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Transition between Hermitian and non-Hermitian Gaussian ensembles

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Abstract
The transition between Hermitian and non-Hermitian matrices of the Gaussian unitary ensemble is revisited. An expression for the kernel of the rescaled Hermite polynomials is derived which expresses the sum in terms of the highest order polynomials. From this Christoffel–Darboux-like formula some results are derived including an extension to the complex plane of the Airy kernel.

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1. Introduction

The Gaussian ensemble of random matrices introduced in late 1950s by Wigner [1] has since then a successful history. It consists of three classes of Hermitian matrices whose elements are real, complex and quaternion, respectively, denoted as the orthogonal (GOE), the unitary (GUE) and the symplectic (GSE) indexed by the number of degrees of freedom $\beta = 1, 2$ and 4 of the matrix elements. The link with the characterization of the manifestation of chaos in quantum mechanics [2] multiplied its applications. By the same time, Ginibre studied the classes of Gaussian matrices obtained by removing the Hermiticity condition [3]. Remaining obscured by the success of the Hermitian ensemble, that model has attracted, in more recent years, great attention, and important contributions extended Ginibre's pioneering work [4]. The transition from Hermitian to non-Hermitian has also been investigated (see, for instance, [5] for developments and applications).

A crucial step in RMT analytical derivations of the eigenvalue statistical measures is to find appropriate orthogonal polynomials in terms of which the joint distribution of matrix elements can be put in a determinantal form. Once this is done, measures can be expressed in terms of the associated polynomial kernel. Next, the polynomial's Christoffel–Darboux formula is used to derive the kernel asymptotics for large order of polynomials, the regime one is usually interested in. This was the procedure followed by Mehta to unravel, in his seminal work, the statistical properties of the Wigner Gaussian ensemble in which case the polynomials are the Hermite ones [6]. In the studies of the transition from GUE to its non-Hermitian Ginibre correspondent, it has been found that, by an appropriate rescaling, the same Hermite polynomials are the ones to be used. However, with the argument rescaled, the Christoffel–Darboux formula of the Hermite polynomials is no longer valid. Despite this, several results have already been obtained. It has been shown that, asymptotically, starting from Wigner's semi-circle law on the real axis, the density of eigenvalues evolves into an elliptic shape distribution as the Hermitian condition is progressively broken, approaching the uniform circular distribution of the Ginibre case [7]. The existence of an important quasi-Hermitian regime with special properties [8] has also been established. The expression for the gap probability at the bulk of the spectrum was derived [9] and also statistics at the spectral edge have been investigated [10]. The structure of the eigenvalue trajectories along the transition has recently been revealed [11].

Important as they are, the above achievements do not make worthless the endeavor to have, for the rescaled polynomials, some equivalent of Christoffel–Darboux formula. Of course, the obtainment of a formula of this kind has in itself an obvious mathematical interest. But also from the side of applications, one might expect that with it, analytical difficulties can be avoided [10] allowing us to derive other results.

In the next section, we show that there is indeed a way to express the sum over polynomials appearing in the kernel associated with the rescaled Hermite polynomials in terms of the higher order ones. Then, in the following section the formula is applied to derive the elliptic shape of the asymptotic density distribution, a known result, and the gap probability at the bulk of the spectrum, a new result. Finally, the extension of the Airy kernel at the edge is obtained, and the presence of different asymptotic regimes is discussed.

2. The Christoffel–Darboux-like formula

Consider a matrix S, of size N, taken from the ensemble of random matrices whose joint distribution of elements is given by

$$P(S) = \exp[-\mathrm{tr}(S^{\dagger}S)], \tag{1}$$

and define a matrix $H(t)$ by the relation

$$H(t) = \left(\frac{S+S^{\dagger}}{2}\right) + t\left(\frac{S-S^{\dagger}}{2}\right), \tag{2}$$

where the parameter t varies from 0 to 1. It can easily be proved that $H(0)$ is a Hermitian GUE matrix [11]. As $S = H(1)$ belongs to the Ginibre ensemble of non-Hermitian matrices, $H(t)$ undergoes a transition from the Wigner to the Ginibre ensemble. With $t > 0$, equation (2) together with its adjoint can be inverted to express S in terms of H and $H^{\dagger}$ as

$$S = \left(\frac{1+t}{2t}\right)H - \left(\frac{1-t}{2t}\right)H^{\dagger}. \tag{3}$$

