

Normal forms and complex periodic orbits in semiclassical expansions of Hamiltonian systems

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Abstract

Bifurcations of periodic orbits as an external parameter is varied are a characteristic feature of generic Hamiltonian systems. Meyer's classification of normal forms provides a powerful tool to understand the structure of phase space dynamics in their neighborhood. We provide a pedestrian presentation of this classical theory and extend it by including systematically the periodic orbits lying in the complex plane on each side of the bifurcation. This allows for a more coherent and unified treatment of contributions of periodic orbits in semiclassical expansions. The contribution of complex fixed points is found to be exponentially small only for a particular type of bifurcation (the extremal one). In all other cases complex orbits give rise to corrections in powers of \hbar and, unlike the former one, their contribution is hidden in the "shadow" of a real periodic orbit.

71 pages, 4 tables, 20 figures

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Proposed running head: Semiclassics in the complex plane.

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$\stackrel{\text{def}}{=}$	is equal by definition to
$\text{sgn}(x)$	sign of x
$\mathcal{A} \setminus \mathcal{B}$	elements of set \mathcal{A} not in \mathcal{B}
\mathcal{A}^\pm	positive or negative numbers of \mathcal{A}
$\mathcal{A} \times \mathcal{B}$	cartesian product of two sets
$\Re(z), \Im(z)$	real, imaginary part of $z \in \mathbb{C}$
$\mathbb{A} \stackrel{\circ}{=} \mathbb{B}$	\mathbb{A} and \mathbb{B} are two equivalent matrices
$O_{\{x_1, \dots, x_n\}}(k)$	(set of) smooth function(s) of $x = \{x_1, \dots, x_n\}$ whose deriva- tives up to order $k - 1$ vanish at $\{0, \dots, 0\}$.
$\mathfrak{o}, \mathfrak{a}, \mathfrak{p}$, etc.	fixed points or periodic orbits
$\mathbb{A}, \mathbb{B}, \mathbb{R}, \mathbb{A}$	2×2 -matrices
$\text{sp}(U)$	spectrum of matrix U
ε^\star	$ \varepsilon $ if $\ell = 3$, $\sqrt{ \varepsilon }$ otherwise

1 Introduction

Semiclassical methods have had many applications in the description of the quantum mechanical behavior of particles whose classical analog is chaotic. These methods are essentially based on the knowledge of the classical periodic orbits. Aside from these classical contributions, there are other ones, usually associated to non-classical effects such as diffraction or tunneling. The main motivation of this work is to better understand a particular type of contribution to trace formulae, i.e. the complex periodic orbits. We are thus interested in semiclassical expansions which generically take the form of an asymptotic series

$$Q(\hbar) \underset{\hbar \rightarrow 0}{\sim} \sum_{\mathbf{p}} A[\mathbf{p}] e^{iW[\mathbf{p}]/\hbar} \left(1 + \sum_{k \geq 1} a_k[\mathbf{p}] \hbar^k \right). \quad (1.0.1)$$

Here, the first sum concerns (periodic) orbits and fixed points of the classical system. Both $A[\mathbf{p}]$ and $W[\mathbf{p}]$ are complex functions of geometrical quantities attached to each \mathbf{p} . The prefactor $A[\mathbf{p}]$ is well defined if the periodic orbits are assumed to be isolated. This is typically the case for fully chaotic systems, as for example the Sinai billiard or the Bunimovich stadium. However, no smooth potential is known to be fully chaotic and the generic scenario is a coexistence of regular and irregular motion in phase space (the so called mixed dynamics). In this case the phase space structure is much more complicated than in the fully chaotic regime because the frontier between regular and irregular motion is not smooth, but fractal, and its details are very sensitive to changes of external parameters controlling the dynamics.

At the level of periodic orbits, this complexity is reflected on the one hand in a rich structure of coexisting stable and unstable orbits and on the other hand by the occurrence of many bifurcations as some parameter is varied. Close to a bifurcation several periodic orbits can lie very close to each other, and exactly at the bifurcation they coalesce. Semiclassically, the distance between periodic orbits is measured in terms of a phase space length-scale fixed by the value of \hbar . For distances smaller than this scale, the prefactor $A[\mathbf{p}]$ in (1.0.1) diverges, and some uniform approximation treating simultaneously the contribution of several orbits has to be implemented. For two-freedom systems, for which a complete classification of generic bifurcations exists [34] (see also table I below), this was done in Refs. [40, 46]. Because most bifurcations imply the creation of new or the disappearance of existent periodic orbits as a parameter is varied, uniform approximations are not only a good way to solve the problem of the divergence of the prefactor, they also take care of the (unphysical) discontinuities that would have occurred in (1.0.1) if suddenly a term is added (or subtracted) to the interferent sum. In

fact, the most natural way to obtain a continuous physical picture through the bifurcation is to extend the classical dynamics (i.e. the coordinates and momenta) into the complex plane. Then it is easy to see, as we will show below, that classically there is no discontinuity (i.e., no creation or destruction) at the bifurcation but rather an exchange of periodic orbits flowing from the complex to the real plane and vice versa as a parameter is varied. (Complex periodic orbits and more generally complex classical trajectories are solutions of the classical equations of motion whose coordinate and/or momenta are complex variables. Time is always considered as a real variable). Exactly at the bifurcation point several orbits coming from the complex and/or the real plane coalesce.

