Fluctuations of internal energy flow in a vibrated granular gas.

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The non-equilibrium fluctuations of power flux in a fluidized granular media have been recently measured in an experiment [Phys. Rev. Lett. 92, 164301, 2004], which was announced to be a verification of the Fluctuation Relation (FR) by Gallavotti and Cohen. An effective temperature was also identified and proposed to be a useful probe for such non equilibrium systems. We explain these results in terms of a two temperature Poisson process. Within this model, supported by independent Molecular Dynamics simulations, power flux fluctuations do not satisfy the FR and the nature of the effective temperature is clarified. In the pursue of a hypothetical global quantity fulfilling the FR, this points to the need of considering other candidates than the power flux.

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Granular gases, i.e. gases of macroscopic grains losing part of their kinetic energy during collisions, have been the subject of ardent investigation in the last decade [1]. They display, in experiments as well as in simulations, a rich and intriguing phenomenology: non-Gaussian statistics, breakdown of energy equipartition, spontaneous symmetry breaking (clustering, shear, convection, surface waves, shocks etc) are the most striking features of this surprising state of matter (see e.g. [2] and references therein). In addition, a driven granular gas is an ideal testing ground for non-equilibrium statistical mechanics of dissipative stationary states. An inelastic gas can be kept in a state of constant average total kinetic energy thanks to an external driving mechanism, e.g. by rapidly vibrating its container. In such a system, energy is flowing from the external source/thermostat into the gas and then from the gas into an irreversibly draining sink, represented by inelastic collisions. As a consequence, the fluctuations of global or microscopic physical observables, such as the total energy or the velocity of particles or all internal currents, do not behave as expected in equilibrium statistical mechanics. It is therefore tempting to give an interpretation of the observed fluctuations in terms of the few known theoretical results in non-equilibrium statistical physics. In a recent experiment on vibrated granular gases [3] it has been argued that the statistics of the power flux fulfilled the Fluctuation Relation (FR) by Gallavotti and Cohen [1,4], even if –as emphasized in [3]– a) the FR holds under conditions that are not met in the situation under study (in particular, microscopic reversibility is required) and b) there is no proven relation between the measured power flux and the entropy production entering the original FR.

We show here that these experimental results can be explained in terms of a simple Poisson process. We argue that the measured “power flux” $Q$ is well reproduced by a sum of random and independent energy amounts characterized by two different typical temperatures. Such a quantity does not verify the FR. We nevertheless demonstrate that a straight line with slope $\beta_{eff}$ can be observed when plotting $\log[f_Q(Q)/f_Q(-Q)]$ vs. $Q$. $f_Q$ being the probability density distribution of $Q$, in the range which is accessible in experiments and simulations. In [3] the predictability of $\beta_{eff}$ remained an open problem and $1/\beta_{eff}$ was interpreted as an effective temperature. Within the two temperature model, the value of $\beta_{eff}$ is obtained and it further appears that interpreting this quantity as an inverse effective temperature is problematic. We validate our analysis by a direct confrontation against Molecular Dynamics (MD) simulations, obtaining very good agreement.

![Figure 1](image1.png)

**FIG. 1:** Left: Snapshot of the system considered for MD simulations, with the inner region marked by the solid rectangle. Right: Corresponding vertical profiles of density ($\Phi(y)$, dashed line) and temperature ($T(y)$, solid line). The dotted lines mark the bottom and top boundaries of the inner region. Here $N=270$ and $\alpha=0.9$. The mean free path is $\sim 5.7d$.

