) = \lambda(w)$ (see (18)), we obtain after separating the variables the following implicit relation between t and τ, which defines the functional $H_t(\tau)$:

$$H_t(\tau) = \int_0^t \frac{dt'}{|z_{32}(t')|^2} = \int_0^\tau \frac{|\lambda'(\tilde{B}_{\tau'})|^2}{1 + |\lambda(\tilde{B}_{\tau'})|^2} \mathrm{Im}^2(\tilde{B}_{\tau'}) d\tau' \tag{41}$$

This expression is exact, but does not allow a straightforward interpretation. Let us first notice that the t–dependence can be easily shown to be, with the probability one:

$$\lim_{t\to\infty} \frac{H_t(\tau)}{\ln t} = C_1 > 0 \tag{42}$$

The sensitive point is the τ–dependence. The integral kernel f

$$f:\ z \longrightarrow \frac{|\lambda'(z)|^2}{1 + |\lambda(z)|^2} \mathrm{Im}^2(z) \tag{43}$$

is invariant under the transformations of the group F_2, and $H_t(\tau)$ is therefore an additive functional of the BM on the quotient surface $\mathbb{H}/F_2$. This motion is ergodic. Therefore if $H_t(\tau)$ is integrable over the fundamental domain of F_2, one can clear up the asymptotic τ–dependence (see in particular a similar approach proposed by Gruet in [26]). In our case $H_t(\tau)$ is not integrable over the domain of F_2; the τ–dependence for this ideal model requires more attention and have to be treated with more care. We expect to consider that question in details separately, while below we express a conjecture based on the fact that the non-integrability does not lead to a growth faster than τ. We indeed believe that the non-integrability gives rise to sub-leading terms in τ and the following limit holds:

$$\lim_{\tau\to\infty} \frac{H_t(\tau)}{\tau} = C_2 \tag{44}$$

Comparing (42) and (44) we arrive at the relation

$$\lim_{t\to\infty} \frac{\tau(t)}{t} = \frac{C_1}{C_2} \tag{45}$$

Using (45) we formulate a central result connecting the hyperbolic BM $\tilde{B}_\tau$ describing the connection of the expectation of our topological invariant $d(w)$ being the distance in the covering space w, with the word length $L(\tilde{B}_\tau)$ of the corresponding random braid. Recall that the distribution of $\tilde{B}_\tau$ is well known, and in particular,

$$\lim_{\tau\to\infty} \frac{d(\tilde{B}_\tau)}{\tau} = 1$$

almost surely. Moreover, if one notices that the word lengths L for $PSL(2,\mathbb{Z})$ and for F_2 are quasi-isometric, we have:

$$\lim_{\tau\to\infty} \frac{\left\langle L(\tilde{B}_\tau)\right\rangle}{\tau \ln \tau} = C_3 > 0 \quad \text{in probability} \tag{46}$$

what is a direct consequence of a theorem by Gruet proved in [26]. Combining this result with (45, we end up with the following leading asymptotics at large time for the expectation of the word length of Brownian 3–strand braids:

$$\langle L_t \rangle \sim \ln(t) \times \ln(\ln t) \quad \text{for } t \to \infty \tag{47}$$

The average topological complexity increases with time, however the growth is very slow. It is limited (as in the case of two particles where this logarithmic scaling also holds) by the angular part of the 2D BM in an infinite space [25, 27]: when two particles are far apart (and the corresponding typical distance grows with time), their relative angle varies very slowly.