The inclusion of complex orbits in semiclassical formulae is thus an important issue and may be considered as part of a general program on *exact semiclassical mechanics*. The aim of this program is to compute exact quantum results from semiclassical analysis (see for example the review by A. Voros [53] . As Balian and Bloch have pointed out [4] , quantum mechanics can be fully reconstructed from classical trajectories if real as well as complex trajectories are included. This program, which for generic mixed systems remains for the moment at a formal level, has however been pursued successfully for one-dimensional autonomous systems [52, 54] as for the free motion of a particle on a compact surface of constant negative curvature [15] . In spite of the fact that the latter is a prototype of fully chaotic motion, the accomplishment of the program in this case heavily relies in the fact that the semiclassical expressions are exact.

The appearance of complex trajectories in exact semiclassical expressions is of course easy to understand. Consider for example the tunneling effect. By definition this effect is classically forbidden and hence is not taken into account in expressions such as Eq. (1.0.1) where only real classical trajectories are included. It is however remarkable that only complex *classical* paths (and not arbitrary complex paths) are needed in order to correctly describe it [4]. Ref. [31], where complex periodic orbits are used in order to include tunneling effects in the computation of the density of states of a mixed system, provides an illustration of this point. Other interesting recent works exploring different aspects of tunneling in the mixed or fully chaotic regimes can be found in Refs.[32, 12, 50, 22, 19, 24].

As mentioned before, the amplitude $A[\mathbf{p}]$ in Eq.(1.0.1) depends on the geometry of the classical flow and is connected to the linearized motion around \mathbf{p} . If \mathbf{p} is isolated, it can be expressed in terms of the eigenvalues of the monodromy matrix \mathbb{M} at \mathbf{p} which is, by definition, the linearized part of the transversal map Φ after one return in the neighborhood of any point \mathbf{o} of \mathbf{p} . In order to find these eigenvalues, one can, for example, diagonalise the quadratic part of the classical Hamiltonian by

constructing a suitable symplectic chart in which the later is made up of only the relevant geometric monomials. This procedure can be generalized to higher orders: it is the object of normal form theory. In hamiltonian physics, it has been introduced by [41] and [8] .

In order to obtain geometrical semiclassical \hbar expansions, one must therefore classify the generic situations which may happen in the neighborhood of periodic orbits as an external parameter is varied. In [34] [36] such a classification was performed for one-parameter hamiltonian systems with two degrees of freedom. In the original paper, the aim was to prove the genericity in the mathematical sense. In a more recent work [37], the authors use the Lie transformation theory developed in the context of many body classical problems [27, 20, and references therein], which is a powerful method for obtaining normal forms. It has also the great advantage of giving a systematic method that can be numerically implemented. However, because our final aim are semiclassical expansions, the classical objects we are interested in are the generating functions – the phases in Feynman integrals – rather than the explicit maps themselves. Especially when the bifurcation is of an extremal or transitional type (see section 4 below), the explicit expression for the generating functions is not trivially obtained from Meyer’s work. We will give a precise expression for them in section 6.

Before, the aim of section 2 is to introduce the one-parameter hamiltonian systems with two degrees of freedom, and of sections 3 and 4 is to give an original presentation of Meyer’s classification. We will neither use the language of singularity theory nor that of Lie transformations; we will only need some elementary algebraic operations and refer to a physicists’ intuitive approach of genericity. We will systematically use Taylor expansions and make generic statements about the coefficients.

In section 5, we make a detailed study of the structure of the geometrical classical skeleton based on the previous results and include systematically the complex fixed points in the analysis of each type of bifurcation. The reader familiar with the theory of normal forms or not interested in the proofs can directly go to this section were the essential ingredients are summarized.

The quantum expansions based on these classical structures are finally introduced in section 7. We go through all the different types of bifurcations encountered in two-dimensional systems and systematically evaluate the contribution of the real as well as the complex periodic orbits involved. The main difficulty is associated with the complex orbits. When one uses the steepest descent method in semiclassical approximations one must take into account global properties of the integration domain to find the possible deformation of the integration path that determines the contribution of the orbit. In particular, the direction in which one crosses the saddle points must be known in order to correctly

compute the Maslov indices of the orbits. When the dynamics is embedded in a smooth family, the choice of critical points may change discontinuously as the external parameters are varied. This produces the well known Stokes'phenomenon extensively studied by BERRY & HOWLS [6, 7] (see also [28]). The difficulties when dealing with this highly non-local properties may be attenuated if one considers only complex orbits near a bifurcation. Aside from these considerations, our interest is to compute the relative weight of the contribution of complex periodic orbits. Several previous studies in the mixed regime [30, 31] have shown that some complex orbits give an exponentially small contribution. Nevertheless we will argue in section 7 that these exponentially small contributions occur in only one type of bifurcation: the extremal case. For all the others the contribution of complex periodic orbits – if they exist – is proportional to a power law in \hbar . In other words, these complex periodic orbits contribute to $a_k \hbar^k$ -like terms in Eq.(1.0.1).