The event driven MD simulations have been performed.
for a system of N inelastic hard disks with restitution coefficient $\alpha$, diameter $d$ and mass $m = 1$. The vertical 2D box of width $L_x = 48d$ and height $L_y = 32d$ is shaken by a sinusoidal vibration with frequency $f$ (period $\tau_{box} = 1/f$) and amplitude $2.6d$. In a collision two particles lose a fraction $1 - \alpha^2$ of their relative kinetic energy while the total momentum is conserved. Collisions with the elastic walls inject energy and allow the system to reach a stationary state. We have checked that possible inelastic collisions with the walls hardly affect the results. Gravity – set to $g = -1.7df^2$ in order to be consistent with the experiment – has a negligible influence on the measured quantities. We have varied the restitution coefficient from 0.8 up to 0.99 (glass beads yield on average $\alpha \approx 0.9$) and the total area coverage from 0.138 (i.e. $N = 270$) up to 0.32 ($N = 620$). In Fig. 1 left a snapshot of the system is shown. During the simulations all the main physical observables are statistically stationary. The local area coverage field $\Phi(x,y)$ and the granular temperature field $T(x,y)$ (defined in 2D as the average kinetic energy per particle) are almost uniform in the horizontal direction, apart from small layers near the side walls. In Fig. 1 right the profiles $\Phi(y)= (1/L_y) \int dx \Phi(x,y)$ and $T(y) = (1/L_y) \int dx T(x,y)$ are shown to be symmetric with respect to the bottom and the top of the box. Following the experimental procedure, we have focussed our attention on a “window” in the center of the box, fixed in the laboratory frame, of width $2L_x/5$ and height $L_y/3$, marked in Fig. 1 left. Apart from the negligible change of potential energy due to gravity, the total kinetic energy of the particles inside the window, changes during a time $\tau$ because of two contributions: $\Delta K_\tau = Q_\tau - W_\tau$ where $Q_\tau$ is the kinetic energy transported by particles through the boundary of the window (summed when going-in and subtracted when going-out) and $W_\tau$ is the kinetic energy dissipated in inelastic collisions during time $\tau$. For several values of $\tau$ we have measured, as in the experiments, $Q_\tau$ which is related to the kinetic contribution to the heat flux (we checked that inclusion of the collisional contribution, even if non small, does not change the picture). With $N = 270$ and $\alpha = 0.9$ the characteristic times are the mean free time $\tau_{col} \approx 0.47 \tau_{box}$, the diffusion time across the window $\tau_{diff} = 0.82 \tau_{box}$ and the mean time between two subsequent crossings of particles (from outside to inside) $\tau_{cross} \approx 0.039 \tau_{box}$.

We define the injected power as $q_\tau = Q_\tau/\tau$ and two relevant probability density functions (pdfs): $f_Q(Q_\tau)$ and $f_q(q_\tau)$. Fig. 2.a) shows $f_q(q_\tau)$ for different values of $\tau$. A direct comparison with Fig. 3 of Ref. 3 suggests a fair agreement between simulations of inelastic hard disks and the experiment. The pdfs are strongly non-Gaussian and asymmetric, becoming narrower as $\tau$ is increased. At small $\tau$ a strong peak in $q_\tau = 0$ is visible. More interestingly, $f_q(q_\tau)$ at small values of $q_\tau$ has two different exponential tails, i.e. $f_q(q_\tau) \sim \exp(\mp \beta_\pm q_\tau)$ when $q_\tau \to \pm \infty$ with $\beta_- > \beta_+$. The peak and the exponential tails at small $\tau$ are observed also in the experiment (see Fig. 3 of Ref. 3). The dashed line has a slope $\beta_eff = 32/\tau_{gran}$ in the inset the same graph is shown for small values of $\tau = (1, 2, 4, 8) \times \tau_{min}$ (from bottom to top).

![Graph](image_url)

**Fig. 2:** a) pdfs of injected power $f_q(q_\tau)$ from MD simulations for different values of $\tau = (1, 2, 4, 8, 16, 32) \times \tau_{min}$ with $\tau_{min} = 0.015 \tau_{box}$. Here $N = 270$ and $\alpha = 0.9$. The distributions are shifted vertically for clarity. The dashed lines put in evidence the exponential tails at small $\tau$. b) In the inset the same graph is shown for small values of $\tau = (1, 2, 4, 8) \times \tau_{min}$ (from bottom to top).
simulations. At area fraction 13.8\% and $\alpha = 0.9$ we have $\beta_{\text{eff}} T_{\text{gran}} \approx 0.23$. At fixed $\alpha$ and increasing area fraction, $\beta_{\text{eff}} T_{\text{gran}}$ slightly increases, as in the experiment. As $\alpha \to 1$ the slope $\beta_{\text{eff}}$ decreases. At $\alpha = 1$ (without gravity and external driving) the distribution of $Q_\tau$ is symmetrical and $\beta_{\text{eff}} = 0$, indicating that $1/\beta_{\text{eff}}$ is not a physically relevant temperature concept. Interestingly, it appears that $\beta_{\text{eff}}$ is a non hydrodynamic quantity: different systems may show the same density and temperature profiles, with very different values of $\beta_{\text{eff}}$. 

