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Non-Abelian Chern-Simons Particles in an External Magnetic Field

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Abstract

The quantum mechanics and thermodynamics of SU(2) non-Abelian Chern-Simons particles (non-Abelian anyons) in an external magnetic field are addressed. We derive the N-body Hamiltonian in the (anti-)holomorphic gauge when the Hilbert space is projected onto the lowest Landau level of the magnetic field. In the presence of an additional harmonic potential, the N-body spectrum depends linearly on the coupling (statistics) parameter. We calculate the second virial coefficient and find that in the strong magnetic field limit it develops a step-wise behavior as a function of the statistics parameter, in contrast to the linear dependence in the case of Abelian anyons. For small enough values of the statistics parameter we relate the N-body partition functions in the lowest Landau level to those of SU(2) bosons and find that the cluster (and virial) coefficients dependence on the statistics parameter cancels.

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I. INTRODUCTION

Identical particles with statistics continuously interpolating between Bose-Einstein and Fermi-Dirac statistics exist in two and one dimensions. Explicit implementations of these ideas can be found in the two-dimensional anyon model [1] and in the one-dimensional Calogero-Sutherland models [2] where a different approach to statistics proposed by Haldane based on a generalized Pauli exclusion principle [3] is realized.

Contrary to the N-body Calogero model, which is solvable, the N-anyon spectrum is still unknown. However, a simplification arises when considering the anyon model in the background of an external magnetic field. By projecting this model onto the lowest Landau level, a procedure which is justified in the strong field-low temperature limit, a complete eigenstate basis, which continuously interpolates between the bosonic and the fermionic basis, can be found in the screening regime where the flux ϕ carried by the anyons is antiparallel to the external magnetic field B. More precisely, when the statistics parameter $\alpha = \phi/\phi_o$, which varies from $\alpha = 0$ to $\alpha = \pm 1$, is such that $\alpha \in [-1, 0]$ if $eB > 0$, or equivalently $\alpha \in [0, 1]$ if $eB < 0$.

In this situation, the statistical mechanical properties of the anyon gas can be studied in a complete and explicit way [4] and they turn out to be quite similar to those of the Calogero model. In the thermodynamic limit both models are microscopical realizations of Haldane statistics [3,5,6]. Recently, various conformal field theories have also been shown to implement exclusion statistics [7].

It is well known that anyons can be thought of as bosons or fermions coupled to Abelian Chern-Simons gauge fields. An interesting generalization occurs when the gauge fields take values in a non-Abelian group and the particles carry internal degrees of freedom associated with a representation of this group. These models describing non-Abelian Chern-Simons particles have already been considered in several contexts. Verlinde [8] argued that they provide an explicit realization of non-Abelian *braiding* statistics, i.e. statistics corresponding to non-Abelian irreducible representations of the braid group [9]. Field theoretical implementations of such models were recently proposed [10] as Ginzburg-Landau Chern-Simons theories for Pfaffian (non Abelian) quantum Hall states. They generalize Abelian Chern-Simons field theories [11] for Laughlin (Abelian) quantum Hall states.

Whether models of non-Abelian Chern-Simons particles restricted to the lowest Landau level can be solved exactly (a natural possibility since they become effectively one-dimensional), and whether their thermodynamics yield a realization of exclusion statistics different from that corresponding to Abelian anyons, are questions of interest. In this paper, we address these questions by studying the simplest case of a non-Abelian symmetry, namely SU(2) Chern-Simons particles in the fundamental representation. The paper is organized as follows.

In Section II the Verlinde model [8], a generalization of the Aharonov-Bohm Hamiltonian in the (anti-)holomorphic gauge to the non-Abelian case, is introduced. We include contact $\delta^2(z_i - z_j)$ interactions to enforce that the wave functions are regular when particles approach each other, and, in addition, a confining harmonic potential to lift the lowest Landau level degeneracy. We redefine the wave function essentially as $\psi = U\psi'$, where U (the non-Abelian generalization of $\prod_{i<j}(z_i - z_j)^\alpha$ for Abelian anyons with statistics parameter α) is a solution of the Knizhnik-Zamolodchikov equation. Our main result is the Hamiltonian acting on ψ', which takes particularly simple form when ψ' is (anti-)holomorphic, that is when the Hilbert space is restricted to the lowest Landau level of the external magnetic field. In the latter case we find a linear dependence of the N-body energy spectrum on the statistics parameter, generalizing the known results for Abelian anyons.

In Section III we revisit both the Abelian and non-Abelian anyon models to present another derivation of the same Hamiltonian acting on ψ' as in Section II, but now starting from free Pauli Hamiltonians. In this approach, contact interactions do not need to be introduced, instead the spin coupling plays a crucial role. We also show that in the redefinition of the wave function the external magnetic field and the magnetic fluxes carried by particles can be treated on the same footing. The latter point of view is more adapted for models that use mean field approximations [12].

In Section IV we discuss the 2-body problem for an arbitrary strength of the external magnetic field. The 2-body problem decomposes into two Abelian anyon problems, which allows one to calculate the second virial coefficient exactly. In the vanishing magnetic field limit our result reduces to that found previously by Hagen [13] in his comment on the paper by Lee [14]. In the strong magnetic field limit we find that the interpolation from Bose to Fermi statistics is different from the Abelian one. In the latter case [4], which yields Haldane's generalization of the Pauli exclusion principle, the second virial coefficient depends linearly on the statistics parameter. Contrary to that, we find that in the non-Abelian case the second virial coefficient develops a step-wise behavior as a function of the statistics parameter.

In Section V we address the N-body problem in the lowest Landau level. We show how to calculate the partition function taking into account properly degeneracies associated with internal isospin degrees of freedom. The N-body energy spectrum for a given isospin is the N-body SU(2) bosonic spectrum plus a a term linear in the statistics parameter. We relate the N-body bosonic partition functions $Z_{N,I}$ with total isospin I to the partition functions associated with Young diagrams. The latter partition functions, which turn out to be generalizations of the Schur functions, are introduced in subsection V B. We propose systematic rules to calculate them. Collecting all these results, we can calculate the cluster and virial coefficients one by one. For small enough values of the statistics parameter we find, somewhat surprisingly, a cancellation of the dependence on the statistics parameter, and thus the same cluster and virial coefficients as for SU(2) bosons. The N-body lowest Landau level thermodynamics of non-Abelian Chern-Simons particles in the entire interval of definition of the statistics parameter is yet an open question.

We conclude in Section VI, and comment also on generalizations to other symmetry groups and relations to one-dimensional integrable models with inverse square interactions. In Appendix A some basic facts used in the paper on the irreducible representations of the symmetric group are collected.

II. THE ABELIAN ANYON MODEL AND NON-ABELIAN VERLINDE MODEL

A. Abelian case

Let us begin with reviewing the formalism used in [4] to calculate the equation of state of Abelian anyons in the lowest Landau level of an external magnetic field. The dynamics of N anyons in the plane is described by a Aharonov-Bohm Hamiltonian in the background of a constant magnetic field B. Here, as in the sequel, the mass m will be set to 1 and complex coordinates notation $z_i = x_i + iy_i$, $\bar{z}_i = x_i - iy_i$ will be used. For our purposes, it will be convenient to work in the *holomorphic* (+) or *anti-holomorphic* (−) gauge where the Aharonov-Bohm Hamiltonian takes the form

$$H_0^{\pm} = -\sum_i (\nabla_i^{\pm}\overline{\nabla}_i^{\pm} + \overline{\nabla}_i^{\pm}\nabla_i^{\pm}) \tag{1}$$

The covariant derivatives are defined as,

$$\nabla_i^{\pm} = \partial_i - iK_{z_i}^{\pm} \quad \overline{\nabla}_i^{\pm} = \bar{\partial}_i - iK_{\overline{z}_i}^{\pm} \tag{2}$$

with

$$K_{z_i}^{+} = -i\sum_{j\neq i}\frac{\alpha}{z_i - z_j} - i\sum_i b\overline{z}_i \quad K_{\overline{z}_i}^{+} = 0 \tag{3}$$

$$K_{\overline{z}_i}^{-} = i\sum_{j\neq i}\frac{\alpha}{\overline{z}_i - \overline{z}_j} + i\sum_i bz_i \quad K_{z_i}^{-} = 0 \quad , \tag{4}$$

The charge e of each anyon is coupled to the Aharonov-Bohm flux ϕ carried by the other anyons and to the external magnetic field B: therefore the couplings $\alpha = e\phi/2\pi$ and $b = eB/2$.

We are interested in a *boson*-based description of anyons, that is, the Hamiltonian (1) is acting on bosonic wave functions $\Psi(z_1, .., z_N; \overline{z}_1, .., \overline{z}_N)$. Then, the statistics parameter $\alpha \in [-1, 1]$ is such that $\alpha = 0$ corresponds to bosons and $\alpha = \pm 1$ to fermions.

It is also possible, via a singular gauge transformation, to formulate the problem in terms of a "free" Hamiltonian, that is without anyonic Aharonov-Bohm fields, at the expense of introducing multivalued wave functions

$$U_+ H_0^+ U_+^{-1} = -\sum_i (\partial_i \bar{\partial}_i + \bar{\partial}_i \partial_i - 2b\overline{z}_i \bar{\partial}_i - b) \tag{5}$$

$$U_- H_0^- U_-^{-1} = -\sum_i (\partial_i \bar{\partial}_i + \bar{\partial}_i \partial_i + 2b z_i \partial_i + b) \tag{6}$$

with $U_\pm$ satisfying the equations

$$\partial_i U_+ = -\sum_{j \neq i} \frac{\alpha}{z_i - z_j} U_+ \qquad \bar{\partial}_i U_+ = 0 \tag{7}$$

$$\bar{\partial}_i U_- = \sum_{j \neq i} \frac{\alpha}{\overline{z}_i - \overline{z}_j} U_- \qquad \partial_i U_- = 0 \tag{8}$$

which can be considered as the Abelian version of Knizhnik-Zamolodchikov equations [15], with solutions

$$U_+ = \prod_{i<j} (z_i - z_j)^{-\alpha} \qquad U_- = \prod_{i<j} (\overline{z}_i - \overline{z}_j)^{\alpha} \tag{9}$$

Note that due to the multivaluedness of U, $\partial_i \bar{\partial}_i U \neq \bar{\partial}_i \partial_i U$.

$H_0^\pm$ have singular terms which arise from

$$\partial_i (\frac{1}{\overline{z}_i}) = \bar{\partial}_i (\frac{1}{z_i}) = \pi \delta^2(z_i) \tag{10}$$

However, a potential accounting for repulsive contact interactions (which are introduced in order to implement the exclusion of the diagonal of the configuration space) and a harmonic well (which is introduced as a regulator to calculate thermodynamical properties) can be added to (1)

$$V = \sum_{i, j \neq i} \lambda \delta^2(z_i - z_j) + \sum_i \frac{\omega^2}{2} \bar{z}_i z_i \quad . \tag{11}$$

If we choose $\lambda = \pi|\alpha|$, the contact term in V cancels exactly the singular terms in H_0^+ (H_0^-) when $\alpha < 0$ ($\alpha > 0$). The total Hamiltonian $H_{\rm AB} = H_0 + V$ becomes

$$H_{\rm AB}^+ = -\sum_i \left(\partial_i\bar{\partial}_i + \bar{\partial}_i\partial_i - 2b\overline{z}_i\bar{\partial}_i - \frac{\omega^2}{2}z_i\overline{z}_i - b\right) + 2\alpha\sum_{i<j}\frac{\bar{\partial}_i - \bar{\partial}_j}{z_i - z_j} \tag{12}$$

$$H_{\rm AB}^- = -\sum_i \left(\partial_i\bar{\partial}_i + \bar{\partial}_i\partial_i + 2bz_i\partial_i - \frac{\omega^2}{2}z_i\overline{z}_i + b\right) - 2\alpha\sum_{i<j}\frac{\partial_i - \partial_j}{\overline{z}_i - \overline{z}_j} \tag{13}$$

The role of contact interactions in the context of anyons has been extensively discussed in the literature. Their relevance has been originally stressed in the study of soliton solutions in Chern-Simons matter systems [16] and anyonic wave functions in the background of an external magnetic field [17] (see also [18]). The issue was subsequently re-considered by several authors, in the perturbative treatment of the Aharonov-Bohm problem [19], perturbative calculations of statistical and thermodynamical quantities [20], self adjoint extensions [21], etc. Notice that in the case under consideration the contact interactions are *repulsive* and that in addition, the orientation of the external magnetic field and the anyonic flux tubes are opposite, a case which do not support solitons [22]. On the other hand, this is precisely the case considered in [4] since it allows for a physical meaningful lowest Landau level reduction.