2 Poincaré reduction

Let us consider an autonomous hamiltonian system with two degrees of freedom. Denote G its hamiltonian, defined on a four-dimensional phase space. Let \mathbf{p} be a periodic orbit of period $T > 0$ lying in the three-dimensional energy shell $\Sigma(E)$. Let us construct another three-dimensional submanifold \mathfrak{S} which is sufficiently small to intersect \mathbf{p} only once. Let $\mathfrak{o} \stackrel{\text{def}}{=} \mathfrak{S} \cap \mathbf{p}$ and $\mathfrak{s}(E) \stackrel{\text{def}}{=} \mathfrak{S} \cap \Sigma(E)$ (see figure 1).

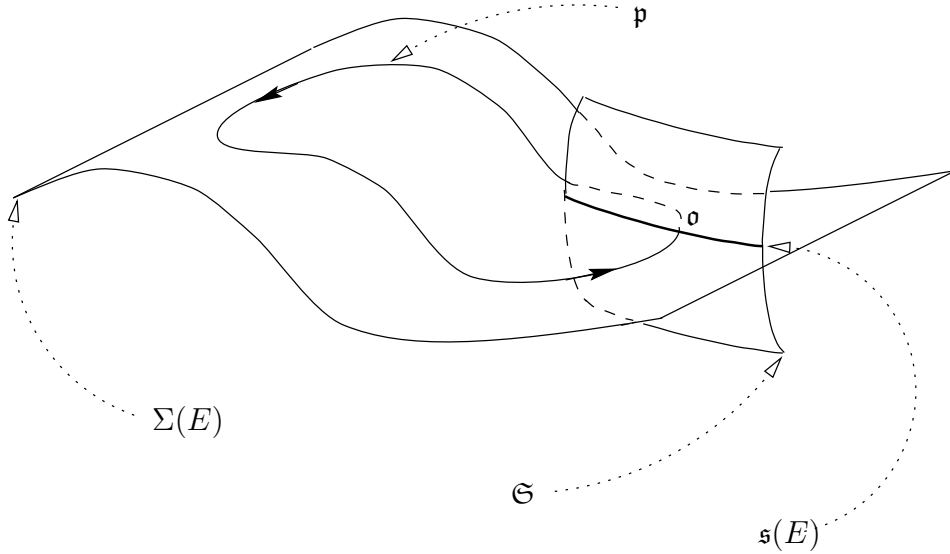


Figure 1: *Poincaré reduction in the neighborhood of a periodic orbit \mathbf{p} .*

Let us call ϑ the angle coordinate defined in a tubular neighborhood of \mathbf{p} . On \mathbf{p} , it represents the arclength coordinate modulo the total length τ of \mathbf{p} counted in a suitable unit system. Moreover there is, by construction, a one-to-one relation between ϑ and the time t since everywhere locally around \mathbf{p} , $d\vartheta/dt \neq 0$. One can thus consider t to be a smooth function of ϑ , $t(\vartheta)$. Let I be the coordinate canonically conjugate to ϑ and (p, q) the other two symplectic coordinates. The hamiltonian G is locally a function of these variables: $G(I, p, \vartheta, q)$. Since $\partial G/\partial I = d\vartheta/dt \neq 0$, one can make use of the implicit function theorem to define a function $H(p, q; \vartheta; E)$ by

$$G(I, p, \vartheta, q) = E \iff I = H(p, q; \vartheta; E) . \quad (2.0.2)$$

From the expression of the partial derivative of the implicit functions we get

$$\frac{dp}{d\vartheta} = \frac{dp/dt}{d\vartheta/dt} = -\frac{\partial G/\partial q}{\partial G/\partial I} = +\frac{\partial H}{\partial q} ; \quad (2.0.3a)$$

$$\frac{dq}{d\vartheta} = \frac{dq/dt}{d\vartheta/dt} = +\frac{\partial G/\partial p}{\partial G/\partial I} = -\frac{\partial H}{\partial p} . \quad (2.0.3b)$$

Therefore, this construction – due to [55, chapter XII, §141], see also [8, chapter VI, §3] – reduces locally the original two freedom system to an equivalent one-dimensional system. The price to pay is that the new system is nonautonomous since the new hamiltonian H depends on the new time ϑ . Nevertheless, the reduced dynamics, now living on $\mathfrak{s}(E)$, is τ -periodic due to the periodicity of ϑ . \mathfrak{o} is an equilibrium point of H obeying, $\forall \vartheta$,

$$\left. \frac{\partial H}{\partial p} \right|_{\substack{p=0 \\ q=0}} = 0 \quad \text{and} \quad \left. \frac{\partial H}{\partial q} \right|_{\substack{p=0 \\ q=0}} = 0 , \quad (2.0.4)$$

if (p, q) is centered on \mathfrak{o} . The reduced dynamics mimics the transversal motion stroboscopically with period τ . The corresponding one-step iteration map $\Phi(E; \tau)$ defined on $\mathfrak{s}(E)$ is the so-called Poincaré map. Exploring what happens around \mathfrak{p} when we go outside $\Sigma(E)$ is equivalent to considering that the reduced dynamics is embedded in a one-parameter (E) unfolding. Since the symplectic chart on $\mathfrak{s}(E)$ is defined in a neighborhood of it, we will adopt a passive point of view by considering that we stay for all E in the same bidimensional phase space \mathcal{P} .