We now adopt a coarse-grained description of the experiment which is able to entirely capture the observed phenomenology. The measured flow of energy is given by

$$Q_\tau = \frac{1}{2} \left( \sum_{i=1}^{n_+} v_{i+}^2 - \sum_{i=1}^{n_-} v_{i-}^2 \right),$$

where $n_-$ ($n_+$) is the number of particles leaving (entering) the window during the time $\tau$, and $v_{i+}$ ($v_{i-}$) are the squared moduli of their velocities. In order to analyze the statistics of the squared moduli of their velocities, we use $Q_\tau$ distributed random variables with average $\mu_\tau$, $\beta_{\text{eff}}$, $\gamma$, $\alpha$, and $\nu_\tau$, respectively. Indeed, compared with the population entering the central region, those particles that leave it have suffered on average more inelastic collisions, so that $T_- < T_+$. Finally we assume Gaussian velocity pdfs $\tilde{f}_Q(x)$. Within such a framework, the distribution $f_Q(Q_\tau)$ of $Q_\tau$ can be studied analytically. Here it is enough to recall that $\sum_{i=1}^{n_i} v_i^2$, in $D$ dimensions, if each component of $v_i$ is independently Gaussian-distributed with zero mean and variance $\tau$, is a stochastic variable $x_\tau$ with a distribution $\chi_{n,T}(x) = f_{1/T,Dn/2}(x)$, where $\chi_{n,T}(x)$ is the Gamma distribution, and whose generating function reads $\chi_{n,T}(x) = (1 - T\tau)^{-Dn/2}$. It is then straightforward to obtain the generating function of $Q_\tau$ in the form $\tilde{f}_Q(z) = \exp[\tau \mu(z)]$ with

$$\mu(z) = \omega \left( -2 + (1 + T\tau)^{-D/2} + (1 - T\tau)^{-D/2} \right),$$

where $\omega$ is the value of $\chi_{n,T}(x)$ at $x = 0$. We observe that $\tilde{f}_Q(z)$ has two poles in $z = \pm 1/T\tau$ and two branch cuts on the real axis for $z \geq 1/T\tau$ and $z < -1/T\tau$. From $\mu(z)$ we immediately obtain the cumulants of $Q_\tau$ through the formula $(Q_\tau)^\gamma = \tau \frac{\partial \mu}{\partial \tau}(0)$.

For $\tau \to \infty$ the large deviation theory states that $f_Q(Q_\tau) \sim \exp(\tau \Pi(Q_\tau/\gamma))$ and $\Pi(q)$ can be obtained from $\mu(z)$ through a Legendre transform, i.e. $\Pi(q) = \max \{\mu(z) - qz\}$. The study of the singularities of $\mu(z)$ reveals the behavior of the large deviation function $\Pi(q)$ for $q \to \pm \infty$. It can be seen that

$$\Pi(q) \sim \frac{q}{T_+} \quad (q \to \infty), \quad \Pi(q) \sim \frac{q}{T_-} \quad (q \to -\infty).$$

We emphasize however that it is almost impossible to appreciate these tails in simulations and in experiments, since the statistics for large values of $q$ and $\tau$ is very poor.