Then, if $\omega = 0$, the ground state of $H_{\rm AB}^+$, which corresponds to $b > 0$ and $\alpha < 0$, is given by analytic wave functions while the ground state of $H_{\rm AB}^-$, associated to $b < 0$ and $\alpha > 0$, is given by anti-analytic wave functions.

As we mentioned before, the harmonic attraction lifts the degeneracy of the ground state. In order to see this, it is convenient to re-define the wave functions as

$$\Psi^\pm(z_1,..,z_N;\overline{z}_1,..,\overline{z}_N) = \prod_i \exp(-\frac{\omega_t \mp b}{2}z_i\overline{z}_i)\psi^\pm(z_1,..,z_N;\overline{z}_1,..,\overline{z}_N) \tag{14}$$

where $\omega_t \equiv \sqrt{\omega^2 + b^2}$. The Hamiltonian acting on $\psi^\pm$ is then

$$\begin{aligned} H^+ &= -\sum_i \Big(\partial_i\bar{\partial}_i + \bar{\partial}_i\partial_i - (\omega_t + b)\overline{z}_i\bar{\partial}_i - (\omega_t - b)z_i\partial_i - \omega_t\Big) \\ &\quad +2\alpha\sum_{i<j}(\frac{\bar{\partial}_i - \bar{\partial}_j}{z_i - z_j} - \frac{\omega_t - b}{2}) \end{aligned} \tag{15}$$

$$\begin{aligned} H^- &= -\sum_i \Big(\partial_i\bar{\partial}_i + \bar{\partial}_i\partial_i - (\omega_t + b)\overline{z}_i\bar{\partial}_i - (\omega_t - b)z_i\partial_i - \omega_t\Big) \\ &\quad -2\alpha\sum_{i<j}(\frac{\partial_i - \partial_j}{\overline{z}_i - \overline{z}_j} - \frac{\omega_t + b}{2}) \end{aligned} \tag{16}$$

Acting on analytic and anti-analytic wave functions

$$\psi^+ = \prod_i z_i^{\ell_i} \qquad \psi^- = \prod_i \overline{z}_i^{\ell_i} \tag{17}$$

the Hamiltonians $H^\pm$ have the spectrum

$$E_N = N\omega_t + (\sum_i \ell_i - \frac{\alpha}{2}N(N-1))(\omega_t - b) \tag{18}$$

$$E_N = N\omega_t + (\sum_i \ell_i + \frac{\alpha}{2}N(N-1))(\omega_t + b) \tag{19}$$

Notice that the exclusion of the diagonal of the configuration space, in view of (9) and our discussion on the sign of α, is realized for the wave functions in the singular (s) gauge

$$\psi^{s+} = \prod_{i<j}(z_i - z_j)^{-\alpha} \prod_i z_i^{\ell_i} \exp(-\frac{\omega_t - b}{2} z_i \overline{z}_i) \tag{20}$$

$$\psi^{s-} = \prod_{i<j}(\overline{z}_i - \overline{z}_j)^{\alpha} \prod_i \overline{z}_i^{\ell_i} \exp(-\frac{\omega_t + b}{2} z_i \overline{z}_i) \tag{21}$$

which do vanish at coinciding points.

B. Non-Abelian case: the Verlinde model

Let us generalize the above construction to SU(2) non-Abelian Chern-Simons particles in the lowest Landau level of an external Abelian magnetic field. We are considering the case where N identical *bosonic* particles are in the isospin 1/2 *fundamental* representation of $SU(2)$. The wave functions of the system $\Psi(z_1, .., z_N, \bar{z}_1, \ldots, \bar{z}_N)$ belongs to the tensor product $\Gamma_1 \times ... \times \Gamma_N$ where Γ_i is the two-dimensional space in which the SU(2) generators,

$$T_i^a = \frac{\sigma_i^a}{2} \quad a = 1, 2, 3 \quad i = 1, .., N \tag{22}$$

act. The non-Abelian generalization of the Aharonov-Bohm Hamiltonian in the holomorphic and anti-holomorphic gauges is given by,

$$H_0^\pm = -\sum_i (\nabla_i^\pm \overline{\nabla}_i^\pm + \overline{\nabla}_i^\pm \nabla_i^\pm) \tag{23}$$

where we the covariant derivatives are defined as before but with non Abelian gauge fields,

$$K_{z_i}^+ = ig \sum_{j \neq i} \frac{T_i^a T_j^a}{z_i - z_j} - i \sum_i b\overline{z}_i \quad K_{\overline{z}_i}^+ = 0 \tag{24}$$

$$K^-_{\overline{z}_i} = -ig \sum_{j\neq i} \frac{T^a_i T^a_j}{\overline{z}_i - \overline{z}_j} + i \sum_i b z_i \quad K^-_{z_i} = 0 \quad , \tag{25}$$

Here g is the Chern-Simons coupling constant ($g = 1/2\pi\kappa$ in [14]) and satisfies the topological constraint $g = 2/n$ with n an integer.

Naturally, other choices such as Coulomb or axial gauges are possible. The connection among these different gauge choices is obtained via $W^{-1}H_0W$ where W is not necessarily unitary. Unlike the Abelian case where W is known, for the non-Abelian case an explicit expression is available only in the 2-body case.

The Hamiltonian acts on totally symmetric wave functions with bosonic interchange conditions, meaning that the wave function is symmetric under the interchange of *both* coordinates and isospin indices,

$$\Psi_{i_1,..,i_m,..,i_l,..,i_N}(z_1,..,z_m,..,z_l,..,z_N) = \Psi_{i_1,..,i_l,..,i_m,..,i_N}(z_1,..,z_l,..,z_m,..,z_N) \tag{26}$$

As in the Abelian case, there exists a Hamiltonian without anyonic gauge fields but acting on wave functions with *non trivial* boundary conditions. This Hamiltonian, $U_\pm H_0^\pm U_\pm^{-1}$, takes the same form than in the Abelian case where $U_\pm$ satisfies the non-Abelian Knizhnik-Zamolodchikov equations

$$\partial_i U_+ = U_+ g \sum_{j\neq i} \frac{T^a_i T^a_j}{z_i - z_j} \qquad \bar{\partial}_i U_+ = 0 \tag{27}$$

$$\bar{\partial}_i U_- = -U_- g \sum_{j\neq i} \frac{T^a_i T^a_j}{\overline{z}_i - \overline{z}_j} \qquad \partial_i U_- = 0 \tag{28}$$

As in the Abelian case, the Hamiltonians $H_0^\pm$ have singular terms that can be eliminated by adding an appropriate potential V. Thus, as a non-Abelian generalization, we are led to consider $H^\pm_{\rm AB} = H_0^\pm + V$ with

$$V = |g|\pi \sum_{i,j\neq i} T^a_i T^a_j \delta^2(z_i - z_j) + \sum_i \frac{\omega^2}{2} \bar{z}_i z_i \tag{29}$$

It can be shown that this potential corresponds to a repulsive interaction in the bosonic sector. We then obtain

$$H^+_{\rm AB} = -\sum_i \left(\partial_i\bar{\partial}_i + \bar{\partial}_i\partial_i - 2b\overline{z}_i\bar{\partial}_i - \frac{\omega^2}{2} z_i\overline{z}_i - b \right) - 2g \sum_{i<j} T^a_i T^a_j \frac{\bar{\partial}_i - \bar{\partial}_j}{z_i - z_j} \tag{30}$$

$$H^-_{\rm AB} = -\sum_i \left(\partial_i\bar{\partial}_i + \bar{\partial}_i\partial_i + 2b z_i\partial_i - \frac{\omega^2}{2} z_i\overline{z}_i + b \right) + 2g \sum_{i<j} T^a_i T^a_j \frac{\partial_i - \partial_j}{\overline{z}_i - \overline{z}_j} \tag{31}$$

and factorizing the Gaussian factor in the wavefunctions as in (14)

$$
\begin{aligned}
H^+ = & -\sum_i \left(\partial_i\bar{\partial}_i + \bar{\partial}_i\partial_i - (\omega_t+b)\overline{z}_i\bar{\partial}_i - (\omega_t-b)z_i\partial_i - \omega_t\right) \\
& -2g\sum_{i<j} T_i^a T_j^a \left(\frac{\bar{\partial}_i - \bar{\partial}_j}{z_i - z_j} - \frac{\omega_t - b}{2}\right)
\end{aligned} \tag{32}
$$

$$
\begin{aligned}
H^- = & -\sum_i \left(\partial_i\bar{\partial}_i + \bar{\partial}_i\partial_i - (\omega_t+b)\overline{z}_i\bar{\partial}_i - (\omega_t-b)z_i\partial_i - \omega_t\right) \\
& +2g\sum_{i<j} T_i^a T_j^a \frac{\partial_i - \partial_j}{\overline{z}_i - \overline{z}_j} - \frac{\omega_t + b}{2})
\end{aligned} \tag{33}
$$

Acting on analytic (anti-analytic) wave functions, H^+ (H^-) takes the form,

$$
H^+ = N\omega_t + \left(\sum_i \ell_i + \hat{\Omega}_{I,N}\right)(\omega_t - b) \tag{34}
$$

$$
H^- = N\omega_t + \left(\sum_i \ell_i - \hat{\Omega}_{I,N}\right)(\omega_t + b) \tag{35}
$$

where

$$
\hat{\Omega}_{I,N} = g\sum_{i<j} T_i^a T_j^a \tag{36}
$$

It can be easily shown that the operator $\hat{\Omega}$ has eigenvalues,

$$
\Omega_{N,I} = \frac{g}{2}\left(I(I+1) - \frac{3}{4}N\right) \tag{37}
$$

where N is as before the total number of particles and I is the total isospin.

For analytic (anti-analytic) wave functions, the spectrum reads

$$
E_{N,I} = N\omega_t + \left(\sum_{j=1}^{N} \ell_j + \Omega_{N,I}\right)(\omega_t - b) \tag{38}
$$

$$
E_{N,I} = N\omega_t + \left(\sum_{j=1}^{N} \ell_j - \Omega_{N,I}\right)(\omega_t + b) \tag{39}
$$

III. THE ABELIAN AND NON-ABELIAN ANYON MODELS REVISITED

In the standard presentation given above of the Abelian and non Abelian Verlinde anyon models, short range $\delta^2(z_i - z_j)$ interactions have been added by hand to the non hermitian Hamiltonians expressed in the holomorphic or anti-holomorphic gauges. We now present another approach to derive the same Hamiltonians (32,33), which starts from free Pauli Hamiltonians, therefore implying spin coupling to the local magnetic field carried by the anyons. In addition, the eigenstates redefinitions with respect to the vortex (short distance) and the magnetic field (long distance) will be treated on an equal footing.