B. Model with a compact domain

We here discuss the physical reasons that justify a regularization of the integral (41) over the fundamental domain of F_2. The divergence of this integral occurs when η approaches the points $\{0, 1, \infty\}$ meaning that either two particles collide, or one particle goes to infinity. It is then quite natural to modify slightly the model in order to avoid these pathological cases, which in fact are not realistic if one describes the physical objects of finite thickness such as polymers. Specifically, we assume that three Brownian particles evolve in a bounded domain of characteristic size R and that they experience hard-core repulsion at distance r. This can be re-expressed in terms of constraints due to hard walls added to the domain of η, hereafter denoted by D:

$$\begin{aligned} &\epsilon(r,R) \leq |\eta| \leq M(r,R) \\ &|\eta(r,R) - 1| \geq \epsilon \end{aligned} \tag{48}$$

The precise shape of the boundary of D is not important. Note that $w(D)$ is compact in $\mathbb{H}$. Restricting our model to this domain, we can now claim the integrability of the functional $H_t^D(\tau)$ over the domain $w(D)$. Within this approximation, we obtain the following τ–dependence from the ergodic property offered above:

$$\lim_{\tau\to\infty} \frac{H_t^D(\tau)}{\tau} = C_4 > 0 \tag{49}$$

In this model the t–dependence is readily changed as follows:

$$\lim_{t\to\infty} \frac{H_t^D(\tau)}{t} = C_5 > 0 \tag{50}$$

Comparing (49) and (50) we arrive at

$$\lim_{t\to\infty} \frac{\tau(t)}{t} = \frac{C_4}{C_5} \tag{51}$$

The expectation of $d(w)$ can now be described asymptotically: $d(w)$ is a hyperbolic BM in the variable $\tau \propto t$ (see (51)). In particular, we have with the probability one:

$$\lim_{t\to\infty} \frac{\langle d(w(\eta)) \rangle}{t} = C_6 > 0 \tag{52}$$

For a compact domain one can straightforwardly check that the topological invariant, d, and the word length, L, are quasi-isometric. The asymptotic behavior of the expectation value of L is therefore similar to the one of $\langle d(w(\eta)) \rangle$:

$$\left\langle L_t^D \right\rangle \sim t \quad \text{for } t \to \infty \tag{53}$$

For the bounded domain, the topological complexity grows much faster, as the infinite space effect discussed above is absent. The scaling $\left\langle L_t^D \right\rangle \propto t$ should be compared to the (commutative) scaling $\langle L_{\text{comm}} \rangle \propto \sqrt{t}$ for two particles. It is noteworthy to stress that (53) is fully consistent with a discrete model considered in [18, 19].

V. CONCLUSION

To conclude, we would like to comment the new physical content of our results. So far the only topological invariant studied in the context of entanglements of fluctuating linear objects was the so-called winding number [1, 2, 4, 6], which is known to be incomplete for more than two linear objects. We here propose a model, which describes exactly the underlying non-Abelian topology of the problem.

We propose a simple geometrical construction of topological invariants of 3–strand Brownian braids viewed as world lines of 3 particles performing independent Brownian motions in the complex plane z. Our construction is based on the properties of conformal maps of doubly-punctured plane z to the universal covering surface. We pay special attention to the case of indistinguishable particles. Our approach is mainly "self made" and its geometrical transparency we consider as the basic advantage. The standard machinery of constructing the nontrivial braid group representations from the Conformal Field Theory is outlined in the Appendix where we mainly review the strategy realized in [29]. The Appendix is added to our paper exclusively to establish some links between the approachs based on the geometry of conformal maps and on CFT.

Our method of conformal maps allow us to investigate the statistical properties of the topological complexity of a bunch of 3–strand Brownian braids and to compute the expectation value of the irreducible braid length.

APPENDIX A: BRAID GROUP REPRESENTATION AND CFT

In this section we try to describe the derivation of the braid group representation from the monodromies of some CFT. We will mainly describe the system under investigation in physical terms, rather than in more rigorous but less transparent algebraic topological setting.

So, consider n ($n = 3$ shall be mainly treated) indistinguishable particles, living in the complex plane $\mathbb{C}$. The "static" (or configurational) phase space is the following set $X_n = Y_n/S_n = \mathbb{C}\times Y_{n-1}/S_n$ where $Y_n = \{\mathbf{z} = (z_1, ..., z_n) \in \mathbb{C}^n;\ i \neq j \Rightarrow z_i \neq z_j\}$ and S_n is the group of permutations of n elements. The above defined decomposition means that one can factor out the coordinate of the center of mass of the system. Thus, we study the topology of X_n, or more precisely of its first homotopy group $\pi_1(X_n)$, describing the classes of equivalence of closed curves on X_n where $B_n = \pi_1(X_n)$.