Substituting equation (3) into equation (1), we obtain the density distribution of the t-dependent matrix elements of H:

$$P(H) = K_N(t)\exp\left\{-\mathrm{tr}\left[\frac{1+t^2}{2t^2}(H^{\dagger}H) - \frac{1-t^2}{4t^2}(HH + H^{\dagger}H^{\dagger})\right]\right\}. \tag{4}$$

Replacing in (4), H by its decomposition $H = QDQ^{-1}$, where D is a diagonal matrix which contains the complex eigenvalues $z = x + \mathrm{i}y$, the traces in (4) can be calculated. The two last ones immediately give $\mathrm{tr}(HH) = \sum z_i^2$ and $\mathrm{tr}(H^{\dagger}H^{\dagger}) = \sum \bar{z}_i^2$, while the trace $\mathrm{tr}(H^{\dagger}H)$ can be dealt with using the method developed by Ginibre or alternatively Dyson's version of it [6]. After some manipulations the joint density distribution

$$P(z_1, z_2, \ldots, z_N) = \text{const} \exp\left[-\sum_{k=1}^{N}\left(x_k^2 + \frac{y^2}{t^2}\right)\right]\prod_{j>i}|z_j - z_i|^2 \tag{5}$$

of the eigenvalues in the complex plane is obtained. The structure of this distribution, namely the exponential of a sum of individual eigenvalue terms multiplied by the product of the differences between each pair of them is typical of the joint density distributions of eigenvalues of the RMT ensembles. To investigate statistical properties of the eigenvalues of $H(t)$, we would like, therefore, to be able to use the powerful RMT method of multidimensional integration. The first step in this method is to find polynomials $p_n(z)$ orthogonal with respect to the weight defined by the exponential factor; then the term with the differences is written as the product of Vandermonde determinants in which rows are put in the form of polynomials. With these definitions, the normalized joint density distribution can be written as

$$P(x_1, y_1, x_2, y_2, \ldots, x_N, y_N) = \frac{1}{N!}\det[K_N(x_i, y_i, x_j, y_j)], \tag{6}$$

where

$$K_N(z_1, z_2) = \sum_{k=0}^{N-1} f_k(\bar{z}_1) f_k(z_2), \tag{7}$$

with

$$f_k(z) = \exp\left[-\frac{1}{2}(x^2 + y^2/t^2)\right] p_k(z). \tag{8}$$

The effect of integrating one eigenvalue in equation (6) is to remove the row and column corresponding to the given eigenvalue with the remaining determinant being multiplied by a constant. Integrating, for instance, all eigenvalues gives the normalization constant $N!$ in equation (6). Integrating keeping a set of n eigenvalues fixed gives the n-point correlation function:

$$R_n(x_1, y_1, \ldots, x_n, y_n) = \frac{N!}{(N-n)!}\int \cdots \int P(x_1, y_1, \ldots, x_N, y_N)\prod_{n+1}^{N} \mathrm{d}x_i\, \mathrm{d}y_i. \tag{9}$$

With $n = 1$, we have the density of eigenvalues

$$R_1(x, y) = K_N(z, z) = \sum_{j=0}^{N-1} f_j(\bar{z}) f_j(z) \tag{10}$$

and, with $n = 2$, the two-point correlation function

$$R_2(x_1, y_1; x_2, y_2) = K_N(z_1, z_2) = \sum_{j=0}^{N-1} f_j(\bar{z}_1) f_j(z_2). \tag{11}$$

In the Ginibre case, $t = 1$, the method applies due to the orthogonality

$$\int \frac{\mathrm{d}x\,\mathrm{d}y}{\pi}\exp(-|z|^2)\bar{z}^i z^j = \mathrm{i}!\delta_{ij} \tag{12}$$

of the product $\bar{z}^i z^j$ with respect to the weight $\exp(-|z|^2)$, such that the polynomials are the monomials

$$p_n(z) = \frac{z^n}{\sqrt{n!}}. \tag{13}$$

As the imaginary part of z appears in equation (5) divided by the parameter t, this orthogonality does not hold for $t \neq 1$. To overcome this difficulty we define, by the Rodrigues formula, the polynomials

$$p_n(z) = \frac{(-1)^n}{2^{n/2}\sqrt{n!}(1+t^2)^{n/2}} e^{x^2+y^2/t^2} \left(\frac{\partial}{\partial x} + it^2 \frac{\partial}{\partial y}\right)^n e^{-x^2-y^2/t^2}. \tag{14}$$