In the singular gauge, let us start with the free N-body Pauli Hamiltonians

$$H^+_{\rm free} = -2\sum_{i=1}^{N} \partial_i \bar{\partial}_i \tag{40}$$

$$H^-_{\rm free} = -2\sum_{i=1}^{N} \bar{\partial}_i \partial_i \tag{41}$$

where the index $\pm$ refers here to the spin degree of freedom.

If one considers the additional coupling to an external magnetic field, then, in the symmetric gauge, $\partial \to \partial - b\bar{z}/2$ and $\bar{\partial} \to \bar{\partial} + bz/2$. Of course one could choose other gauges for the B field, as the holomorphic or anti-holomorphic gauges, in which case in (40) $\partial \to \partial - b\bar{z}, \bar{\partial} \to \bar{\partial}$, and in (41), $\bar{\partial} \to \bar{\partial} + bz, \partial \to \partial$, which yield basically the Hamiltonians (5,6) discussed above. However, we insist at this point on using the symmetric gauge, since it is the natural gauge to work with in the presence of the singular Aharonov-Bohm flux tubes.

Indeed, the anyon model is defined via the non trivial monodromy of the N-body eigenstates of $H_{\rm free}$

$$\psi_{\rm free}(z_1, z_2, \ldots, z_N; \bar{z}_1, \bar{z}_2, \ldots, \bar{z}_N) = e^{-i\alpha \sum_{k<l} \theta_{kl}} \Psi(z_1, z_2, \ldots, z_N; \bar{z}_1, \bar{z}_2, \ldots, \bar{z}_N) \tag{42}$$

where $\sum_{k<l} \theta_{kl}$ is the sum of the relative angles between pairs of particles. As said before, $\psi(z_1, z_2, \ldots, z_N; \bar{z}_1, \bar{z}_2, \ldots, \bar{z}_N)$ is by convention bosonic in the regular gauge with the statistics parameter $\alpha = 0$ for Bose statistics, and $\alpha = \pm 1$ for Fermi statistics.

Looking at (42) as a singular gauge transformation, one would obtain, in the symmetric gauge, a N-anyon Aharonov-Bohm Hamiltonian in the background of the external magnetic field with $\mp\pi\alpha \sum_{i<j} \delta^2(z_i - z_j)$ interactions and $\mp \sum_i b$ shifts, induced by the spin up or spin down coupling to the local magnetic field of the vortices and the homogeneous background magnetic field. The parameter α represents as usual the Aharonov-Bohm flux carried by the anyons in units of the quantum of flux.

The short range (contact) interactions have to implement the exclusion of the diagonal of the configuration space (Pauli exclusion), and thus have to be repulsive. So, depending of the sign of α, the spin up Hamiltonian (40) ($\alpha \in [-1,0]$) or spin down Hamiltonian (41) ($\alpha \in [0,1]$) have to be used.

To materialize the short range repulsion in the eigenstates, one proceeds by redefining

$$\Psi(z_1, z_2, \ldots, z_N; \bar{z}_1, \bar{z}_2, \ldots, \bar{z}_N) = \prod_{i<j} |z_i - z_j|^{\mp\alpha} \tilde{\psi}(z_1, z_2, \ldots, z_N; \bar{z}_1, \bar{z}_2, \ldots, \bar{z}_N) \qquad (43)$$

This is nothing but saying that the eigenstates do vanish as quickly as $r_{ij}^{\mp\alpha}$ when particles i and j come close together (again the $\mp$ sign has been chosen accordingly to the sign of α). At this point, (42,43) together give

$$\psi_{\text{free}} = \prod_{k<l} (z_k - z_l)^{-\alpha} \tilde{\psi}_+ \qquad \alpha \in [-1,0] \qquad (44)$$

$$\psi_{\text{free}} = \prod_{k<l} (\bar{z}_k - \bar{z}_l)^{\alpha} \tilde{\psi}_- \qquad \alpha \in [0,1] \qquad (45)$$

where both $\prod_{k<l}(z_k - z_l)^{-\alpha}$ and $\prod_{k<l}(\bar{z}_k - \bar{z}_l)^{\alpha}$ are solutions of the holomorphic and anti-holomorphic Knizhnik-Zamolodchikov equations in the Abelian case. The non hermitian Hamiltonian acting on $\tilde{\psi}$ rewrites

$$\tilde{H}^+ = -2\sum_{i=1}^{N} \left[\partial_i \bar{\partial}_i - (\frac{b}{2})^2 z_i \bar{z}_i + \frac{b}{2}(z_i \partial_i - \bar{z}_i \bar{\partial}_i) \right] + 2\alpha \sum_{i<j} (\frac{\bar{\partial}_i - \bar{\partial}_j}{z_i - z_j} + \frac{b}{2}) - \sum_i b \qquad (46)$$

$$\tilde{H}^- = -2\sum_{i=1}^{N} \left[\bar{\partial}_i \partial_i - (\frac{b}{2})^2 z_i \bar{z}_i + \frac{b}{2}(z_i \partial_i - \bar{z}_i \bar{\partial}_i) \right] - 2\alpha \sum_{i<j} (\frac{\partial_i - \partial_j}{\bar{z}_i - \bar{z}_j} - \frac{b}{2}) + \sum_i b \qquad (47)$$

The external magnetic field did not yet plaid any role in the eigenstates redefinition, thus the Hamiltonians (46, 47) expressed in the holomorphic or anti-holomorphic gauges with respect to the vortices, but in the symmetric gauge with respect to the external magnetic field. Also, there is no such singular gauge as (42) for a homogeneous magnetic field. Let us however extract from the eigenstates the Landau exponential factor $\exp(\pm\frac{b}{2}\sum_i z_i \bar{z}_i)$. It should be considered on the same footing as $\prod_i |z_i - z_j|^{\pm\alpha}$ in (43), as one can easily realize by considering the 2-dimensional identity

$$\int dr' \ln|\vec{r} - \vec{r}'| = \pi r^2/2 \tag{48}$$

It means that a magnetic field can be regarded as the average of a distribution of vortices [12]. Let us redefine

$$\psi_{\text{free}} = \prod_{k<l}(z_k - z_l)^{-\alpha} \exp(-\frac{1}{2}b\sum_i z_i\bar{z}_i)\psi_+ \tag{49}$$

$$\psi_{\text{free}} = \prod_{k<l}(\bar{z}_k - \bar{z}_l)^{\alpha} \exp(\frac{1}{2}b\sum_i z_i\bar{z}_i)\psi_- \tag{50}$$

where the prefactors $\psi_{\text{free}} = U^{\pm}\psi_{\pm}$ are now solutions of the holomorphic and anti-holomorphic Knizhnik-Zamolodchikov equations in presence of the external B field

$$(\partial_i + \frac{b}{2}\bar{z}_i + \alpha\sum_{j\neq i}\frac{1}{z_i - z_j})U^+ = 0 \quad (\bar{\partial}_i + \frac{b}{2}z_i)U^+ = 0 \tag{51}$$

$$(\bar{\partial}_i - \frac{b}{2}z_i - \alpha\sum_{j\neq i}\frac{1}{\bar{z}_i - \bar{z}_j})U^- = 0 \quad (\partial_i - \frac{b}{2}\bar{z}_i)U^- = 0 \tag{52}$$

In order for (49,50) to be physically meaningful, i.e. short distance vanishing eigenstates due to the repulsive vortices, and long distance exponential damping due to the magnetic field, one concludes that in (49), $\alpha < 0, b > 0$, whereas in (50), $\alpha > 0, b < 0$. In both cases, the local vortices carried by the anyons are antiparallel to the external magnetic field, i.e. a screening regime. The resulting Hamiltonian in the holomorphic and anti-holomorphic gauges acting on $\psi_\pm$ narrows down to

$$H^+ = -2\sum_{i=1}^{N}\left[\partial_i\bar{\partial}_i - b\bar{z}_i\bar{\partial}_i\right] + 2\alpha\sum_{i<j}\frac{\bar{\partial}_i - \bar{\partial}_j}{z_i - z_j} \tag{53}$$

$$H^- = -2\sum_{i=1}^{N}\left[\bar{\partial}_i\partial_i + bz_i\partial_i\right] - 2\alpha\sum_{i<j}\frac{\partial_i - \partial_j}{\bar{z}_i - \bar{z}_j} \tag{54}$$

Any analytic function of the variables z_i is an eigenstate of the holomorphic Hamiltonian (53), whereas a anti-analytic function of $\bar{z}_i$ is an eigenstate of (54). From (49,50), one infers the infinitely degenerate ground state with zero energy[†]

[†]The lowest Landau level spectrum has been shifted downward by the spin induced shift.

$$\psi_{\rm free} = \prod_{i<j}(z_i - z_j)^{-\alpha}\prod_i z_i^{\ell_i}\exp(-\frac{1}{2}b\sum_i z_i\bar z_i), \qquad \ell_i \geq 0 \tag{55}$$

$$\psi_{\rm free} = \prod_{i<j}(\bar z_i - \bar z_j)^{\alpha}\prod_i \bar z_i^{\ell_i}\exp(\frac{1}{2}b\sum_i z_i\bar z_i), \qquad \ell_i \geq 0 \tag{56}$$

where the orbital quantum numbers $\ell_i = 0, 1, ...\infty$ are such that the eigenstates are symmetric. If one leaves aside the anyonic prefactors $\prod_{i<j}(z_i - z_j)^{-\alpha}$ and $\prod_{i<j}(\bar z_i - \bar z_j)^{\alpha}$, the N-anyon ground state is a symmetrised product of 1-body Landau ground states with orbital angular momentum ℓ_i, as for ideal one-dimensional bosons.

To compute its thermodynamical properties, one has to regularize the system at long distance. Adding a harmonic well confinement $\sum_i \frac{1}{2}\omega^2 \bar z_i z_i$ to the Hamiltonians (40,41), an N-body anyonic eigenstate in the lowest Landau level of an external magnetic field is still entirely characterized by a product 1-body eigenstates with a given orbital quantum number ℓ_i.

If $b > 0$, $b = +\omega_c$ where ω_c is is half the cyclotron frequency, and $\alpha \in [-1, 0]$. If $b < 0$, $b = -\omega_c$, and $\alpha \in [0, 1]$. In the presence of the harmonic well, the eigenstates are still given by (55,56), but now with $\omega_{\rm c} \to \omega_{\rm t} = \sqrt{\omega_{\rm c}^2 + \omega^2}$. It follows that (49,50) should rewrite as

$$\psi_{\rm free} = \prod_{k<l}(z_k - z_l)^{-\alpha}\exp(-\frac{1}{2}\omega_t\sum_i z_i\bar z_i)\psi_+ \tag{57}$$

$$\psi_{\rm free} = \prod_{k<l}(\bar z_k - \bar z_l)^{\alpha}\exp(-\frac{1}{2}\omega_t\sum_i z_i\bar z_i)\psi_- \tag{58}$$

to get the holomorphic and anti-holomorphic Hamiltonians in the presence of the harmonic well

$$\begin{aligned} H^+ = & -2\sum_{i=1}^N\left[\partial_i\bar\partial_i - \frac{\omega_t + \omega_c}{2}\bar z_i\bar\partial_i - \frac{\omega_t - \omega_c}{2}z_i\partial_i\right] \\ & +2\alpha\sum_{i<j}\left[\frac{\bar\partial_i - \bar\partial_j}{z_i - z_j} - \frac{\omega_t - \omega_c}{2}\right] + \sum_i(\omega_t - \omega_c) \end{aligned} \tag{59}$$

$$\begin{aligned} H^- = & -2\sum_{i=1}^N\left[\partial_i\bar\partial_i - \frac{\omega_t - \omega_c}{2}\bar z_i\bar\partial_i - \frac{\omega_t + \omega_c}{2}z_i\partial_i\right] \\ & -2\alpha\sum_{i<j}\left[\frac{\partial_i - \partial_j}{\bar z_i - \bar z_j} - \frac{\omega_t - \omega_c}{2}\right] + \sum_i(\omega_t - \omega_c) \end{aligned} \tag{60}$$

These Hamiltonians are identical, up to a constant energy shift, to the Hamiltonians (15,16) obtained above since, when acting on regular wavefunctions, $\partial_i\bar{\partial}_i = \bar{\partial}_i\partial_i$.