A Gallavotti-Cohen-type relation $\Pi(q) - \Pi(-q) = \beta q$ for any $q$ and an arbitrary value of $\beta$ would imply $\mu(z) = \beta(\mu(z) - qz)$. One can see that such a $\beta$ does not exist, i.e. the fluctuations of $Q_\tau$ do not satisfy a Gallavotti-Cohen-like relation. The observed linearity of the graph $\log f_Q(q_\tau)/f_Q(-q_\tau)/\tau = \pi(q_\tau) - \pi(-q_\tau)$ vs. $q_\tau$ can be explained by the following observation: at large values of $\tau$ it is extremely difficult, in simulations as well as in experiments, to reach large values of $q$, while for small $q$, $\pi(q) - \pi(-q) \approx 2\pi'(0)q + o(q^3)$, i.e. a straight line with a slope $\beta_{\text{eff}} = 2\pi'(0)$ is likely to be observed. It has been already shown that in dissipative systems deviations from the FR can be hidden by insufficient statistics at high values of $q$. The knowledge of $\mu(z)$ is useful to predict this slope. At large $\tau$, $\gamma(\tau \approx \Pi(0) = -z^* \approx 0$ where $z^*$ is the value of $z$ for which $\mu(z) - qz$ is extremal. This gives

$$\beta_{\text{eff}} = \frac{2\delta^2 - 1}{\frac{2}{\gamma^2} + 1} \quad \text{with} \quad \gamma = \frac{T_+}{T_\tau}; \quad \delta = \frac{2}{2 + D}.$$ 

When $\gamma = 1$ (i.e. $\alpha = 1$) $\beta_{\text{eff}} = 0$. As $\alpha$ decreases, $\gamma$ increases, since the collisions dissipate more energy, and $\beta_{\text{eff}} T_\tau$ grows, reaches a maximum and subsequently decreases asymptotically toward 0 as $\gamma \delta^{-1}$. We emphasize that $\beta_{\text{eff}}$ does not depend upon $\omega$. We have compared these predictions with the numerical and experimental results, measuring the temperatures $T_\tau$ and $T_-$ in the simulation. To bypass the problem of temperature anisotropy (discussed below), we have used the horizontal component of the temperature, obtaining (with $\alpha = 0.9$ and area fraction 13.8\%) $T_\tau \approx 141d^2f^2$ and $T_- \approx 91d^2f^2$, i.e. $\gamma = 1.55$ and, from Eq. 4 in $D = 2$, $\beta_{\text{eff}} = 0.00193$ in very good agreement with the measured value 0.0022. It should be noted that the temperature ratio can also be approximated by $(\alpha^2)^{-m}$ where $m$ is the average number of collisions undergone by a particle between the moments of entering and leaving the observation region. In our numerical simulations (as well as in the experiment) $m \approx 2$, which corresponds to $\gamma \approx 1.5$.

What happens for small values of $\tau$? We note that $f_Q(z)$ has the form $\exp(\tau \mu(z))$ for any value of $\tau$ and not only for large $\tau$. Therefore at small $\tau$ one can expand the exponential, obtaining $f_Q(z) \sim 1 + \omega \tau (-2 + (1 - T\tau)^{-D/2} + (1 - T\tau)^{-D/2})$. This immediately leads to an analytical expression for $f_Q(Q_\tau) = \chi_{1,T_\tau}(Q_\tau) + \chi_{1,T_\tau}(-Q_\tau)$, which fairly accounts for the strong peak which is observed in the experiment and in the simulations, and predicts exponential tails for $f_Q(Q_\tau)$: $\chi_{1,T_\tau}(x) \approx x^{D/2-1} \exp(-x/T)$.
obtained by fitting the tails in the hard disks simulation, \( T_+ \) and \( T_- \). This suggests an experimental test of this theoretical approach: the measure at small values of \( \tau \) of the slopes of the exponential tails of \( f_{Q}(Q_{\pm}) \) should coincide with a direct measure of \( T_+ \) and \( T_- \). However, we point out that the values of \( \beta_+ \) and \( \beta_- \) obtained by fitting the tails in the hard disks simulation, using values as small as \( \tau = 0.00015 \tau_{\text{max}} \) yield estimates of \( T_+ \) and \( T_- \) which are larger (by a factor \( \sim 1.6 \)) than those found by a direct measure. This disagreement brings the limits of such a simple two-temperature picture to the fore. In the simulation and in the original experiment the measured injected energy is indeed the sum of several different contributions, namely \( Q_{\pm} \approx Q_{x}^{yy} + Q_{y}^{yy} + Q_{x}^{xx} + Q_{y}^{xx} \) where \( Q_{ij}^{ij} \) is the kinetic energy transported by the \( i \) component of the velocity by particles crossing the boundary through a wall perpendicular to direction \( j \). Two main differences with the simplified interpretation given above arise: a) there are two couples of temperatures, i.e. \( T_+^{\Box}, T_-^{\Box} \) as well as \( T_+^{\odot}, T_-^{\odot} \) [11]; b) the diagonal contributions \( Q_{xx}^{yy} \) are sums of squares of velocities whose distribution is not a Gaussian but is \( \sim \exp(-v^2/T) \), since the probability of crossing is biased by the velocity itself. The calculation of \( f_{\beta}(Q_{\pm}) \) is still feasible, with qualitatively similar results [12].