Expanding the power operator, equation (14) can be written as

$$p_n(z) = \frac{1}{2^{n/2}\sqrt{n!}(1+t^2)^{n/2}} \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} \left(e^{x^2} \frac{d^k}{dx^k} e^{-x^2}\right) \times (it)^{n-k} \left(e^{y^2/t^2} t^{n-k} \frac{d^{n-k}}{dy^{n-k}} e^{-y^2/t^2}\right), \tag{15}$$

such that the Rodrigues formula of the Hermite polynomials, namely $H_n(x) = (-1)^n \exp(x^2) d^n/dx^n \exp(-x^2)$, can be used to express the polynomials $p_n(z)$ in terms of the Hermite ones as

$$p_n(z) = \frac{1}{2^{n/2}\sqrt{n!}(1+t^2)^{n/2}} \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} H_k(x)(it)^{n-k} H_{n-k}\left(\frac{y}{t}\right). \tag{16}$$

Using this expression and the orthogonality relations

$$\int_{-\infty}^{\infty} \exp(-x^2) H_m(x) H_n(x) = \sqrt{\pi} 2^n n! \delta_{mn} \tag{17}$$

of the Hermite polynomials, the orthonormality relations

$$\int \frac{dx\,dy}{t\pi} \exp(-x^2 - y^2/t^2) p_m(\bar{z}) p_n(z) = \delta_{mn} \tag{18}$$

for the polynomials $p_n(z)$ are straightforwardly proved. Multiplying now equation (16) by $w^n/\sqrt{n!}$ and summing from zero to infinite we obtain

$$\sum_{n=0}^{\infty} \frac{w^n}{\sqrt{n!}} p_n(z) = \sum_{n=0}^{\infty} \frac{w^n}{2^{n/2}(1+t^2)^{n/2}} \sum_{k=0}^{n} \frac{1}{k!(n-k)!} H_k(x)(it)^{n-k} H_{n-k}\left(\frac{y}{t}\right), \tag{19}$$

which, after inverting the order of summations and the introduction of the new index $l = n-k$, becomes the product of independent sums as

$$\sum_{n=0}^{\infty} \frac{w^n}{\sqrt{n!}} p_n(z) = \sum_{k=0}^{\infty} \left(\frac{w}{\sqrt{2(1+t^2)}}\right)^k \frac{H_k(x)}{k!} \sum_{l=0}^{\infty} \frac{1}{l!} \left(\frac{itw}{\sqrt{2(1+t^2)}}\right)^l H_l\left(\frac{y}{t}\right). \tag{20}$$

By an appropriate definition, in each case, of the variable q of the generating function

$$\exp(2qx - q^2) = \sum_{n=0}^{\infty} \frac{q^n}{n!} H_n(z) \tag{21}$$

of the Hermite polynomials, these two sums can be resummed to give

$$\sum_{n=0}^{\infty} \frac{w^n}{\sqrt{n!}} p_n(z) = \exp\left[\frac{2wz}{\sqrt{2(1+t^2)}} - \frac{(1-t^2)w^2}{2(1+t^2)}\right]. \tag{22}$$

Moreover, by rearranging the argument in this exponential, it turns out that, with an appropriate definition of the variable q, again it can be put in the form of the generating function

$$\sum_{n=0}^{\infty} \frac{w^n}{\sqrt{n!}} p_n(z) = \exp\left[\frac{2w\sqrt{1-t^2}}{\sqrt{2(1+t^2)}} \frac{z}{\sqrt{1-t^2}} - \frac{(1-t^2)w^2}{2(1+t^2)}\right] \tag{23}$$

of the Hermite polynomials. Expanding it, a comparison between terms of the same power in the two series leads to the explicit expression

$$p_n(z) = \frac{1}{\sqrt{2^n n!}}\left(\frac{1-t^2}{1+t^2}\right)^{n/2} H_n\left(\frac{z}{\sqrt{1-t^2}}\right) \tag{24}$$

of the rescaled Hermite polynomials. Apart from the fact that we are using a different parameterization, this expression is known; nevertheless, the present derivation furnishes for them, as byproducts, a Rodrigues formula and a generation function.