The N-body spectrum is nothing else but the sum of the 1-body spectra $\epsilon_{\ell_i} = (\omega_t - \omega_c) + \ell_i(\omega_{\rm t} - \omega_{\rm c})$, shifted by the 2-body statistical energy $-\frac{1}{2}N(N-1)(\omega_{\rm t} - \omega_{\rm c})\alpha$. So, the virtue of the harmonic confinement has been to partially lift the degeneracy with respect to the ℓ_i's, and to dress the spectrum with an explicit α dependence

$$E_N = N(\omega_t - \omega_c) + (\sum_i \ell_i \mp \frac{1}{2}N(N-1)\alpha)(\omega_t - \omega_c) \tag{61}$$

Let us now turn to the non-Abelian case : The non-Abelian Knizhnik-Zamolodchikov equations become in place of (51) and (52)

$$(\partial_i + \frac{b}{2}\bar{z}_i)U^+ - U^+ g\sum_{j\neq i}\frac{T_i^a T_j^a}{z_i - z_j} = 0 \quad (\bar{\partial}_i + \frac{b}{2}z_i)U^+ = 0 \tag{62}$$

$$(\bar{\partial}_i - \frac{b}{2}z_i)U^- + U^- g\sum_{j\neq i}\frac{T_i^a T_j^a}{\bar{z}_i - \bar{z}_j} = 0 \quad (\partial_i - \frac{b}{2}\bar{z}_i)U^- = 0 \tag{63}$$

One can proceed exactly in the same way as in the Abelian case, i.e. start from the free Pauli Hamiltonians $H^{\pm}_{\rm free}$ in (40,41), eventually coupled to the external magnetic field, and redefine the free eigenstates according to

$$\psi_{\rm free}(z_1, z_2, \ldots, z_N; \bar{z}_1, \bar{z}_2, \ldots, \bar{z}_N) = U^{\pm}\psi_{\pm}(z_1, z_2, \ldots, z_N; \bar{z}_1, \bar{z}_2, \ldots, \bar{z}_N) \tag{64}$$

The non Hermitian Hamiltonian, in the presence of an harmonic well rewrites in the holomorphic and anti-holomorphic gauges as

$$\begin{aligned} H^+ = & -2\sum_{i=1}^{N}\left[\partial_i\bar{\partial}_i - \frac{\omega_t + \omega_c}{2}\bar{z}_i\bar{\partial}_i - \frac{\omega_t - \omega_c}{2}z_i\partial_i\right] \\ & -2g\sum_{i<j}T_i^a T_j^a\left[\frac{\bar{\partial}_i - \bar{\partial}_j}{z_i - z_j} - \frac{\omega_t - \omega_c}{2}\right] + \sum_i(\omega_t - \omega_c) \end{aligned} \tag{65}$$

$$\begin{aligned} H^- = & -2\sum_{i=1}^{N}\left[\partial_i\bar{\partial}_i - \frac{\omega_t - \omega_c}{2}\bar{z}_i\bar{\partial}_i - \frac{\omega_t + \omega_c}{2}z_i\partial_i\right] \\ & +2g\sum_{i<j}T_i^a T_j^a\left[\frac{\partial_i - \partial_j}{\bar{z}_i - \bar{z}_j} - \frac{\omega_t - \omega_c}{2}\right] + \sum_i(\omega_t - \omega_c) \end{aligned} \tag{66}$$

a generalization of (59,60) which coincides with (32,33). Accordingly the spectrum of (65) (respectively(66)) acting on analytic (respectively anti-analytic) eigenstates is

$$E_{N,I} = N(\omega_t - \omega_c) + (\sum_i \ell_i \pm \Omega_{N,I})(\omega_t - \omega_c) \tag{67}$$

where Ω has been defined in (37).

The cases $b > 0, \alpha \in [-1, 0]$ and $b < 0, \alpha \in [0, 1]$ are equivalent. So in the sequel one will assume without loss of generality $b \equiv \omega_c > 0, \alpha \in [-1, 0]$, i.e. the holomorphic gauge.

IV. THE THERMODYNAMICS OF TWO NON-ABELIAN CHERN-SIMONS PARTICLES IN AN EXTERNAL MAGNETIC FIELD

A. Virial expansion

In this section, we study the thermodynamic quantities in which anyonic statistics manifests itself. For this purpose the virial coefficients a_k, which result from the expansion of the pressure P in terms of the particle density ρ, can be used

$$\frac{1}{V} \ln \Xi = \beta P = \sum_{k=1}^{\infty} a_k \rho^k \tag{68}$$

Using the cluster expansion,

$$\ln \Xi = \sum_{k=1}^{\infty} b_k z^k \ , \tag{69}$$

where $z = e^{\beta\mu}$ is the fugacity, it is possible to express the k-th virial coefficient in terms of partition functions Z_j with $j \leq k$. Indeed, from

$$\Xi = \sum_{N=0}^{\infty} Z_N z^N \tag{70}$$

it follows that,

$$\begin{aligned} b_1 &= Z_1 \ , \\ b_2 &= Z_2 - \frac{1}{2} Z_1^2 \ , \\ b_3 &= Z_3 - Z_2 Z_1 + \frac{1}{3} Z_1^3 \ , \end{aligned} \tag{71}$$

and

$$\begin{aligned} \tilde{a}_1 &= 1 \ , \\ \tilde{a}_2 &= -\tilde{b}_2 \ , \\ \tilde{a}_3 &= -2\tilde{b}_3 + 4\tilde{b}_2^2 \ , \end{aligned} \tag{72}$$

where $\tilde{a}_k = a_k/V^{k-1}$ and $\tilde{b}_k = b_k/b_1^k$.

Considering a harmonic potential as a regulator, the thermodynamic quantities in a box of infinite volume V can be calculated from those in a harmonic well of frequency ω by using the following "thermodynamic limit prescription": in the thermodynamic limit, the cluster coefficients b_k are obtained from those in the harmonic well b_k^ω by (depending on the cluster coefficient order k) [23]

$$\frac{1}{(k\beta^2\omega^2)^{d/2}} \to \frac{V}{\lambda_T^d} , \tag{73}$$

where d is the spatial dimension. In two dimensions, in the presence of an external magnetic field, (73) becomes

$$\frac{1}{k\beta(\omega_t - \omega_c)} \to V\rho_L , \tag{74}$$

where $\rho_L = \frac{eB}{2\pi}$ is the Landau level degeneracy per unit volume.

B. 2-body Abelian anyon case

Let us consider two non-Abelian Chern-Simons particles in an external magnetic field. This problem can be solved exactly since it decomposes into two 2-body Abelian anyon problems. We start by discussing the Abelian anyon problem paying special attention to the peculiarities of the strong magnetic field limit which are important for understanding the non-Abelian case.

The spectrum for the relative motion of two Abelian anyons in an external magnetic field B and a harmonic well ω is

$$E_{nm} = (2n + 1 + |m - \alpha|)\omega_t - (m - \alpha)\omega_c , \tag{75}$$

$$\psi_{nm} = e^{i(m-\alpha)\theta} r^{|m-\alpha|} L_n(\omega_t z\bar{z}) . \tag{76}$$

The spectrum and eigenstates are periodic with period 2 in α since m has to be chosen to be an even (odd) integer for boson (fermion) based anyons. So one can always restrict $\alpha \in [-1, 1]$.

In the thermodynamic limit $\omega \to 0$, the spectrum and eigenstates rewrite as

$$m \geq \alpha : \quad E_{nm} = (2n + 1)\omega_c \tag{77}$$

$$\psi_{nm} = z^{m-\alpha} L_n(\omega_c z\bar{z}) \tag{78}$$

or

$$m \leq \alpha: \quad E_{nm} = (2n + 1 + 2(\alpha - m))\omega_c \tag{79}$$

$$\psi_{nm} = \bar{z}^{\alpha - m} L_n(\omega_c z\bar{z}) \tag{80}$$

The projection onto the lowest Landau level corresponds to $n = 0$. This implies that the wave functions are analytic (anti-analytic) for $m \geq \alpha$ ($m \leq \alpha$). Notice nevertheless that the projection is not well defined at the bosonic end (when $\alpha \to 0^+$ for boson based anyons or $\alpha \to -1^+$ for fermion based anyons). Indeed, let us look at the ground state $n = 0$ in the the boson based description. If $\alpha \in [-1, 0]$, the analytic ground state basis (78) is complete since then the $m = 0$ state belongs to this basis. However, if $\alpha \in [0, 1]$, the analytic ground state basis is incomplete since the $m = 0$ state is anti-analytic and has the energy which varies linearly with α, joining the ground state basis when $\alpha \to 0^+$, as represented in Fig. 1.

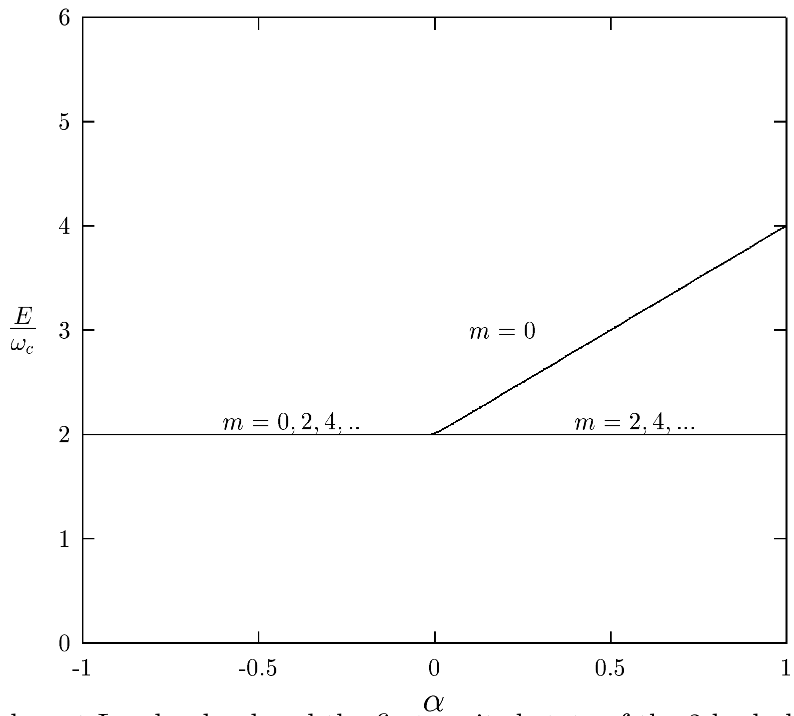

FIG. 1. The lowest Landau level and the first excited state of the 2-body boson based anyon problem.

The 2-anyon second virial coefficient in a magnetic field has been computed in [18] (b and f stand for boson based and fermion based anyons, respectively)

$$a_2^{b,f} = \frac{\lambda_T^2}{x}\Big(\mp \frac{1}{4}\tanh x - \frac{1}{2}\alpha - \frac{e^x(e^{-2x\alpha}-1)}{4}(\frac{1}{\sinh x} \pm \frac{1}{\cosh x}) + \frac{1\pm 1}{2}(e^{x(|\alpha|-\alpha)}-1)\Big) \tag{81}$$

where $x = \beta\omega_c$ and $\lambda_T = \sqrt{2\pi\beta}$ is the thermal wavelength. It is depicted in the Fig. 2 both for low magnetic fields (low x) and for high magnetic fields (large x).