3 Normal form for an individual hamiltonian

From the preceding arguments, changing slightly the notation, let us start from a smooth Hamiltonian $H(p, q; t)$ defined on a bidimensional phase space \mathcal{P} and which is τ -periodic in time ($\tau > 0$). We will assume that the origin is an isolated fixed point.

If we denote $w \stackrel{\text{def}}{=} \begin{pmatrix} p \\ q \end{pmatrix}$ and $\nabla_w \stackrel{\text{def}}{=} \begin{pmatrix} \partial_p \\ \partial_q \end{pmatrix}$, the equations of motion can be written in the compact form

$$\dot{w}(t) = \mathbb{J} \nabla_w H(w(t); t) \quad (3.0.5)$$

where “ $\dot{\cdot}$ ” $\stackrel{\text{def}}{=} d/dt$ and $\mathbb{J} \stackrel{\text{def}}{=} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

3.1 Elimination of the time-dependence in the linearized motion

The classification of the dynamics in the neighborhood of an isolated fixed point is based on the linearization of the motion around it.

Linearizing equation (3.0.5) around the origin it follows

$$\dot{w}(t) = \mathbb{A}(t) w(t) \quad (3.1.1)$$

where

$$\mathbb{A}(t) \stackrel{\text{def}}{=} \mathbb{J} \text{Hess}[H(w; t)]|_{w=\binom{0}{0}} = \mathbb{J} \begin{pmatrix} \partial_{p,p}^2 H(w; t) & \partial_{p,q}^2 H(w; t) \\ \partial_{q,p}^2 H(w; t) & \partial_{q,q}^2 H(w; t) \end{pmatrix} |_{w=\binom{0}{0}} . \quad (3.1.2)$$

It is natural to expect that the linearized dynamics (3.1.1) is roughly akin to the true dynamics provided we are close enough to the origin. We will come back to this point at the end of the subsection.

Recall that a symplectic (complex) matrix \mathbb{S} is defined by the condition

$${}^t\mathbb{S} \mathbb{J} \mathbb{S} = \mathbb{J} \quad ({}^t\mathbb{S} \text{ denotes the transpose of } \mathbb{S}) \quad (3.1.3)$$

In two dimensions, this is equivalent to

$$\det \mathbb{S} = 1 . \quad (3.1.4)$$

By definition [1, definition 3.1.13] a complex matrix \mathbb{K} is an infinitesimal symplectic matrix if it verifies

$$\mathbb{J} \mathbb{K} = -{}^t\mathbb{K} \mathbb{J} . \quad (3.1.5)$$

Then it can easily be checked that $\mathbb{A}(t)$ is an infinitesimal symplectic matrix. As its name suggests, the exponential of an infinitesimal symplectic matrix is symplectic.

Let $\mathbb{U}(t)$ be the evolution matrix associated with $\mathbb{A}(t)$. In other words, $w(t) = \mathbb{U}(t) w_0$ is the unique solution of equation (3.1.1) such that $w(0) = w_0$. We have

$$\dot{\mathbb{U}}(t) = \mathbb{A}(t) \mathbb{U}(t) \quad \text{with} \quad \mathbb{U}(0) = \mathbb{1} \stackrel{\text{def}}{=} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} . \quad (3.1.6)$$

One often writes the formal solution of the latter equation as a time ordered product

$$\mathbb{U}(t) = \mathcal{T} e^{\int_0^t \mathbb{A}(t') dt'} . \quad (3.1.7)$$

It is straightforward to show that $\mathbb{U}(t)$ is symplectic for all t . Let \mathbb{L} be any time-independent infinitesimal symplectic matrix. Then, being the product of two symplectic matrices, $\mathbb{F}(t) \stackrel{\text{def}}{=} e^{t\mathbb{L}} [\mathbb{U}(t)]^{-1}$ is also symplectic. If we make the symplectic, time-dependent change of coordinates $\tilde{w} = \mathbb{F}(t) w$, then equation (3.1.1) becomes $\dot{\tilde{w}}(t) = \mathbb{L} \tilde{w}(t)$. All the information of the original linearized dynamics has been hidden in the change of variables. Let us keep in mind that $H(p, q; t)$ is a reduced hamiltonian and hence that \mathcal{P} is a (local) Poincaré surface of section centered at one point of an isolated periodic orbit \mathbf{p} of a higher dimensional system. Then, the time-dependence of the flow defined by equation (3.0.5) can be seen as a transversal projection of the non-reduced autonomous flow as we move along \mathbf{p} . The time periodicity of the reduced flow comes from the fact that after one cycle along \mathbf{p} we are back on \mathcal{P} . Hence we should consider only real τ -periodic symplectic changes of coordinates, $\mathbb{F}(\tau) = \mathbb{1}$. \mathbb{L} must then be chosen such that $e^{\tau\mathbb{L}} = \mathbb{M} \stackrel{\text{def}}{=} \mathbb{U}(\tau)$, the so-called monodromy matrix. $\tau\mathbb{L}$ exists and can be any logarithm of \mathbb{M} since \mathbb{M} has no zero eigenvalues as a consequence of equation (3.1.4). Nevertheless, it may happen that no *real* logarithm of \mathbb{M} exists. In this case one possibility is to relax the constraint on $\mathbb{F}(t)$ by retaining only the intersection with the Poincaré surface of section every second crossing. That means that $\mathbb{F}(t)$ must only be 2τ -periodic. Because $\mathbb{U}(2\tau) = [\mathbb{U}(\tau)]^2$ and using the fact that every real matrix that has a square root has a real logarithm [37, theorem 2 of II.E], we can indeed choose the infinitesimal symplectic matrix \mathbb{L} to be real without loss of generality. This is the statement of the Floquet theorem [23] (see also the work of CHERRY [17, 16, 18]). There exists a symplectic coordinate system in which the linearized motion (3.1.1) is governed by a constant real infinitesimal symplectic matrix \mathbb{L} . Equivalently, in a well chosen symplectic chart, the hamiltonian can be written