From equation (7), the key quantity in deriving the spectral statistical properties of the ensemble is the kernel function

$$S_n(a,z,t) = \sum_{k=0}^{n-1} p_k(\bar{a})p_k(z) \tag{25}$$

associated with these polynomials. More precisely, we are interested in its asymptotic expression in the limit of large matrix sizes. In order to be able to deduce this asymptotics, a relation expressing the sum in terms of higher order polynomials is needed. To derive this relation, consider the quantity

$$G_n(a,z) = -\frac{1+t^2}{4t^2}[(1+t^2)p_n(\bar{a})p_n'(z) - (1-t^2)p_n'(\bar{a})p_n(z)] \tag{26}$$

and the recurrence relations

$$p_n' = \sqrt{\frac{2n}{1+t^2}}p_{n-1} \tag{27}$$

and

$$\sqrt{2n(1+t^2)}p_n = 2zp_{n-1} - (1-t^2)p_{n-1}', \tag{28}$$

obtained from the recurrence relations $H_n'(x) = 2nH_{n-1}(x)$ and $H_n = 2xH_{n-1} - 2(n-1)H_{n-2}$ of the Hermite polynomials. Now, if the derivatives in equation (26) are replaced using equation (27), and the polynomials of order n are expressed in terms of the polynomials of order $n-1$ and their derivatives using equation (28) then, by rearranging terms, $G_n(a,z)$ can be written as

$$\begin{aligned} G_n(a,z) = &-\frac{1}{2t^2}[(1+t^2)\bar{a} - (1-t^2)z]p_{n-1}(\bar{a})p_{n-1}(z) \\ &- \frac{1-t^2}{4t^2}[(1-t^2)p_{n-1}(\bar{a})p_{n-1}'(z) - (1+t^2)p_{n-1}'(\bar{a})p_{n-1}(z)]. \end{aligned} \tag{29}$$

Multiplying this equation by the exponential $e^{-\frac{(1+t^2)}{2t^2}\bar{a}z + \frac{(1-t^2)}{4t^2}z^2}$, an integration by parts between arbitrary limits leads to the recurrence relation

$$\begin{aligned} \int_{z_0}^{z} dv \exp\left(-\frac{(1+t^2)}{2t^2}\bar{a}v + \frac{(1-t^2)}{4t^2}v^2\right) G_n(a,v) = \exp\left(-\frac{(1+t^2)}{2t^2}\bar{a}z + \frac{(1-t^2)}{4t^2}z^2\right) \\ \times p_{n-1}(\bar{a})p_{n-1}(z)|_{z_0}^{z} \\ + \int_{z_0}^{z} dv \exp\left(-\frac{(1+t^2)}{2t^2}\bar{a}v + \frac{(1-t^2)}{4t^2}v^2\right) G_{n-1}(a,v). \end{aligned} \tag{30}$$

A recurrent application of this relation, and the fact that $G_0(a,z) = 0$, gives for the polynomial kernel the formula

$$\begin{aligned} S_n(a,z,t) = \exp\left(\frac{(1+t^2)}{2t^2}\bar{a}z - \frac{(1-t^2)}{4t^2}z^2\right)\Bigg[S_n(a,z_0,t)\exp\left(-\frac{(1+t^2)}{2t^2}\bar{a}z_0 + \frac{(1-t^2)}{4t^2}z_0^2\right) \\ + \int_{z_0}^{z} dv \exp\left(-\frac{(1+t^2)}{2t^2}\bar{a}v + \frac{(1-t^2)}{4t^2}v^2\right) G_n(a,v)\Bigg], \end{aligned} \tag{31}$$

which is a main result of the present paper. By expressing the sum in the kernel over the polynomials up to a given order in terms of an integral which contains the highest order ones, it has, for our polynomials, the same meaning of the Christoffel–Darboux formula. Of course, it provides, as the Christoffel–Darboux formula does, a way of using the asymptotics of higher order polynomials. Asymptotics which in the case of the Hermite polynomials are well known in the whole complex plane [14].

The integration limit z_0 in equation (31) remains indefinite and this freedom can be used to address calculations in the two main regions of interest in the complex plane, namely the bulk and the edge of the eigenvalue distribution. For the bulk, the appropriate interval of integration is from the origin to z and therefore $z_0 = 0$; on the other hand, for the edge, the integral is from z to infinity, and $z_0 = \infty$.