In the limit of the strong magnetic field, one obtains for boson based anyons

$$a_2^b = \frac{1}{2\rho_L}(-1-2\alpha) \quad \alpha \in [-1,0]\ , \tag{82}$$

$$a_2^b = \frac{1}{2\rho_L}\left(-1-2\alpha+4(1-e^{-2\alpha x})\right) \quad \alpha \in [0,1], \tag{83}$$

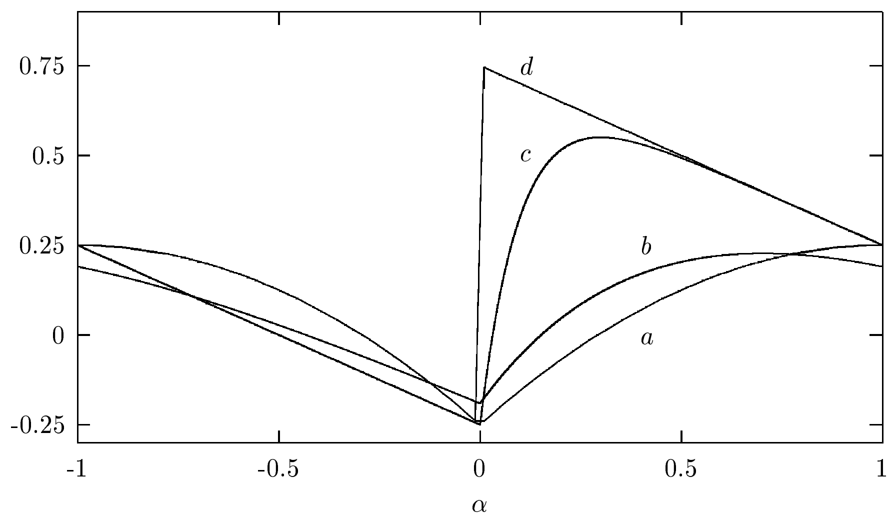

FIG. 2. Second virial coefficient as a function of the statistics parameter α: (a) $\frac{a_2}{\lambda_T^2}$ for zero magnetic field, (b) $\frac{a_2\rho_L}{2}$ for $x=1$, (c) $\frac{a_2\rho_L}{2}$ for $x=5$, (d) $\frac{a_2\rho_L}{2}$ for $x\to\infty$.

For fermion based anyons, the resulting expression is

$$a_2^f = \frac{1}{2\rho_L}\left(1-2\alpha-4e^{-2x(1+\alpha)}\right)\ , \tag{84}$$

As the exponential factor is only relevant for $\alpha \to -1$, (84) can be rewritten as,

$$a_2^f = \frac{1}{2\rho_L}\left(1 - 2\alpha - 4e^{-2x(1+\alpha)}\right) , \quad \alpha \in [-1, 0] , \tag{85}$$

$$a_2^f = \frac{1}{2\rho_L}(1 - 2\alpha) , \quad \alpha \in [0, 1], \tag{86}$$

Note that as far as x is finite, the virial coefficients are continuous functions of α. These results should be compared to the ones that would arise if one projects into the lowest Landau level at the very beginning,

$$a_2^b = \frac{1}{2\rho_L}(-1 - 2\alpha) \quad \alpha \in [-1, 0] , \tag{87}$$

$$a_2^b = \frac{1}{2\rho_L}(-1 - 2\alpha + 4) \quad \alpha \in [0, 1], \tag{88}$$

$$a_2^f = \frac{1}{2\rho_L}(1 - 2\alpha) , \tag{89}$$

a_2^b has now a jump at $\alpha = 0$ while $a_2^f(-1) \neq a_2^f(1)$. These discontinuities are a direct consequence of the projection onto the lowest Landau level which ignores the $m = 0$ ($m = 1$) state leaving the ground state -or in other words the vanishing of the gap- in the boson (fermion) based description for $\alpha = 0$ ($\alpha = -1$). These states, when properly taken into account, smooth the discontinuities as can be seen in Fig. 2.

C. 2-body non-Abelian problem

Consider now the 2-body non-Abelian SU(2) case. The isospin is either $I = 1$ (triplet)

$$\chi_{1,1} = |++> \qquad \chi_{1,0} = \frac{1}{\sqrt{2}}(|+-> + |-+>) \qquad \chi_{1,-1} = |--> \tag{90}$$

or $I = 0$ (singlet)

$$\chi_{0,0} = \frac{1}{\sqrt{2}}(|+-> - |-+>) . \tag{91}$$

The wave function is

$$\psi_{1,m_I} = \phi^s_{m_I}(z_1, z_2)\, \chi_{1,m_I} , \quad \psi_{0,0} = \phi^a(z_1, z_2)\, \chi_{0,0} . \tag{92}$$

As we have seen above, the action of the Hamiltonian on the direct sum of two sectors $\Gamma_{I=1} \oplus \Gamma_{I=0}$ is diagonal in each sector. The eigenvalues of the operator Ω are given by

(39). Thus, with respect to the coordinate part of the wave functions, there is a "bosonic" sector for $I = 1$ and a "fermionic" sector for $I = 0$, with statistics parameters $-g/4$ and $3g/4$, and degeneracies 3 and 1, respectively. It follows that the second virial coefficient can be expressed in terms of the second virial coefficients for Abelian anyons as

$$a_2^{\rm total} = \frac{1}{4}\left[3a_2^b(-\frac{g}{4}) + a_2^f(\frac{3g}{4})\right] . \tag{93}$$

This result is a generalization for $B \neq 0$ to the one obtained in [13]. Remember that $0 < g < 2$: in the fermionic sector, $3g/4$ can fall outside the interval of definition $[-1, 1]$. In this case, the Abelian second virial coefficient should be extended periodically, which means that (93) can still be used but with $3g/4 \to 3g/4 - 2$. As a consequence, a cusp appears in the second virial coefficient for $g = 4/3$, as shown in Fig. 3.

In the vanishing magnetic field limit, the second virial coefficient interpolates between its end values for bosons and fermions with two internal degrees of freedom when g varies from $g = 0$ to $g = 2$.

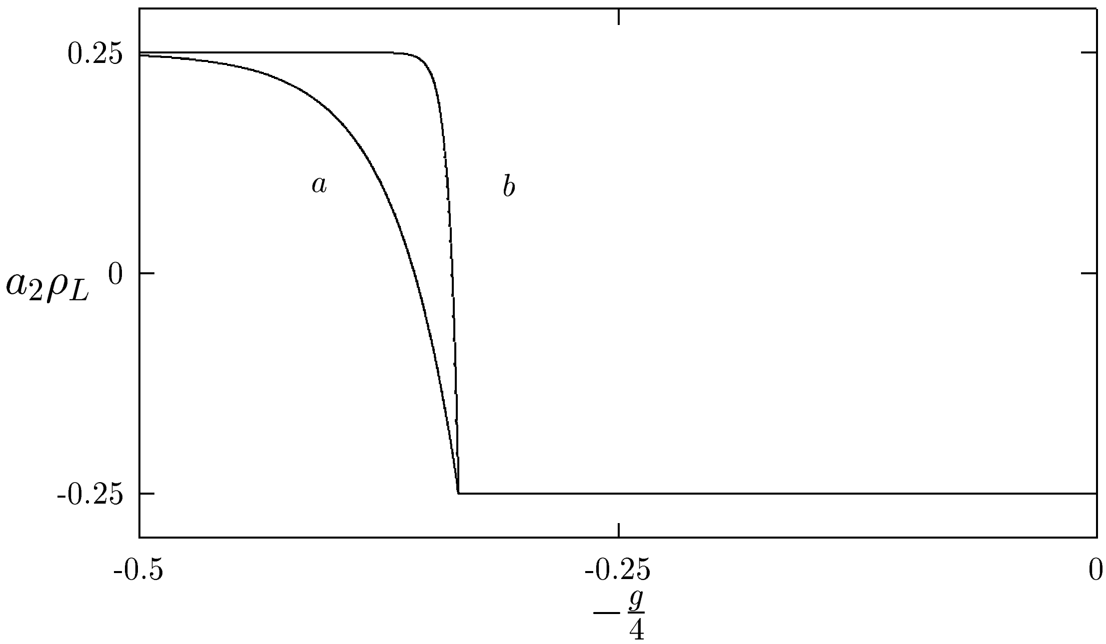

FIG. 3. Second virial coefficient for the non-Abelian problem as a function of the statistics parameter of the bosonic sector $-\frac{g}{4}$ for (a) $x = 10$ and (b) $x = 70$.

In the strong magnetic field limit, the lowest Landau level second virial coefficient reads if $0 < g < 4/3$,

$$a_2(g) = -\frac{1}{4\rho_L} \tag{94}$$

and if $4/3 < g < 2$,

$$a_2(g) = \frac{1}{4\rho_L}\left(1 - 2e^{-2x\left(\frac{3g}{4}-1\right)}\right) . \tag{95}$$

It is close to a step function, changing abruptly near $g = 4/3$ on an interval of width $\sim 1/(\beta\omega_c)$, from its bosonic end value $-1/4\rho_L$ to its fermionic end value $1/4\rho_L$.

V. THE THERMODYNAMICS FOR A N-BODY NON-ABELIAN PROBLEM IN A STRONG MAGNETIC FIELD

We now ask the question about a possible generalization of the 2-body analysis to the N-body case.

Let us first remind that in the Abelian case, from the lowest Landau level spectrum (39,39) one can deduce [4] an equation of state which coincides with Haldane's statistics thermodynamics for a gas of particles whose 1-body spectrum is reduced to a single level of energy $E = \omega_c$. The virial coefficients are

$$a_n = (-\frac{1}{\rho_{\rm L}})^{n-1}\frac{1}{n}\{(1+\alpha)^n - \alpha^n\} \tag{96}$$

The critical filling $\nu_{\rm cr} = -1/\alpha$ where the pressure diverges describes a non degenerate ground state with all the ℓ_i's null. In the *singular gauge*,

$$\psi' = \prod_{i<j} z_{ij}^{-\alpha} \exp(-\frac{\omega_{\rm c}}{2}\sum_i^N z_i\bar{z}_i) \tag{97}$$

When $\alpha = -1$, one recovers a Vandermonde determinant built from 1-body Landau eigenstates. Incidentally, in the Haldane statistics point of view, when $\alpha = -m$, the non degenerate ground state coincides with the Laughlin eigenstates at the critical filling $\nu_{\rm cr} = 1/m$.

A. Lowest Landau level non-Abelian case

For particles with isospin 1/2, there is a one-to-one correspondence between the total isospin of the system and the Young diagram for the isospin wave function [24]. The corresponding Young diagrams may have at most two raws, of length $\frac{N}{2}+I$ and $\frac{N}{2}-I$ for total isospin I and will be d denoted as $[(\frac{N}{2}+I)(\frac{N}{2}-I)]$. Since the total wave function is symmetric under interchange of particles, the symmetry properties of the isospin and coordinate wave functions are determined by the same Young diagram (see Appendix A).

The space of solutions of the N-body problem is decomposed into a direct sum of sectors with given values of the total isospin:

$$\underbrace{\frac{1}{2}\otimes\cdots\otimes\frac{1}{2}}_{N}=\Gamma_{I=\frac{N}{2}}\oplus\underbrace{\Gamma_{I=\frac{N}{2}-1}\oplus\cdots\oplus\Gamma_{I=\frac{N}{2}-1}}_{(N-1)!}\oplus\cdots\oplus\underbrace{\Gamma_{I=\frac{N}{2}-\left[\frac{N}{2}\right]}\oplus\cdots\oplus\Gamma_{I=\frac{N}{2}-\left[\frac{N}{2}\right]}}_{d_{N,\frac{N}{2}-\left[\frac{N}{2}\right]}} \tag{98}$$

The number of times the subspace with total isospin I appears in the decomposition (98) is given by

$$d_{N,I}=d_{[(\frac{N}{2}+I)(\frac{N}{2}-I)]}=\frac{N!(2I+1)}{(\frac{N}{2}+I+1)!(\frac{N}{2}-I)!} \tag{99}$$

which is also equal to the dimension of the representation $[(\frac{1}{2}N+I)(\frac{1}{2}N-I)]$.