$$H(p, q; t) = \underbrace{\frac{1}{2} \mathbf{c} p^2 + \mathbf{d} p q + \frac{1}{2} \mathbf{e} q^2}_{\stackrel{\text{def}}{=} h^{(2)}(p, q)} + h^{(\geq 3)}(p, q; t) \quad (3.1.8)$$

in a neighborhood of the origin after having subtracted the irrelevant value of $H(0, 0; t)$. $(\mathbf{c}, \mathbf{d}, \mathbf{e}) \in \mathbb{R}^3$ and $h^{(\geq 3)}$ is a τ -periodic function whose first and second derivatives with respect to p or q vanish at the origin for all t . In this case $\mathbb{A}(t)$ is a constant matrix \mathbb{A} and, from definition (3.1.2)

$$\mathbb{L} = \mathbb{A} = \begin{pmatrix} -\mathbf{d} & -\mathbf{e} \\ \mathbf{c} & \mathbf{d} \end{pmatrix}, \quad (3.1.9)$$

and the monodromy matrix \mathbb{M} is such that

$$\mathbb{M}^{|\varsigma|} = e^{\varsigma\tau\mathbb{L}} \quad \text{with} \quad \begin{cases} \varsigma = \pm 1 & \text{if } \mathbb{M} \text{ has a real logarithm;} \\ \varsigma = \pm 2 & \text{if } \mathbb{M} \text{ has no real logarithm.} \end{cases} \quad (3.1.10)$$

The choice of sign of ς depends on the choice of the arrow of time. The linearized motion, even though it gives an idea of the nature of the true dynamics, cannot reproduce its fine structure. The time elimination makes the quadratic part of H integrable even though we have started from an a priori non-integrable one. The chaotic complexity is hidden in the higher-order terms and the aim of normal form theory is to retain only the main actors of this rich structure.

3.2 Normal form for the quadratic part of the hamiltonian

After having eliminated the time-dependence in the quadratic part of the hamiltonian, we are now ready to carry on the simplification of $h^{(2)}(p, q)$ by constructing a more suitable symplectic chart (p', q') . The general classification of the normal forms of quadratic hamiltonians in any number of dimensions has been done a long time ago by [56]. In our case, since we are dealing with a bidimensional phase space, we will give a more hand-waving demonstration which however illustrates the main guidelines for deriving Meyer's classification in a pedestrian way.

Let us characterize the corresponding symplectic transformation by a generating smooth function $S(p', q)$. The change of coordinates $(p, q) \mapsto (p', q')$ is implicitly defined by

$$\begin{cases} p = \partial_q S(p', q) ; & (3.2.1a) \\ q' = \partial_{p'} S(p', q) & (3.2.1b) \end{cases}$$

in the neighborhood of the origin, provided that

$$\partial_{p',q}^2 S(0,0) \neq 0. \quad (3.2.2)$$

Since without loss of generality the transformation can be assumed to leave the origin invariant, $S(p', q)$ can be written as

$$S(p', q) = \frac{1}{2} \gamma p'^2 + \delta p'q + \frac{1}{2} \eta q^2 + O_{p',q}(3) \quad (3.2.3)$$

where $(\gamma, \eta) \in \mathbb{R}^2$ and $\delta \in \mathbb{R} \setminus \{0\}$, by virtue of condition (3.2.2). It is thus straightforward from equations (3.2.1) to obtain the linear part of the symplectic change of coordinates and deduce the new

expression of the quadratic hamiltonian

$$\begin{aligned}
h^{(2)}(p', q') = & \frac{1}{2} \underbrace{\left[\frac{\gamma^2}{\delta^2} (c \eta^2 + 2d \eta + e) - 2\gamma(d + c \eta) + c \delta^2 \right]}_{\stackrel{\text{def}}{=} c'} p'^2 \\
& + \underbrace{\left[-\frac{\gamma}{\delta^2} (c \eta^2 + 2d \eta + e) + d + c \eta \right]}_{\stackrel{\text{def}}{=} d'} p' q' \\
& + \frac{1}{2} \underbrace{\left[\frac{1}{\delta^2} (c \eta^2 + 2d \eta + e) \right]}_{\stackrel{\text{def}}{=} e'} q'^2 + O_{p', q'}(3) \quad (3.2.4)
\end{aligned}$$