Before passing to applications, we remark that equation (31) is the solution of the first-order non-homogeneous differential equation:

$$\frac{\partial S_n(a,z,t)}{\partial z} - \frac{1}{2t^2}[(1+t^2)\bar{a} - (1-t^2)z]S_n(a,z,t) = G_n(a,z). \tag{32}$$

In particular, putting $t = 1$, it reduces to

$$\frac{\partial S_n(a,z,1)}{\partial z} - \bar{a}S_n(a,z,1) = -p_n(\bar{a})p_n'(z), \tag{33}$$

which, with $p_n(z)$ given by equation (13), is satisfied by

$$S_n(a,z,1) = \sum_{k=0}^{n-1} \frac{(\bar{a}z)^k}{k!}. \tag{34}$$

3. Applications

Starting with the bulk region with $z_0 = 0$, equation (31) becomes

$$S_n(a,z,t) = \exp\left(\frac{(1+t^2)}{2t^2}\bar{a}z - \frac{(1-t^2)}{4t^2}z^2\right) \times \left[S_n(a,0,t) + \int_0^z \mathrm{d}v\, \exp\left(-\frac{(1+t^2)}{2t^2}\bar{a}v + \frac{(1-t^2)}{4t^2}v^2\right)G_n(a,v)\right], \tag{35}$$

where using equations (25) and (35) itself, the 'initial' condition $S_n(a,0,t)$ can be determined to be given by

$$S_n(a,0,t) = S_n(0,\bar{a},t) = \exp\left(-\frac{(1-t^2)}{4t^2}\bar{a}^2\right)\left[S_n(0,0,t) + \int_0^{\bar{a}} \mathrm{d}v\, \exp\left(\frac{(1-t^2)}{4t^2}v^2\right)G_n(0,v)\right]. \tag{36}$$

This leads to the expression

$$S_n(a,z,t) = \exp\left(\frac{(1+t^2)}{2t^2}\bar{a}z - \frac{(1-t^2)}{4t^2}z^2 - \frac{(1-t^2)}{4t^2}\bar{a}^2\right)\Bigg\{S_n(0,0,t) + \int_0^{\bar{a}} \mathrm{d}v\, \exp\left(\frac{(1-t^2)}{4t^2}v^2\right)G_n(0,v) + \exp\left(\frac{(1-t^2)}{4t^2}\bar{a}^2\right)\int_0^z \mathrm{d}v\, \exp\left(-\frac{(1+t^2)}{2t^2}\bar{a}v + \frac{(1-t^2)}{4t^2}v^2\right)G_n(a,v)\Bigg\}, \tag{37}$$

where

$$S_n(0,0,t) = \sum_{k=0}^{n-1} p_k^2(0) = \sum_{l=0}^{[(n-1)/2]} \frac{(2l)!}{2^{2l}(l!)^2}\left(\frac{1-t^2}{1+t^2}\right)^{2l}, \tag{38}$$

which for large values of n can be approximated as

$$S_n(0,0,t) \sim \sum_{l=0}^{\infty} \frac{(2l)!}{2^{2l}(l!)^2}\left(\frac{1-t^2}{1+t^2}\right)^{2l} = \left[1-\left(\frac{1-t^2}{1+t^2}\right)^2\right]^{-\frac{1}{2}} = \frac{1+t^2}{2t}. \tag{39}$$

3.1. The density and the two-point correlation function

The asymptotic of the density can be derived using equation (37) together with equation (10). Since

$$x^2 + y^2/t^2 = \frac{(1+t^2)}{2t^2}|z|^2 - \frac{(1-t^2)}{4t^2}(\bar{z}^2 + z^2) \tag{40}$$

we obtain

$$K_n(z,z,t) = S_n(0,0,t) + \int_0^{\bar{z}} \mathrm{d}v \exp\left[\frac{(1-t^2)}{4t^2}v^2\right] G_n(0,v) + \exp\left[\frac{(1-t^2)}{4t^2}\bar{z}^2\right]\int_0^{z} \mathrm{d}v \exp\left[-\frac{(1+t^2)}{2t^2}\bar{z}v + \frac{(1-t^2)}{4t^2}v^2\right] G_n(z,v). \tag{41}$$

To proceed, we write

$$G_n(a,z) = -\frac{1+t^2}{4t^2}\left[(1+t^2)p_n(\bar{a})p_n'(z) - (1-t^2)\sqrt{\frac{n}{n+1}}p_{n-1}(\bar{a})p_{n+1}'(z)\right] \tag{42}$$

using equation (27). Performing an integration by parts produces an integral which can also be integrated by parts using again equation (27). Repeating this procedure, a series in inverse powers of n is generated in which polynomials appear in increasing order. The first dominant term of this series is

$$K_n(z,z,t) = S_n(0,0,t) - \left(\frac{1+t^2}{4t^2}\right)\left[(1+t^2)|f_n(z)|^2 - (1-t^2)\sqrt{\frac{n}{n+1}}f_{n-1}(\bar{z})f_{n+1}(z)\right], \tag{43}$$