The energy spectrum (38) can be represented in the form

$$E_{N,I}=(\omega_t-\omega_c)\Omega_{N,I}(g)+E_{N,I}^{\text{bosons}}(\ell_1,\ldots,\ell_N), \tag{100}$$

where

$$E_{N,I}^{\text{bosons}}(\ell_1,\ldots,\ell_N)=N\omega_t+(\omega_t-\omega_c)\sum_{j=1}^{N}\ell_j \tag{101}$$

is (up to a constant) the spectrum of N SU(2) bosons in a harmonic well of frequency $(\omega_t-\omega_c)$ in the isospin I sector(s); the numbers ℓ_i are discussed in detail in the next subsection. This implies that N-particle partition function is

$$Z_N=\sum_{I=\frac{N}{2}-\left[\frac{N}{2}\right]}^{\frac{N}{2}} e^{-\beta(\omega_t-\omega_c)\,\Omega_{N,I}(g)}\,Z_{N,I}^{\text{bosons}} \tag{102}$$

where $Z_{N,I}^{\text{bosons}}$ is the partition function of SU(2)-bosons determined by the spectrum (101).

For $g=0$, (102) reduces to

$$Z_N=\sum_{I=\frac{N}{2}-\left[\frac{N}{2}\right]}^{\frac{N}{2}} Z_{N,I}^{\text{bosons}}=\sum_{k=0}^{N}Z_{N-k}^{b}Z_k^{b}\,, \tag{103}$$

where Z_k^b is the usual N-particle partition function for bosons without internal degrees of freedom. The latter equality in (103) follows from the fact that the grand partition function for isospin 1/2 bosons is the square of the grand partition function of bosons without internal degrees of freedom.

The partition functions for SU(2) bosons $Z_{N,I}^{\text{bosons}}$ are discussed in the next subsection, where these are related to partition functions associated with Young diagrams (see (117)).

B. Partition functions associated with Young diagrams

We start with a system of identical particles without internal degrees of freedom in a harmonic well. The single-particle energy spectrum is

$$\varepsilon_\ell = (\omega_t - \omega_c)\ell \tag{104}$$

with $\ell = 0, 1, 2, \ldots$, where we have left aside the constant ground state energy shift ω_t which has no effect on the thermodynamics.

The 2-body wave function of arbitrary permutation symmetry can be represented as a linear combination of symmetric and antisymmetric wave functions. This corresponds to decomposition of the set of all possible values of the quantum numbers (ℓ_1, ℓ_2), $\ell_1, \ell_2 = 0, 1, 2, \ldots$ for two particles into

$$\ell_1 \leq \ell_2 \qquad \text{and} \qquad \ell_1 < \ell_2 \tag{105}$$

Correspondingly, the Boltzmann partition function is the sum of the partition functions for bosons and fermions ($q \equiv e^{-\beta(\omega_t - \omega_c)}$)

$$Z_2^{\mathrm{Boltz}} \equiv \frac{1}{(1-q)^2} = Z_2^{\mathrm{b}} + Z_2^{\mathrm{f}} = Z_{\Box\Box} + Z_{\begin{smallmatrix}\Box\\\Box\end{smallmatrix}} \tag{106}$$

where

$$Z_{\Box\Box} = \frac{1}{(1-q)(1-q^2)}\ , \quad Z_{\begin{smallmatrix}\Box\\\Box\end{smallmatrix}} = \frac{q}{(1-q)(1-q^2)}\ . \tag{107}$$

In the 3-body case, a wave function of an arbitrary permutation symmetry can be decomposed into a sum of wave functions with three types of symmetry, bosonic, fermionic and mixed. Note that there are two (identical) representations of the mixed symmetry and that these representations are two-dimensional[‡]. To discuss decomposition of all possible values of the quantum numbers (ℓ_1, ℓ_2, ℓ_3), for three particles, one should consider standard Young *tableaux* labeling all possible symmetries of wave functions (see Appendix A).

With each standard Young tableau we associate a ℓ-Young tableau which is obtained by the change $i \to \ell_i$ for all the numbers inside the standard Young tableau. In the 3-body case, we then obtain four ℓ-Young tableaux

$$\begin{array}{|c|c|c|}\hline \ell_1 & \ell_2 & \ell_3 \\ \hline \end{array}\ , \quad \begin{array}{|c|c|}\hline \ell_1 & \ell_2 \\ \hline \ell_3 \\ \cline{1-1} \end{array}\ , \quad \begin{array}{|c|c|}\hline \ell_1 & \ell_3 \\ \hline \ell_2 \\ \cline{1-1} \end{array}\ , \quad \text{and} \quad \begin{array}{|c|}\hline \ell_1 \\ \hline \ell_2 \\ \hline \ell_3 \\ \hline \end{array}\ .$$

FIG. 4. ℓ-Young tableaux for three particles

[‡]The above decomposition of the wave function is a decomposition of the *regular* representation of S_3 into irreducible representations.

The second and third tableaux are associated with the two standard Young tableaux for the states of mixed symmetry.

All possible values of ℓ_1, ℓ_2, and ℓ_3 can be arranged as

$$\ell_1 \le \ell_2 \le \ell_3 \ , \tag{108}$$
$$\ell_1 \le \ell_2 < \ell_3 \ , \tag{109}$$
$$\ell_1 < \ell_2 \le \ell_3 \ , \tag{110}$$
$$\ell_1 < \ell_2 < \ell_3 \ , \tag{111}$$

where (108) and (111), corresponding to bosons and fermions, respectively lead to the partition functions

$$\begin{aligned} Z_{\square\square\square} &= \frac{1}{(1-q)(1-q^2)(1-q^3)} \ , \\ Z_{\begin{smallmatrix}\square\\\square\\\square\end{smallmatrix}} &= \frac{q^3}{(1-q)(1-q^2)(1-q^3)} \ . \end{aligned} \tag{112}$$

Straightforward calculations show that (109) and (110) lead to the partition functions $2q/[(1-q)(1-q^2)(1-q^3)]$ and $2q^2/[(1-q)(1-q^2)(1-q^3)]$, respectively. We associate these contributions with the second and third ℓ-Young tableaux, respectively, in Fig. 4. Thus the partition function of the Young diagram of mixed symmetry is

$$Z_{\begin{smallmatrix}\square\square\\\square\end{smallmatrix}} = \frac{2(q+q^2)}{(1-q)(1-q^2)(1-q^3)} \ . \tag{113}$$

The Boltzmann partition function rewrites as

$$Z_3^{\rm Boltz} \equiv \frac{1}{(1-q)^3} = Z_{\square\square\square} + Z_{\begin{smallmatrix}\square\square\\\square\end{smallmatrix}} + Z_{\begin{smallmatrix}\square\\\square\\\square\end{smallmatrix}} \tag{114}$$

as it should.

We are now in position to propose general rules to write down partition functions associated with Young diagrams in the N-body case. To this aim, we represent the partition function associated with a Young diagram Y_N with N boxes as

$$Z_{Y_N} = d_{Y_N} \frac{P_{Y_N}(q)}{(q)_N} \ , \quad (q)_n \equiv \prod_{k=1}^{n} (1-q^k) \ . \tag{115}$$

where d_{Y_N} is the dimension of the irreducible representation of the symmetric group S_N corresponding to the Young diagram Y_N (equal to the number of standard Young tableaux for Y_N), and P_{Y_N} is a polynomial in q. Note that the partition functions for N Abelian bosons and fermions read $Z_N^b = 1/[(q)_N]$ and $Z_N^f = q^{N(N-1)}/[(q)_N]$, respectively.

To evaluate $P_{Y_N}(q)$, to each ℓ-Young tableau is associated a *chain* of inequalities for the integer numbers $\{\ell_1 \leq \ell_2 \leq \ldots \leq \ell_N\}$ in the following way: If the row containing the box with ℓ_{j+1} is located below the raw containing the box with ℓ_j, then $\ell_j < \ell_{j+1}$; otherwise, $\ell_j \leq \ell_{j+1}$. Let $s_1, s_2, \ldots$ be the positions of the signs '$<$' in this chain counted from the right. Then the contribution to the polynomial P_{Y_N} from a given ℓ-Young tableau is $q^{\sum_j s_j}$. Finally, P_{Y_N}, associated with a given Young diagram, is the sum of the contributions from all ℓ-Young tableaux for this diagram.

For instance, for (109), $s_1 = 1$, and the contribution to $P_{[21]}$ is q while for (110), $s_1 = 2$, and its contribution is q^2. For the 4 and 5-body cases, explicit expressions for P_{Y_N} are given in Appendix A.

Notice a relation between the partition functions for Y_N and its conjugated Young diagram $\bar{Y_N}$: if $P_{Y_N} = \sum_{k=1}^{k_{\max}} c_{Y_N}(k) q^k$ then $P_{\bar{Y_N}} = \sum_{k=1}^{k_{\max}} c_{Y_N}(k) q^{k_{\max}-k}$.

As a consistency check, all possible partition functions in the N-body case should sum to the N-body Boltzmann partition function, generalizing (106) and (114)

$$Z_N^{\text{Boltz}} = \frac{1}{(1-q)^N} = \sum_{Y_N} Z_{Y_N} \; . \tag{116}$$

Using the partition functions above we checked using *Mathematica*, up to the 6-body case, that (116) is indeed valid.

Now we consider SU(2) bosons. The decomposition of the space of solution of the N-body problem into sectors with different isospins was discussed in the previous section. Taking into account the multiplicity $2I+1$ for isospin I states, the partition function for SU(2) bosons in the isospin I sector(s) can be related to the partition function associated with the Young diagram corresponding to isospin I as follows

$$Z_{N,I}^{\text{bosons}} = (2I+1) Z_{[(\frac{N}{2}+I)(\frac{N}{2}-I)]} \; . \tag{117}$$

Note that $Z_{[(\frac{N}{2}+I)(\frac{N}{2}-I)]}$ receives contributions from $d_{[(\frac{N}{2}+I)(\frac{N}{2}-I)]}$ ℓ-Young tableaux. This corresponds to taking into account all $d_{[(\frac{N}{2}+I)(\frac{N}{2}-I)]}$ sectors of given I in the decomposition of the space of solutions of N SU(2) bosons in calculating their partition functions. One can check, using (115) and the expressions for the polynomials given in this Section and in Appendix B, that (117) leads to the correct expression for the total partition functions for N bosons (103).