In physical terms we can rephrase the said above as follows. A quantum mechanical description of this system requires to define the wave functions, or in other words an n fold tensor–valued functions $\Psi : X_n \to V^{\otimes n}$ over this configuration space. These functions should represent the internal quantum numbers such as a spin for each particles. The most general case consists in taking Ψ as the so-called G–modules, where G is a Lie group. The obtained multi–valued (in fact, tensor–valued) structure is called a fiber bundle. The main difficulty in such construction is as follows: the functions $\Psi(\mathbf{z})$ and $\Psi(\mathbf{z}' \neq \mathbf{z})$ belong to different spaces, say $h_{\mathbf{z}}$ and $h_{\mathbf{z}'}$, and therefore can not be directly compared. To overcome this difficulty one has to define a map $T_{\gamma_{\mathbf{z}\mathbf{z}'}} : h_{\mathbf{z}} \to h_{\mathbf{z}'}$ that "transports" the wave function from $\mathbf{z}$ to $\mathbf{z}'$ along the path γ.

Now let us define the holonomy operator, $T_{\gamma_{\mathbf{z}\mathbf{z}'}}$, and a one–form $\omega = dT_{\mathbf{z}\mathbf{z}}$, called "flat connection" over X_n. The important point is that the flatness of the connection

$$d\omega + \omega \wedge \omega = 0$$

is a necessary and sufficient condition for the holonomy group $P_{\mathbf{z}}$ at $\mathbf{z}$ to give rise to a *monodromy representation* of the fundamental group $\pi_1(X_n)$. Recall that the holonomy group is the group of $T_{\gamma_{\mathbf{z}\mathbf{z}}}$ of all closed paths $\gamma_{\mathbf{z}\mathbf{z}}$ with the natural path composition as an internal law. We are therefore lead by this statement to a study of flat connections over X_n.

What is the physical meaning to be extracted from such consideration? The basic idea, put forward by the discovery of the non-Abelian Bohm–Aharnov effect [31], is that a closed trajectory in the phase space can affect internal quantum numbers of the system giving rise to topological interaction which depend only on the homotopy class of this closed path. The flat connections usually considered in the literature are the following matrix valued 1–forms, or the so-called Knizhnik-Zamolodchikov (KZ) connections:

$$\omega^{KZ} = \sum_{1\leq<a<b\leq n} \Omega_{ab} d\log z_{ab} \tag{A1}$$

where $\Omega_{ab} \in \mathrm{End}(V^{\otimes n})$ is the Casimir invariant (not explicitly written here), and $z_{ab} = z_a - z_b$. The reader is referred to [30] for a more detailed description of this aspect. This connection was introduced in 2D conformal field theory in the context of the Wess–Zumino–Novikov–Witten model and is related to the study of chiral current algebras [32, 33]. In this model, a primary field is covariant under two kinds of transformations: local gauge transformations generated by the current J and conformal reparameterization generated by the stress-energy tensor T. These T and J are linked by the Sugawara formula describing consistency between two covariances, and leads hence to the KZ equation for an n–points correlator Ψ:

$$h\, d\Psi = \omega\Psi \tag{A2}$$

h being a complex parameter. This formulation suggests a geometric interpretation: Ψ is covariantly constant for this connection. The equation (A2) being a first order linear differential equation, can be treated in the frameworks of the analytic theory of differential equations. Any solution of (A2) can be represented by a linear combination of its $n-1$ fundamental solutions. The formal (and in practice not very useful) description of the holonomy group:

$$T_{\gamma_{\mathbf{zz}}} = P\exp\int_{\gamma_{\mathbf{zz}}} \omega^{KZ} \tag{A3}$$

then reduces to a matrix representation. It is shown in [33] that equation (A2) can be reduced in the case $n = 3$ to a system of ordinary differential equations for the $SU(N)$–invariant amplitude $F\left(\eta = \dfrac{z_{23}}{z_{13}}\right)$ defined by

$$\Psi(z_1, z_2, z_3) = z_{13}^{-\frac{3}{4h}} (\eta(1-\eta))^{-\frac{N+1}{Nh}} F(\eta) \tag{A4}$$