where the functions $f_n(z)$ are localized in the complex plane. Their modulus reaches a maximum, which defines a curve in the complex plane. To deduce the defining equation of this curve, we resort to the asymptotic form [14]

$$\exp\left(-\frac{x^2}{2}\right)H_n(x) = \frac{2^{\frac{n}{2}-\frac{3}{4}}}{(n\pi)^{\frac{1}{4}}}\sqrt{\frac{n!}{\sinh\theta}}\exp\left[\left(\frac{n}{2}+\frac{1}{4}\right)(2\theta - 2\sinh 2\theta)\right] \tag{44}$$

of the Hermite polynomials, where

$$x = \sqrt{2n}\cosh\theta. \tag{45}$$

For the complex argument of the f_n, writing $\theta = u + \mathrm{i}v$ we have

$$z = \sqrt{2n(1-t^2)}(\cosh u\cos v + \mathrm{i}\sinh u\sin v) \tag{46}$$

which corresponds to a transformation from Cartesian to elliptic coordinates. The modulus $|f_n(z)|$ contains a rapidly varying exponential term and a slowly varying coefficient. As a consequence, the points where the argument of the exponential is maximum correspond to the locus where $|f_n(z)|$ reaches, in the complex plane, its maximum values. In terms of (u, v), this argument is

$$2n\left(t^2\cosh^2 u\cos^2 v - \frac{\sinh^2 u\sin^2 v}{t^2} + u - \sinh u\cosh u\cos 2v\right), \tag{47}$$

which for a fixed value of v has, as a function of u, a parabolic shape. At the maxima of this family of parabolas the derivatives with respect to u and v of the above function vanish. This condition leads to the pair

$$\sinh 2u\left(t^2\cos^2 v - \frac{\sin^2 v}{t^2}\right) + 1 - \cosh 2u\cos 2v = 0 \tag{48}$$

and

$$-\sin 2v\left(t^2\cosh^2 u + \frac{\sinh^2 u}{t^2}\right) + 2\sinh u\cosh u\sin 2v = 0 \tag{49}$$

of coupled equations. The second equation has the solution

$$\tanh u = t^2 \tag{50}$$

which substituted in the first equation makes it vanish for any value of v. As a consequence, the curve is an ellipse of axes

$$a = \sqrt{\frac{2n}{1+t^2}} \tag{51}$$

and

$$b = t^2\sqrt{\frac{2n}{1+t^2}} \tag{52}$$

and area

$$\pi ab = \frac{2nt^2}{1+t^2}, \tag{53}$$

which satisfies the condition $S_n(0, 0, t)\pi ab/(\pi t) = n$.

Turning to correlations, asymptotically we can assume that, at the bulk of the distribution, the polynomial kernel can be approximated by the first term in equation (37) such that the expression

$$K(a, z, t) = \frac{1+t^2}{2t}\exp\left[\frac{1-t^2}{8t^2}(\bar{a}^2 - z^2) - \frac{1+t^2}{4t^2}(|a|^2 + |z|^2)\right] \times \exp\left[\frac{1-t^2}{8t^2}(a^2 - \bar{z}^2) + \frac{1+t^2}{2t^2}\bar{a}z\right] \tag{54}$$

for the kernel is obtained. As a matter of fact, this expression can also directly be derived by solving the differential equation (32) neglecting its non-homogeneity. By taking the modulus of equation (54) the two-point correlation function

$$|K(a, z, t)|^2 = \left(\frac{1+t^2}{2t}\right)\exp\left(-\frac{1+t^2}{2t^2}|a-z|^2\right) \tag{55}$$

is obtained, which is just a rescaling of the Ginibre result. As a consequence, at the bulk, other statistical measures too are expected to be rescaling those of the Ginibre limit ($t = 1$) [8].

3.2. The gap probability

Turning to eigenvalue spacings, we want to calculate the probability $E(0, s)$ that a disc of radius s be empty. In the limit when $n \to \infty$, this is given by the Fredholm determinant [12]

$$E(0, s) = \lim_{\lambda \to 1} \det(1 - \lambda K), \tag{56}$$

which has the expansion

$$\log[\det(1 - \lambda K)] = -\sum_{n=1}^{\infty} \frac{\lambda^n}{n} \int_{C_s} d^2 z_1 d^2 z_2 \cdots d^2 z_n K(z_1, z_2) K(z_2, z_3) \cdots K(z_n, z_1). \tag{57}$$