To conclude this section, we comment on the relation between the above partition functions and the Schur functions. Schur functions naturally arise for the system of non interacting particles whose 1-body spectrum is represented by a finite set of levels $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_M$ [25]. One defines $x_1 = e^{-\beta\varepsilon_1}$, ..., $x_M = e^{-\beta\varepsilon_M}$. Then the Boltzmann partition function can be represented as a sum over all irreducible representations λ of S_N

$$
\begin{aligned}
Z_N^{\text{Boltz}}(x_1,\ldots,x_M) &\equiv (x_1+\cdots+x_M)^N \\
&= \sum_{\substack{\lambda \\ |\lambda|=N}} d(\lambda) S_\lambda(x_1,\ldots,x_M) \quad ,
\end{aligned} \tag{118}
$$

where $S_\lambda(x_1,\ldots,x_M)$ are the Schur functions associated with the irreducible representations λ of the symmetric group S_N. The case discussed above corresponds to $\varepsilon_k = (k-1)(\omega_t - \omega_c)$ and $M \to \infty$. We thus observe that the partition functions discussed in this section are a generalization of the Schur functions to an infinite number of arguments corresponding to an infinite set of equally spaced 1-body energy levels. These functions can be used in studies of the statistical mechanics of systems with parastatistics [25,26]

C. On the virial expansion for non-Abelian Chern-Simons particles in a strong magnetic field

Unlike the Abelian case, an explicit expression for the N-body partition functions is not available. Nevertheless, it is possible to construct iteratively the cluster coefficients b_k^ω of order k for any $k \le N$. The cluster coefficients in the thermodynamic limit are obtained as

$$
b_k = V k \rho_L \beta \lim_{\omega \to 0} (\omega_t - \omega_c) b_k^\omega \tag{119}
$$

Using the partition functions (102) with (117), we find the cluster coefficients of lowest orders

$$
b_2^\omega = (-1 + \Omega_{2,0} + 3\Omega_{2,1}) \frac{1}{\beta(\omega_t - \omega_c)} + \ldots \tag{120}
$$

$$
\begin{aligned}
b_3^\omega = &\left(\Omega_{2,0} + 3\Omega_{2,1} - \frac{2}{3}(\Omega_{3,\frac{1}{2}} + \Omega_{3,\frac{3}{2}})\right) \frac{1}{\beta^2(\omega_t - \omega_c)^2} + \\
&\left(\frac{2}{9} + \Omega_{2,0}(1 - \frac{1}{2}\Omega_{2,0}) + 6\Omega_{2,1}(1 - \frac{1}{4}\Omega_{2,1}) - \right. \\
&\left. \Omega_{3,\frac{1}{2}}(1 - \frac{1}{3}\Omega_{3,\frac{1}{2}}) - 2\Omega_{3,\frac{3}{2}}(1 - \frac{1}{6}\Omega_{3,\frac{3}{2}})\right) \frac{1}{\beta(\omega_t - \omega_c)} + \ldots
\end{aligned} \tag{121}
$$

In the thermodynamic limit, b_k^ω should behave as $(\omega_t - \omega_c)^{-1}$, i.e. for small ω b_k should be proportional to the volume. It is easy indeed to verify that $\Omega_{N,I}$ in (37) is such that the correct thermodynamic limit is obtained.

Moreover, $\Omega_{N,I}$ is such that, in the cluster expansion, the leading term $(\omega_t - \omega_c)^{-1}$ *does not depend on g* what we checked up to the fifth virial coefficient. This implies that the cluster coefficients (and consequently the virial coefficients) do not depend on the statistics parameter. It would be interesting to understand a possible symmetry underlying this cancellation.

The above results agree with the exact second virial coefficient analysis if g is sufficiently small, $|g| \leq 4/3$. However, at $g = 4/3$ there is a jump in the second virial coefficient. It follows that the N-body spectrum (38) should not be considered as the correct spectrum for arbitrary g. As we have seen in the 2-body case, a shift in the statistics parameter had to be done, which in turn lead to the jump in the second virial coefficient. We have not found a generalization of this procedure to the N-body case.

In analogy with the Abelian case, we can argue that the spectrum of anyons is indeed given by (38) if $\Omega_{N,I}$ are less than one. Under this assumption, it is easy to show that the cluster coefficients b_k are independent of g when

$$0 \leq g \leq \frac{8}{k(k-1)} , \qquad k \geq 3 . \tag{122}$$

In this range the cluster coefficients are the same as those of SU(2) bosons. The behavior of the cluster (and virial) coefficients in the entire interval of the statistics parameter is yet an open question.

VI. CONCLUDING REMARKS

We have developed a formalism to study the thermodynamics of non-Abelian Chern-Simons particles in the lowest Landau level of an external magnetic field. We expect that the thermodynamics of non-Abelian Chern-Simons particles described by Hamiltonians of the type (32,33) and (65,66) with other groups, as well as other irreducible representations, can be analyzed using the same ideas with minor modifications. The hope is that under some appropriate choices of the non-Abelian symmetry one can obtain non-trivial thermodynamics in the lowest Landau level.

In this respect, an attractive possibility would be to use non-Abelian symmetries proposed recently in Ginzburg-Landau Chern-Simons theories for non-Abelian quantum Hall states [10]. An argument in favor of this choice is the new exclusion statistics thermodynamics found for non-Abelian quantum Hall states quasiparticles on the edge [27] and for their bulk counterparts [28]. We plan to address this issue in future publications.

Another observation is that the same form of the spectrum, namely a bosonic N-body spectrum plus a linear term in the statistics parameter, arises in one-dimensional integrable models with inverse square interaction [29]. In those models, the coupling to the Chern-Simons field is replaced by the coupling of the inverse square interaction. The spectrum is then valid in the entire interval of definition of the coupling parameter. It follows that that the cluster and virial coefficients do not depend on the coupling parameter in the entire interval of definition of the interaction parameter. It would certainly be rewarding to find a symmetry principle underlying this cancellation.

ACKNOWLEDGMENTS

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APPENDIX A: YOUNG DIAGRAMS AND IRREDUCIBLE REPRESENTATIONS

For completeness we include in this appendix a brief description of the relation between Young diagrams and representations of the permutation group of N particles S_N. In particular we show how wave functions of arbitrary permutation symmetry can be constructed using Young operators.

The irreducible representations of S_N can be associated with the partitions of N in positive integers λ_i,

$$\lambda_1 + \lambda_2 + ... + \lambda_m = N \ , \quad \lambda_1 \geq \lambda_2 ... \geq \lambda_m \ . \tag{A1}$$

The different partitions (A1) can be depicted graphically by means of *Young diagrams* in which each number λ_i is represented by a row of λ_i cells. The partitions (A1) are usually denoted by $[\lambda] = [\lambda_1 \lambda_2 ... \lambda_m]$ where the power is used to indicate the repeated appearance of the same integer. Thus for two particles we have, $[\lambda] = [2]$ and $[\lambda] = [1^2]$ associated with the first two diagrams in Fig A, while for three particles we have $[\lambda] = [3]$, $[\lambda] = [21]$, and $[\lambda] = [1^3]$ corresponding to the last three diagrams in Fig.A.

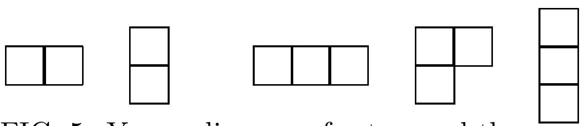

FIG. 5. Young diagrams for two and three particles

The number of inequivalent irreducible representations of S_N is equal to the number of different Young diagrams. In our examples, S_2 and S_3 have two and three inequivalent representations respectively. Using Young diagrams it is also possible to determine the dimensions of such representations as well as to construct explicitly representations $\Gamma^{[\lambda]}(P)$ of the elements $P \in S_N$ and the basis vectors. To do that, a *standard* Young tablueax is defined in which the numbers from 1 to N are distributed among the boxes of a Young tableau in such a way that they increase from left to right along the same row and from top to bottom along the the same columns. Then the dimension of the representation is equal to the number of standard Young tableaux. For example, there is only one standard Young tableau for each of the diagrams in Fig A, except for [21], and so they correspond to one-dimensional representations of the permutation group. The diagram [21] has two standard Young tableaux shown in Fig 6, corresponding to a two-dimensional representation.

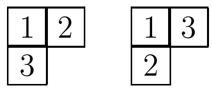

FIG. 6. Standard Young tableaux for the [21] representation

Since any element of S_N can be written as a the product of $N-1$ transpositions $P_{i-1,i}$, it is sufficient to specify the matrices $\Gamma^{[\lambda]}(P_{i-1i})$ representing these transpositions. The following rules define the *Young-Yamanouchi standard* orthogonal representation [30] where superscripts r, t run over all standard Young tableaux for the Young diagram $[\lambda]$.

Diagonal elements

1. $\Gamma^{[\lambda]}_{rr}(P_{i-1i}) = 1$ if in the tableau r, i and and $i-1$ are on the same row.

2. $\Gamma^{[\lambda]}_{rr}(P_{i-1i}) = -1$ if in the tableau r, i and and $i-1$ are on the same column.

3. $\Gamma^{[\lambda]}_{rr}(P_{i-1i}) = \pm\frac{1}{d}$, otherwise. Here d is the number of vertical and horizontal steps that one must take to move from $i-1$ to i, and the upper (lower) signs applies when the row containing the number i is above (below) the one containing $i-1$.

Non-diagonal elements

1. $\Gamma^{[\lambda]}_{rt}(P_{i-1i}) = (1-\frac{1}{d^2})^{\frac{1}{2}}$, if the tableaux r and t differ only by a permutation of numbers i and $i-1$ and where d is defined as above.

For instance, for the two-dimensional representation [21] we only need to find $\Gamma^{[21]}(P_{12})$ and $\Gamma^{[21]}(P_{23})$ (as $P_{13} = P_{12}P_{23}P_{12}$, $P_{123} = P_{12}P_{23}$ and $P_{132} = P_{123}^{-1}$). The application of the rules above gives

$$\Gamma^{[21]}(P_{12}) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} , \qquad \Gamma^{[21]}(P_{23}) = \begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix} . \tag{A2}$$

To construct the basis for the representation [λ], $N!$ *Young operators* are introduced as

$$\omega_{rt}^{[\lambda]} = (\frac{f_\lambda}{N})^{\frac{1}{2}} \sum_P \Gamma_{rt}^{[\lambda]}(P)P . \tag{A3}$$

where f_λ is the dimension of the representation. These f_λ operators with a fixed second index transform into each other under permutations and form a basis for the irreducible representation $\Gamma^{[\lambda]}$.

A basis more convenient in physical applications can be obtained by applying the Young operators to a product of "one-particle wave functions" in the following way. Consider a set of N orthogonal functions $\psi_a(i)$, where i stands for the sets of variables on which the ψ_a depend, and form the product wave function Ψ_0,

$$\Psi_0 = \psi_1(1)\psi_2(2)\cdots\psi_N(N) . \tag{A4}$$

One can then obtain a basis for the orthogonal representations by applying the Young operators to this function

$$\Psi_{rt}^{[\lambda]} = \omega_{rt}^{[\lambda]}\Psi_0 . \tag{A5}$$

where P is understood as acting on the *arguments* of Ψ_a. Then, the functions $\Psi_{rt}^{[\lambda]}$ with a fixed *second* index transform into each other under permutation and form a basis for a irreducible representation of the permutation group. The Young tableau r that corresponds to the first index, enumerates the different wave functions in a given basis and characterizes the symmetry under the interchange of arguments. The second index t enumerates the different basis for $\Gamma^{[\lambda]}$ and it can be shown that it characterizes the symmetry of the wave function under the interchange of the function ψ_a.