Generically none of the coefficients (c, d, e) vanish. Nevertheless, we will also deal with situations where one real scalar like c, d, e or $\text{tr}(\mathbb{M}^2) - 2$ can vanish a priori, since we are going to study not only individual hamiltonians but one-parameter unfoldings. Actually, we can choose (δ, γ, η) in order to cancel some of the coefficients (c', d', e') . For example, the cancellation of e' implies solving an algebraic equation of degree at most two in η whose discriminant is

$$d^2 - ce = -\det(\mathbb{L}). \quad (3.2.5)$$

The simplified form we want to obtain depends essentially on the nature of the eigenvalues of the monodromy matrix which governs the qualitative regime about the fixed point.

Let us then briefly review the spectrum $\text{sp}(\mathbb{M})$ of \mathbb{M} . Since the monodromy matrix is symplectic, from condition (3.1.4) we have

$$\det(\mathbb{M}) = 1 \quad (3.2.6)$$

Moreover \mathbb{M} is real and hence $\text{sp}(\mathbb{M})$ is symmetric with respect to the real axis: $v \in \text{sp}(\mathbb{M}) \Rightarrow v^* \in \text{sp}(\mathbb{M})$.

Let us now introduce the usual equivalence relation between matrices. Two matrices \mathbb{A} and \mathbb{B} will be equivalent – we will denote $\mathbb{A} \stackrel{\circ}{=} \mathbb{B}$ – if by definition they represent the same linear map but expressed in two different bases; in other words there exists an invertible matrix \mathbb{P} such that $\mathbb{A} = \mathbb{P}^{-1} \mathbb{B} \mathbb{P}$. Without loss of generality \mathbb{P} can be chosen with determinant one and thus symplectic. The Jordan

decomposition of \mathbb{M} asserts that [51, chapter VI]

$$\mathbb{M} \doteq \begin{pmatrix} \mu & 0 \\ \tilde{\mu} & \frac{1}{\mu} \end{pmatrix} \quad \text{where} \quad \mu \in \mathbb{R} \cup U(1) \setminus \{0\} \quad \text{and} \quad \tilde{\mu} \in \{0, 1\}. \quad (3.2.7)$$

It is possible to distinguish three different cases:

- (i). $\text{sp}(\mathbb{M}) \subset \mathbb{R} \setminus \{-1, 0, 1\} \iff |\text{tr}(\mathbb{M})| > 2$ and \mathfrak{o} is said to be an *unstable* fixed point;
- (ii). $\text{sp}(\mathbb{M}) \subset \{-1, 1\} \iff |\text{tr}(\mathbb{M})| = 2$ and \mathfrak{o} is said to be a *marginal* fixed point¹;
- (iii). $\text{sp}(\mathbb{M}) \subset U(1) \setminus \{-1, 1\} \iff |\text{tr}(\mathbb{M})| < 2$ and \mathfrak{o} is said to be a *stable* fixed point².

Generically, in the absence of any symmetry but if the dynamics is embedded in a one parameter family, the two eigenvalues of \mathbb{M} may be degenerate (the marginal case). It would require at least two parameters to get $\mathbb{M} \in \{1, -1\}$ and, following our philosophy, we discard from the discussion all sets of codimension greater than two. In other words we will have $\tilde{\mu} = 1$ in the generic marginal case, which can therefore be divided in two sub cases

- (ii)' $\text{tr}(\mathbb{M}) = +2$ and generically $\mathbb{M} \doteq \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$;
- (ii)'' $\text{tr}(\mathbb{M}) = -2$ and generically $\mathbb{M} \doteq \begin{pmatrix} -1 & 0 \\ 1 & -1 \end{pmatrix}$.

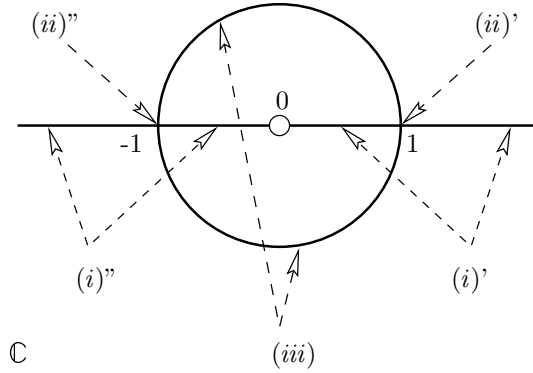


Figure 2: *Geometrical representation of the locus of $\text{sp} \mathbb{M}$ in the \mathbb{C} plane.*

Moreover, case (i) is usually also subdivided in the following way

- (i)' $\text{tr}(\mathbb{M}) > +2$ and \mathfrak{o} is said to be a *hyperbolic* fixed point;
- (i)'' $\text{tr}(\mathbb{M}) < -2$ and \mathfrak{o} is said to be an *inverse hyperbolic* fixed point.