Substituting the kernels $K(z_i, z_j)$ by its expression given by equation (54), their product in equation (57) assumes the simpler expression

$$K(z_1, z_2) \cdots K(z_n, z_1) = \left(\frac{1+t^2}{2t}\right)^n \exp\left[-\frac{1+t^2}{2t^2} \sum_{i=1}^{n} |z_i|^2\right]$$
$$\times \exp\left[\frac{1+t^2}{2t^2} (\bar{z}_1 z_2 + \bar{z}_2 z_3 + \cdots + \bar{z}_n z_1)\right]. \tag{58}$$

Replacing equation (58) into equation (57), the integrations are better performed using the complex form $z_i = r_i \xi_i$ with $\xi_i = e^{i\phi_i}$ and writing the elements of area as $d^2 z_i = r_i dr_i d\xi_i / i\pi \xi_i$. The integrations in the radial variables are from 0 to s and in the unit circle for the angle variables. Expanding the n exponentials $\exp(\bar{z}_i z_j)$), the integrations in the angle variables become

$$\sum_{l_1=0} \cdots \sum_{l_n=0} \oint \frac{d\xi_1}{i\pi \xi_1} \cdots \oint \frac{d\xi_n}{i\pi \xi_n} (r_1 r_2 \bar{\xi}_1 \xi_2)^{l_1} (r_2 r_3 \bar{\xi}_2 \xi_3)^{l_2} \cdots (r_1 r_2 \bar{\xi}_n \xi_1)^{l_n}, \tag{59}$$

which can also be written as

$$(r_1)^{l_n + l_1} \cdots (r_n)^{l_{n-1} + l_n} \sum_{l_1=0} \cdots \sum_{l_n=0} \oint \frac{d\xi_1}{i\pi \xi_1} \cdots \oint \frac{d\xi_n}{i\pi \xi_n} (\bar{\xi}_1 \xi_2)^{l_1} \cdots (\bar{\xi}_n \xi_1)^{l_n}. \tag{60}$$

It is easy to convince ourselves that the above integrations vanish unless all the indices l_i are equal, that is, we must have $l_1 = l_2 = \cdots = l_n = l$. In this case, each integration gives a factor of 2. The integrations in the radial variables of the nth term in equation (57) are all equal to the same incomplete gamma function such that the expression of the Fredholm determinant becomes

$$\log[\det(1 - K)] = -\sum_{n=1}^{\infty} \frac{1}{n} \sum_{l=0}^{\infty} \left(\frac{\lambda}{l!} \int_0^{\frac{1+t^2}{2t^2} s^2} dv\, e^{-v} v^l\right)^n. \tag{61}$$

Performing the summation in the index n we arrive, with $\lambda = 1$, at the expression

$$E(0, s) = \prod_{l=0}^{\infty} \left(1 - \frac{1}{l!} \int_0^{\frac{1+t^2}{2t^2} s^2} dv\, e^{-v} v^l\right) \tag{62}$$

of the gap function as an infinite product. As predicted, this is just a rescaling of the Ginibre result $(t = 1)$ [13]. In figure 1, the exactness of this formula is shown.

3.3. The complex Airy kernel

Let us now consider the problem of investigating statistical properties at the asymptotic region far from the origin. At this region the polynomial kernel $S_n(a, z)$ is given by

$$S_n(a, z, t) = -\exp\left(\frac{(1+t^2)}{2t^2} \bar{a} z - \frac{(1-t^2)}{4t^2} z^2\right)$$
$$\times \int_z^{\infty} dv \exp\left(-\frac{(1+t^2)}{2t^2} \bar{a} v + \frac{(1-t^2)}{4t^2} v^2\right) G_n(a, v) \tag{63}$$

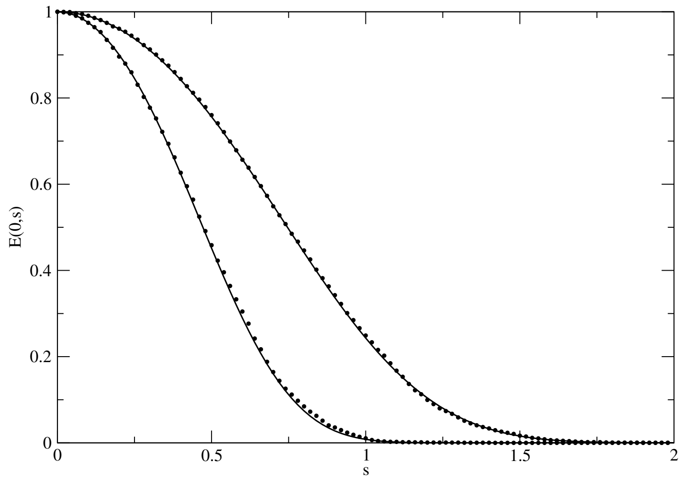

Figure 1. The expression for the gap probability $E(0, s)$, equation (62), for $t = 1$ and $t = 0.5$ is compared with numerical simulation.