This parameterization yields

$$\left(h\frac{d}{d\eta} + \frac{K_{12}}{1-\eta} - \frac{K_{23}}{\eta}\right) F(\eta) = 0 \tag{A5}$$

where $K_{12},\ K_{23} \in \mathrm{End}(V^{\otimes n})$. Denoting then by I_0, I_1 the basis of $SU(N)$ invariant tensors in $V^{\otimes 3}$ such that

$$K_{23}I_0 = 0,\ K_{12}I_1 = 0 \tag{A6}$$

one can write

$$F(\eta) = (1-\eta)f^0(\eta)I_0 + \eta f^1(\eta)I_1 \tag{A7}$$

and then reduce (A2) to the following system of ordinary differential equations [29]:

$$\begin{cases} h(1-\eta)\dfrac{df^0}{d\eta} &= (h-2)f^0 + f^1 \\ h\eta\dfrac{df^1}{d\eta} &= (2-h)f^1 - f^0 \end{cases} \tag{A8}$$

which admits the form of ordinary 2nd order Riemann differential equation with branching points at $\eta = 0, 1$

$$\eta(\eta-1)\frac{d^2 f^j}{d\eta^2} + \left(\left(3 - \frac{4}{h}\right) - 1 - j + \frac{2}{h}\eta\right)\frac{df^j}{d\eta} + \left(1 - \frac{1}{h}\right)\left(1 - \frac{3}{h}\right) f^j = 0 \qquad (j = 0, 1) \tag{A9}$$

The basis of fundamental solutions of (A9) can be written in terms of standard hypergeometric functions

$$\begin{aligned} f_0^j(\eta) &= \frac{B(h^{-1},j+h^{-1})}{B(h^{-1},2h^{-1})}(1-\eta)^{2h^{-1}+j-1}F(j-h^{-1},j+h^{-1},j+2h^{-1},1-\eta) \\ f_1^j(\eta) &= \frac{B(h^{-1},1-j+h^{-1})}{B(h^{-1},2h^{-1})}\eta^{2h^{-1}-j}F(1-j-h^{-1},1-j+h^{-1},1-j+2h^{-1},\eta) \end{aligned} \tag{A10}$$

Using (A10) and the integral representations of the hypergeometric functions, the authors of [29] have directly compute the action of B_3 generators $\sigma_1,\ \sigma_2$ in the basis of solutions. Choosing a path in the homotopy class of σ_i $(i=1,2)$:

$$\sigma_i:\ z_{\sigma_i}(t)=\frac{z_i+z_{i+1}}{2}+\frac{1}{2}z_{i,i+1}e^{-i\pi t},\ z_{j\neq i,i+1}=\text{const} \tag{A11}$$

corresponding to an elementary move σ_i, they end up with the following braid relations:

$$\sigma_i:\ f_k^j\to [B_i]_k^l f_l^j \tag{A12}$$

where the monodromy matrices are:

$$B_1=\bar{q}^{1/N}\begin{pmatrix} q & 1 \\ 0 & -\bar{q}\end{pmatrix};\ B_2=q^{-1/N}\begin{pmatrix} -\bar{q} & 0 \\ 1 & q\end{pmatrix} \tag{A13}$$

with $q=e^{-\frac{i\pi}{h}}$. This is a 2–dimensional representation of B_3.

Following the general method developed at the length of the Sections II and III, we can conjecture that

$$w(\eta)=\frac{f_0^j(\eta)}{f_0^j(\eta)}$$

is a topological invariant of the group B_3. In particular, we expect that $\dfrac{dw(\eta)}{d\eta}$ gives the corresponding explicit expression of the non-Abelian flat connection.