We now want to transpose this classification onto the spectrum $\text{sp}(\mathbb{L})$ of \mathbb{L} . Recall that the relation between \mathbb{L} and \mathbb{M} is given by (3.1.10). It can be easily verified that in cases (i)'' and (ii)'' the monodromy matrix has no real logarithm and hence $\varsigma = \pm 2$ while in all others cases – (i)', (ii)', (iii) – we can take $\varsigma = \pm 1$.

Since \mathbb{L} is infinitesimal symplectic, property (3.1.5) implies that its spectrum $\text{sp}(\mathbb{L})$ is symmetric with respect to zero: $v \in \text{sp}(\mathbb{L}) \Rightarrow -v \in \text{sp}(\mathbb{L})$. Moreover, because \mathbb{L} is real, $\text{sp}(\mathbb{L})$ is symmetric with respect to the real axis: $v \in \text{sp}(\mathbb{L}) \Rightarrow v^* \in \text{sp}(\mathbb{L})$. Let

$$\mathbb{L} \doteq \begin{pmatrix} \lambda & 0 \\ \tilde{\lambda} & -\lambda \end{pmatrix} \quad \text{where} \quad \lambda \in \mathbb{R} \cup (i\mathbb{R}) \quad \text{and} \quad \tilde{\lambda} \in \{0, 1\}, \quad (3.2.8)$$

be the Jordan decomposition of \mathbb{L} . Then, relation (3.1.10) yields

$$\mathbb{M}^{|\varsigma|} \doteq \begin{pmatrix} \mu^{|\varsigma|} & 0 \\ \tilde{\mu} [2 - |\varsigma| + (|\varsigma| - 1)(\mu + \frac{1}{\mu})] & \frac{1}{\mu^{|\varsigma|}} \end{pmatrix} \doteq \begin{pmatrix} e^{\varsigma\tau\lambda} & 0 \\ \frac{\tilde{\lambda}}{\lambda} \sinh(\varsigma\tau\lambda) & e^{-\varsigma\tau\lambda} \end{pmatrix}, \quad (3.2.9)$$

which implies that $e^{2\tau\lambda} = \mu^2$ with $\lambda \in \mathbb{R} \cup (i\mathbb{R})$. Therefore, in the unstable case, we have $\lambda \in \mathbb{R} \setminus \{0\}$ and then $-\det(\mathbb{L}) > 0$. In the stable case, we have $\lambda\tau \in i(\mathbb{R} \setminus \pi\mathbb{Z})$ and then $-\det(\mathbb{L}) < 0$. In both cases $\tilde{\mu} = 0$ and $\tilde{\lambda} = 0$. In the marginal case, we have $\mathbb{M}^2 = 1 + \mathbb{X}$ where, generically,

$$\mathbb{X} \in \left\{ \begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ -2 & 0 \end{pmatrix} \right\} \quad \text{modulo} \doteq . \quad (3.2.10)$$

Therefore, possibly changing the arrow of time, $\tau \mapsto -\tau$, we can choose

$$\mathbb{L} \doteq \frac{1}{2\tau} \begin{pmatrix} 0 & 0 \\ -2 & 0 \end{pmatrix}; \quad (3.2.11)$$

then $\lambda = 0$ and $\det(\mathbb{L}) = 0$. Other choices, say $\tilde{\mathbb{L}}$, of real infinitesimal symplectic matrices obeying relation (3.1.10) are possible and not necessarily with $\det(\tilde{\mathbb{L}}) = 0$. But following what we have done in subsection 3.1, we can make a (linear and $|\varsigma|\tau$ -periodic) symplectic change of variables corresponding to $\mathbb{F}(t) \stackrel{\text{def}}{=} e^{-t\tilde{\mathbb{L}}} e^{-t\mathbb{L}}$ in order to deal with an \mathbb{L} whose determinant vanishes. Summarizing

- (i). unstable case $\lambda \in \mathbb{R} \setminus \{0\}$ and $-\det(\mathbb{L}) > 0$;
- (ii). marginal case $\lambda = 0$ and $-\det(\mathbb{L}) = 0$;
- (iii). stable case $\lambda \in \tau^{-1}i(\mathbb{R} \setminus \pi\mathbb{Z})$ and $-\det(\mathbb{L}) < 0$.

We can now go back to the reduction of the quadratic part of the hamiltonian (3.2.4).