**FIG. 3:** Color online: Pdf of transverse \((xy)\) injected power \( f_{\beta}(q_{\tau}) \) for low values of \( \tau = (1, 2, 4, 8) \times \tau_{\text{min}} \) with \( \tau_{\text{max}} = 0.00015 \tau_{\text{max}} \) (resp. circle, squares, diamond and triangles) from MD simulations with \( N = 270 \) and \( \alpha = 0.9 \). The distributions are shifted vertically for clarity. The (red) solid lines show the solution of the “two temperatures” model. The dashed lines indicate exponential tails \( \sim \exp(\mp \beta_{+} q_{\tau}) \). Left inset: same plot for a large value \( \tau = 6400 \tau_{\text{min}} \). Right inset: plot of \((1/\tau) \log(f_{\beta}(Q_{\tau})/f_{\beta}(-Q_{\tau}))\) vs. \( q = Q/\tau \) from MD simulations (same parameters as above) at large values of \( \tau = (1, 2, 4, 8) \times \tau_{\text{min}} \) with \( \tau_{\text{min}} = 3200 \tau_{\text{min}} \), together with a dashed line of slope \( \beta_{\text{eff}} \), predicted by Eq. [11] and the dotted line of slope \( \beta_{\text{gran}} = 1/\tau_{\text{gran}} \).

Here we focus on the proposed two-temperature model, showing its ability to explain the statistics of internal energy currents. To this extent we have repeated the MD simulations discussed above, but measuring only the transversal kinetic energy current \( Q_{xy}^{yy} \) through the bottom boundary of the central region. For new measurements, shown in Fig. 3, we have also improved the time resolution of the measure, using a minimum \( \tau = 0.00015 \tau_{\text{cross}} \). The coefficients of the exponential tails of \( f_{\beta}(q_{\tau}) \) at low \( \tau \) are in perfect agreement with the predicted \( \tau/\tau_{\pm} \) and the slope of the graph \( \log(f_{\beta}(q_{\tau})/f_{\beta}(-q_{\tau}))\) vs. \( q_{\tau} \), shown in the inset, is accurately recovered by Eq. [11] with \( D = 1 \). More remarkably, the two-temperature model (solved by numerical inversion of \( f_{\beta} \)) fully accounts for the pdf \( f_{\beta}(q_{\tau}) \) at any value of \( \tau \), as evidenced in the same figure.

In conclusion we have implemented MD simulations of inelastic hard disks fluidized in a vibrated box. We have studied the statistics of the power flux through a closed perimeter— which effectively separates a high temperature region from a low temperature one—thereby accurately reproducing the experimental measurements of [8]. Inspired by these results, we have put forward a simple two-temperature Poissonian model, which entirely reproduces the phenomenology of the system but for which the fluctuations of the energy flow do not satisfy a Gallavotti-Cohen relation. Moreover, our approach puts forward a kinetic interpretation for the measured slope \( \beta_{\text{eff}} \), a question previously left open. We conclude here that there are serious evidence against the power flux as a potential candidate for extending the FR to a granular gas, a system for which the validity of such a relation is therefore still open.

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[8] While non-Gaussian tails are quite common in granular gases [see T.P.C. van Noije and M. H. Ernst, Granular Matter 1, 57 (1998)] we have checked that they play a negligible role here.