One can apply the above general discussion to the case of the coordinate wave function of a 3-body system. Consider a coordinate wave function $\phi(123) \equiv \phi(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3)$ (eventually the product of single particle wave functions, $\phi(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3) = \phi_\alpha(\mathbf{r}_1)\phi_\beta(\mathbf{r}_2)\phi_\gamma(\mathbf{r}_3)$). Then in addition to the totally symmetric and totally antisymmetric wave functions,

$$\phi^{[3]} = \frac{1}{\sqrt{6}}\Big(\phi(123) + \phi(123) + \phi(132) + \phi(321) + \phi(231) + \phi(312)\Big) ,$$
$$\phi^{[1^3]} = \frac{1}{\sqrt{6}}\Big(\phi(123) - \phi(123) - \phi(132) - \phi(321) + \phi(231) + \phi(312)\Big) , \quad \text{(A6)}$$

one obtains for the mixed symmetry wave functions for the *two* 2-dimensional representation [21]

$$\phi^{[21]}_{11} = \frac{1}{\sqrt{12}}\Big(2\phi(123) + 2\phi(213) - \phi(132) - \phi(321) - \phi(231) - \phi(312)\Big) ,$$
$$\phi^{[21]}_{21} = \frac{1}{2}\Big(\phi(132) - \phi(321) - \phi(231) + \phi(312)\Big) , \quad \text{(A7)}$$

$$\phi^{[21]}_{22} = \frac{1}{\sqrt{12}}\Big(2\phi(123) - 2\phi(213) + \phi(132) + \phi(321) - \phi(231) - \phi(312)\Big) ,$$
$$\phi^{[21]}_{12} = \frac{1}{2}\Big(\phi(132) - \phi(321) + \phi(231) - \phi(312)\Big) , \quad \text{(A8)}$$

For a system of bosons with two internal degrees of freedom, the total wave functions for a given total isospin I (symmetric under interchange of particles) can be written as

$$\psi_{\alpha\beta} = \sum_\gamma \phi^{[\lambda]}_{\gamma\alpha}\chi^{[\lambda]}_{\gamma\beta} , \quad \text{(A9)}$$

where $\phi^{[\lambda]}_{\gamma\alpha}$ and $\chi^{[\lambda]}_{\gamma\beta}$ are the coordinate and isospin wave functions of the permutation symmetry $[\lambda] = [(N/2+I)(N/2-I)]$.

Again in the 3-body case, the eight-dimensional internal space can be written as the sum of a $I = 3/2$ four-dimensional space, associated with wave functions which are completely symmetric under the interchange of isospin indices, plus two $I = 1/2$ two-dimensional spaces associated with states of mixed symmetry. The basis of the $I = 3/2$ subspace is therefore

$$\chi_{\frac{3}{2},\frac{3}{2}} = |+++> ,$$
$$\chi_{\frac{3}{2},\frac{1}{2}} = \frac{1}{\sqrt{3}}(|++-> + |+-+> + |-++>) ,$$
$$\chi_{\frac{3}{2},-\frac{1}{2}} = \frac{1}{\sqrt{3}}(|--+> + |-+-> + |+-->) ,$$
$$\chi_{\frac{3}{2},-\frac{3}{2}} = |---> \ . \quad \text{(A10)}$$

For $I = 1/2$ there are *two* two-dimensional representations, one generated by

$$\chi^{(1)}_{\frac{1}{2},\frac{1}{2}} = \frac{1}{\sqrt{6}}(2|++-> - |+-+> + |-++>) ,$$
$$\chi^{(1)}_{\frac{1}{2},-\frac{1}{2}} = \frac{1}{\sqrt{6}}(2|--+> - |-+-> + |+-->) \quad \text{(A11)}$$

and the other by

$$
\begin{aligned}
\chi^{(2)}_{1,\frac{1}{2}} &= \frac{1}{\sqrt{2}}(|+-+> - |-++>) , \\
\chi^{(2)}_{1,-\frac{1}{2}} &= \frac{1}{\sqrt{2}}(|-+-> - |+-->) .
\end{aligned} \tag{A12}
$$

The superscript in these formulas is related to the permutation symmetry. For a fixed total isospin projection, say $I_z = 1/2$, $\chi^{(1)}_{\frac{1}{2},\frac{1}{2}}$ and $\chi^{(2)}_{\frac{1}{2},\frac{1}{2}}$ form a basis for a two-dimensional representation of S_3. More precisely, they correspond to $\Psi^{[21]}_{1a}$ and $\Psi^{[21]}_{2a}$, with $a = 1$ or $a = 2$ having taken $\Psi_0(1,2,3) = |+-+>$. Notice that since there are only two available states, $+$ and $-$, $a = 1$ and $a = 2$ give the *same* results so that only two independent wave functions can be formed. The same argument applies to the $I_z = -1/2$ sector starting from $\Psi_0(1,2,3) = |-+->$.

According to (A9), the total wave functions for the system of bosons are obtained by multiplying coordinate and isospin wave functions of appropriate symmetries. For the sector $I = 3/2$, one has four wave functions

$$
\psi_{\frac{3}{2},m_I} = \phi_{\frac{3}{2},m_I}(z_1,z_2,z_3)\chi_{\frac{3}{2},m_I} , \tag{A13}
$$

where $\phi_{\frac{3}{2},m_I}(z_1,z_2,z_3)$ is a totally symmetric wave function (A6). For two representations in the isospin $I = 1/2$ (mixed symmetry) sector, we obtain four basis wave functions

$$
\psi^{(1)}_{\frac{1}{2},m_I} = \phi^{(11)}_{\frac{1}{2},m_I}\chi^{(1)}_{\frac{1}{2},m_I} + \phi^{(21)}_{\frac{1}{2},m_I}\chi^{(2)}_{\frac{1}{2},m_I} , \tag{A14}
$$

$$
\psi^{(2)}_{\frac{1}{2},m_I} = \phi^{(12)}_{\frac{1}{2},m_I}\chi^{(1)}_{\frac{1}{2},m_I} + \phi^{(22)}_{\frac{1}{2},m_I}\chi^{(2)}_{\frac{1}{2},m_I} , \tag{A15}
$$

where $\phi^{(ab)}_{1,m_I}$ with $a,b = 1,2$ are the same as $\phi^{[21]}_{ab}$ in (A7-A8).

APPENDIX B: POLYNOMIALS $P_{Y_N}(Q)$ FOR FOUR AND FIVE PARTICLES

Following the rules of subsection V B, one obtains the following expressions for the polynomials $P_{Y_n}(q)$ in Eq. (115) for four particles

$$
\begin{aligned}
P_{[4]}(q) &= 1 \\
P_{[31]}(q) &= q + q^2 + q^3 \\
P_{[22]} &= q^2 + q^4 \\
P_{[211]} &= q^3 + q^4 + q^5 \\
P_{[1111]} &= q^6
\end{aligned} \tag{B1}
$$

and for five particles

$$\begin{aligned}
P_{[5]}(q) &= 1 \\
P_{[41]}(q) &= q + q^2 + q^3 + q^4 \\
P_{[32]} &= q^2 + q^3 + q^4 + q^5 + q^6 \\
P_{[31^2]} &= q^3 + q^4 + 2q^5 + q^6 + q^7 \\
P_{[2^21]} &= q^4 + q^5 + q^6 + q^7 + q^8 \\
P_{[21^3]} &= q^6 + q^7 + q^8 + q^9 \\
P_{[1^5]} &= q^{10} \ .
\end{aligned} \tag{B2}$$

Note that the sum of all the coefficients of $P_{Y_n}(q)$ is equal to the dimension d_{Y_N} of the associated respresentation of S_N.

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[1] J. M. Leinaas and J. Myrheim, Nuovo Cimento **B37**, 1 (1977); G. A. Goldin, R. Menikoff and D. H. Sharp, J. Math. Phys. **22**, 1664 (1981); F. Wilczek, Phys. Rev. Lett. **48**, 1144 (1982); **49**, 957 (1982)

[2] F. Calogero, J. Math. Phys. **10**, 2191 (1969); **12**, 419 (1971); B. Sutherland, Phys. Rev. **A4**, 2019 (1971); **A5**, 1372 (1972)

[3] F. D. M. Haldane, Phys. Rev. Lett. **67**, 937 (1991)

[4] A. Dasnières de Veigy and S. Ouvry, Phys. Rev. Lett. **72**, 600 (1994)

[5] S. B. Isakov, Mod. Phys. Lett. B **8**, 319 (1994); Y.-S. Wu, Phys. Rev. Lett. **73**, 922 (1994); A. K. Rajakopal, Phys. Rev. Lett. **74**, 1048 (1995).

[6] S. B. Isakov, Int. J. Mod. Phys. A **9**, 2563 (1994); D. Bernard and Y.-S. Wu, cond-mat/9404025; Z. N. C. Ha, Phys. Rev. Lett. **73**, 1574 (1994); M. V. N. Murthy and R. Shankar, Phys. Rev. Lett. **73**, 3331 (1994).

[7] K. Schoutens, Phys. Rev. Lett. **79**, 2608 (1997)

[8] E. Verlinde, "A note on braid statistics and the non-Abelian Aharonov-Bohm effect", in

Modern Quantum Field Theory (World Scientific, Singapore, 1991).

[9] Goldin, R. Menikoff and D. H. Sharp, Phys. Rev. Lett. **54**, 603 (1985).

[10] E. Fradkin, C. Nayak, and K. Schoutens "Landau-Ginzburg theories for non-Abelian quantum Hall states", cond-mat/9811005

[11] S. C. Zhang, Int. J. Mod. Phys. **B6**, 25 (1992).

[12] J. Desbois, C. Furtlehner, and S. Ouvry, Nuclear Physics B **453** [FS] 759 (1995).

[13] C. R. Hagen, Phys. Rev. Lett. **76**, 4086 (1996).

[14] T. Lee, Phys. Rev. Lett. **74**, 4967 (1995); T. Lee and P. Oh, Ann. Phys. **235**, 413 (1994); see also H.-K. Lo, Phys. Rev. D **48**, 4999 (1993).

[15] V. G. Knizhnik and A. B. Zamolodchikov, Nucl. Phys. **B 247**, 83 (1984).

[16] R. Jackiw and Y. S. Pi, Phys. Rev. Lett. **64**, 2969 (1990); Phys. Rev. D **42**, 3500 (1990); Phys. Rev. Lett. **67**, 415 (1991); Phys. Rev. D **44**, 2524 (1991).

[17] Girvin et al. Phys. Rev. Lett **65**, 1671 (1990).

[18] M. D. Johnson and G. S. Canright, Phys. Rev. **B41**, 6870, 1990.

[19] O. Bergamn and G. Lozano, Ann. Phys. **229**, 416 (1994); D. Z. Freedman, G. Lozano, and N. Rius, Phys.Rev. D **49**, 1054 (1994).

[20] A. Dasnières de Veigy and S. Ouvry, Nucl. Phys. **B388** [FS], 715 (1992). M. A. Valle Basagoiti, Phys. Lett. B **306**, 307 (1993); R. Emparan and M. A. Basagoiti, Mod. Phys. Lett A **8**, 3291 (1993).

[21] C. Manuel and R. Tarrach, Phys. Lett. B **268**, 222 (1991); Phys. Lett. B **328**, 113 (1994); G. Amelino Camelia, Phys. Lett. B **299**, 83 (1993); **326**, 282, 1994; Phys. Rev. D **51**, 2000 (1995); S. J. Kim and C. Lee, Phys. Rev. D **55**, 2227 (1997); P. Giacconi, F. Maltoni, and R. Soldati, hep-th 9706198

[22] Z. F. Ezawa, M. Hotta, A. Iwazaki, Phys. Rev. D **44**, 452 (1991).

[23] A. Comtet, Y. Georgelin and S. Ouvry, J. Phys. **A**: Math. Gen. **22**, 3917 (1989); K. Olaussen, "On the harmonic oscillator regularization of partition function", Trondheim Univ. preprint No. 13 (1992), cond-mat/9207005.

[24] See, e.g. L. D. Landau and E.M. Lifschitz, *Quantum Mechanics: Non-Relativistic* (Pergamon, 1977).

[25] See e.g., S. Chaturvedi, Phys. Rev. E **54**, 1378 (1996).

[26] A. P. Polychronakos, Nucl. Phys. B **474**, 529 (1996).

[27] K. Schoutens, Phys. Rev. Lett. **81**, 1929 (1998).

[28] S. B. Isakov, in preparation.

[29] A. P. Polychronakos, private communication.

[30] See, e.g. I. G. Kaplan, *Symmetry in Many-Electron Systems* (Academic Press, 1975).