---

[1] C. Bélisle, Ann. Prob. **4**, 1377 (1989)
[2] A. Comtet, J. Desbois, C. Monthus, J. Stat. Phys. **73**, 433 (1993)
[3] D.R. Nelson, Phys. Rev. Lett. **60**, 1973 (1988); D.R. Nelson, H. Seung, Phys. Rev. B **39**, 9153 (1989)
[4] D.R. Nelson, A. Stern, Proceedings of the XIV Sitges Conference, in *Complex Behavior of Glassy Systems*, June 10–14 (1996)
[5] M. Doi and S.F. Edwards *Theory of Polymer Dynamics* (Academic Press: NY, 1986)
[6] B. Drossel, M. Kardar, Phys. Rev. E **53**, 5861 (1996)
[7] L. Treloar, *The Physics of Rubber Elasticity*, 3rd ed., (Clarendon Press: Oxford, 1975); M. Gottlieb, R.J. Gaylord, Polymer, **24**, 1644 (1983); A.R. Khokhlov, F.F. Ternovskii, Sov. Phys. JETP, **63**, 728 (1986)
[8] S. Obukhov, M. Rubinstein, Phys. Rev. Lett. **65**, 1279 (1990)
[9] Epstein, *Word Processing in Groups* (Barlett and sons: Boston, 1992)
[10] S. Nechaev, A. Grosberg, A. Vershik, J. Phys. A, **29**, 2411 (1996)
[11] P.Dehornoy, *Braids and self-distributivity* (Birkhauser, 2000)
[12] J. Birman, *Braids, Links and Mapping Class Groups* (Ann. Math. Studies **82**, Princeton Univ. Press, 1976)
[13] M. Gromov, *Hyperbolic groups* (Springer: New York, 1987)
[14] *Higher Transcendental Functions* eds.: H. Bateman, A. Erdelyi (McGraw Hill, 1955)
[15] *Handbook of mathematical functions*, eds.: M. Abramowitz and I. Stegun (Dover Publications, Inc.: New York, 1970)
[16] A.B. Venkov, Preprint LOMI P/2/86
[17] N. Ikeda, H. Matsumoto, J. Funct. Ann., **163**, 63 (1999)
[18] R. Voituriez, Nucl. Phys. B **621**, 675 (2002)

[19] S. Nechaev, R. Voituriez, J. Phys. A, **36**, 43 (2003)
[20] V.V. Golubev *Lectures on Analytic Theory of Differential Equations* (Moscow: GITTL, 1950)
[21] E. Hille, *Analytic Function Theory* (Ginn & Co.: Boston, 1962)
[22] J. Harnad, J. McKay, Proc. Roy. Soc. Lond. **456** 261 (2000)
[23] D.R. Hofstadter, Phys. Rev. B, **14**, 2239 (1976)
[24] A. Comtet, Ann. Phys., **173**, 185 (1987)
[25] K. Itô, H.P. McKean, *Diffusion Processes and their Sample Paths* (Springer: Berlin, 1974)
[26] J.C. Gruet, Prob. Theor. Rel. Fields **111**, 489 (1998)
[27] S.F. Edwards, Proc. Phys. Soc. London **91**, 513 (1967)
[28] D. Revuz, M. Yor, *Continuous Martingales and Brownian Motion* (Springer: Berlin, 2001)
[29] I. Todorov, L. Hadjiivanov, Phys. Atom. Nucl. **64**, 2059 (2001)
[30] A. Tsuchiya, Y. Kanie, Lett. Math. Phys., **13** 303 (1987)
[31] E. Verlinde, A note on braid statistics and the nonabelian Aharonov-Bohm effect, in *Modern Quantum Field Theory*, (World Scientic: Singapore, 1991)
[32] E. Witten, Coomm. Math. Phys., **92**, 455 (1094)
[33] D.G. Knizhnik, A.B. Zamoodchikov, Nucl. Phys., **B247**, 83 (1984)