obtained by performing the integration in equation (30) from z to infinity. As the argument of the exponential in the integrand is supposed to be large, an asymptotic series whose first dominant term is

$$S_n(a, z, t) = \frac{1+t^2}{2} \frac{p_n(\bar{a})p_n'(z) - p_n'(\bar{a})p_n(z) + t^2[p_n(\bar{a})p_n'(z) + p_n'(\bar{a})p_n(z)]}{\bar{a} - z + t^2(\bar{a}+z)} \tag{64}$$

can be generated integrating by parts. Expressing the polynomials in terms of the rescaled Hermite ones, equation (64) can be written as

$$S_n(a, z, t) = \frac{1+t^2}{2^{n+1}n!}\left(\frac{1-t^2}{1+t^2}\right)^n \frac{A_- + t^2A_+}{\bar{a} - z + t^2(\bar{a}+z)}, \tag{65}$$

where

$$A_\pm = H_n\left(\frac{\bar{a}}{\sqrt{1-t^2}}\right)H_n'\left(\frac{z}{\sqrt{1-t^2}}\right) \pm H_n'\left(\frac{\bar{a}}{\sqrt{1-t^2}}\right)H_n\left(\frac{z}{\sqrt{1-t^2}}\right). \tag{66}$$

Immediately, by taking the limit $t \to 0$, the Christoffel–Darboux formula at the real axis is recovered.

At the edge, the Hermite polynomials can be replaced by their asymptotic approximation [14]

$$\exp\left(-\frac{z^2}{2}\right)H_n(z) = \pi^{-3/4}2^{(2n+1)/4}\sqrt{n!}\mathrm{Ai}(\xi), \tag{67}$$

where $\mathrm{Ai}(\xi)$ is the Airy function and

$$z = \sqrt{2n+1} - \frac{\xi}{\sqrt{2}n^{1/6}}. \tag{68}$$

At the same order of approximation, the polynomial derivative can be replaced by

$$\exp\left(-\frac{z^2}{2}\right)H_n'(z) = \sqrt{2n}\pi^{-3/4}2^{(2n+1)/4}\sqrt{n!}[\mathrm{Ai}(\xi) - n^{-1/3}\mathrm{Ai}'(\xi)]. \tag{69}$$

Using these approximations in equation (65), the complete kernel, that is equation (7), omitting exponential terms becomes

$$K_n(\xi_1, \xi_2, t) \sim \frac{\mathrm{Ai}(\bar{\xi}_1)\mathrm{Ai}'(\xi_2) - \mathrm{Ai}'(\bar{\xi}_1)\mathrm{Ai}(\xi_2) + 2t^2 n^{1/3}\mathrm{Ai}(\bar{\xi}_1)\mathrm{Ai}(\xi_2)}{\bar{\xi}_2 - \xi_1 - t^2(4n^{2/3} + \bar{\xi}_1 + \xi_2)}. \quad (70)$$

For $t \neq 0$, this expression extends the Airy kernel [15] to the complex plane. Taking now the usual asymptotic limit $n \to \infty$ keeping t fixed, the kernel factorizes implying an uncorrelated regime. Assuming that the parameter t scales with n as $t \sim n^{-\alpha}$ with α positive such that it approaches zero as n gets larger, then $\alpha = 1/3$ is a critical value above which the behavior at the edge is governed by the standard Airy kernel, while below it an uncorrelated regime prevails.

4. Conclusion

We have obtained for the rescaled Hermite polynomials, which are the main ingredient in the formalism describing the transition between the Hermitian (Wigner) and the totally non-Hermitian (Ginibre) Gaussian unitary ensembles, a Christoffel–Darboux-like formula. It expresses the sum of polynomials up to an order in terms of an expression that contains only the higher order ones. Combining this formula with asymptotics of the Hermite polynomials in the complex plane, we have been able to derive the level density, the two-point correlation function and the gap probability at the bulk of the 2D spectrum. On the other hand, asymptotics at the edge provides an extension to the complex plane of the Airy kernel. A Rodrigues formula and a generating function for the rescaled Hermite polynomials also have been established.

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