(i) Unstable case: We can choose at least one real value of η which makes e' vanish and, provided we take $\gamma = \pm c \delta^2 / [2\sqrt{-\det(\mathbb{L})}]$, we can also cancel c' . At last, we get $h^{(2)}(p', q') = \pm \sqrt{-\det(\mathbb{L})} p' q'$. The global sign of the hamiltonian becomes irrelevant by a time reversal transformation and we finally find a symplectic chart in which the quadratic Hamiltonian can be written as

$$h^{(2)}(p, q) = \lambda pq \quad \text{with} \quad \lambda \in \mathbb{R}^+ \setminus \{0\} \quad (3.2.12)$$

after having relabeled the new coordinates. $h^{(2)}(p, q)$ corresponds to an integrable dynamics whose trajectories lie on hyperbolae of center \mathfrak{o} and whose asymptotes, the p and q axes, are the two stable and unstable manifolds respectively (see the right column of table 1). Recall (cf Eq.(3.1.10)) that in case (i)" the hamiltonian $h^{(2)}(p, q)$ describes the stroboscopic view of the real motion with a period twice the original physical period τ . Technically such a prescription misses a -1 factor in the equations of motion (3.1.1) and what effectively happens is that the system goes from one branch of a hyperbola to the other branch between $t = 0$ and $t = \tau$. Then, between $t = \tau$ and $t = 2\tau$, it goes back to the original branch. The 2τ -stroboscopic view does not see this flipping.

(ii) Marginal case: If we have $\mathfrak{d} \neq 0$ and hence $c \neq 0$ we can choose $\eta = -\mathfrak{d}/c$ to get $c\eta^2 + 2\mathfrak{d}\eta + e = 0$. Then both \mathfrak{d}' and e' vanish. Moreover, we take $\delta = |\tau c|^{-1/2}$ to obtain $|c'| = 1/\tau$. By changing possibly t to $-t$ we get

$$h^{(2)}(p, q) = \frac{1}{2\tau} p^2 \quad (3.2.13)$$

after having relabeled the new coordinates. $h^{(2)}(p, q)$ corresponds to an integrable dynamics whose trajectories are straight lines parallel to the q axis (see the middle column of table 1). The distinction between cases (ii)' and (ii)" is analogous to that of the unstable cases. In (ii)", during one period τ , the system flips back and forth between a couple of lines symmetric with respect to \mathfrak{o} .

(iii) Stable case: We have $c\eta^2 + 2\mathfrak{d}\eta + e \neq 0$ for every real η and then e' can never be cancelled. Nevertheless we can choose $\gamma = \delta^2(c\eta + \mathfrak{d}) / (c\eta^2 + 2\mathfrak{d}\eta + e)$ in order to have $\mathfrak{d}' = 0$. Besides, we can make $e' = c'$ by choosing $\delta = |(c\eta^2 + 2\mathfrak{d}\eta + e)|^{1/2} \det(\mathbb{L})^{-1/4}$. If we define $\omega \stackrel{\text{def}}{=} |\lambda|$ and possibly change the arrow of time we find the following normal form

$$h^{(2)}(p, q) = \frac{1}{2} \omega (p^2 + q^2) \quad \text{with} \quad \omega \in \frac{1}{\tau} (\mathbb{R}^+ \setminus \pi\mathbb{Z}) \quad (3.2.14)$$

after having relabeled the new coordinates. $h^{(2)}(p, q)$ corresponds to an integrable dynamics whose trajectories are circles centered on \mathfrak{o} and whose angular speed is ω (see the right column of table 1).

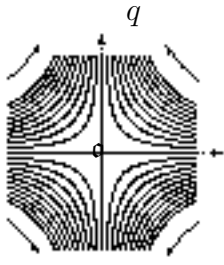
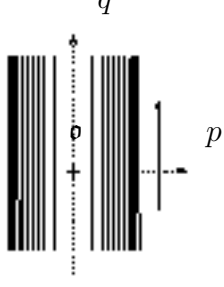
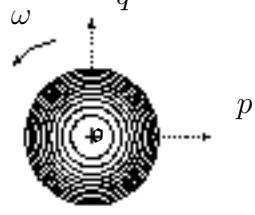
	(i) unstable case		(ii) marginal case		(iii) stable case
	(i)'	(i)''	(ii)'	(ii)''	
M: monodromy matrix	$\text{sp}(\mathbb{M}) \subset \mathbb{R} \setminus \{-1, 0, 1\}$		$\mathbb{M} \doteq \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$	$\mathbb{M} \doteq \begin{pmatrix} -1 & 0 \\ 1 & -1 \end{pmatrix}$	$\text{sp}(\mathbb{M}) \subset U(1) \setminus \{-1, 1\}$
	$\text{tr}(\mathbb{M}) > 2$	$\text{tr}(\mathbb{M}) < -2$			$-2 < \text{tr}(\mathbb{M}) < 2$
$\varsigma \in \{1, \pm 2\}$	$\varsigma = 1$	$ \varsigma = 2$	$\varsigma = 1$	$\varsigma = -2$	$\varsigma = 1$
$\mathbb{M}^{ \varsigma } = e^{\varsigma\tau\mathbb{L}}$	$-\det(\mathbb{L}) > 0$		$\mathbb{L} \doteq \frac{1}{2\tau} \begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix}$		$-\det(\mathbb{L}) < 0$
$\text{sp}(\mathbb{L}) = \{\lambda, -\lambda\}$	$\lambda \in \mathbb{R}^+ \setminus \{0\}$				$\omega = -i\lambda \in \frac{1}{\tau}(\mathbb{R}^+ \setminus \pi\mathbb{Z})$
$h^{(2)}(p, q) =$	λpq		$\frac{1}{2\tau} p^2$		$\frac{1}{2}\omega(p^2 + q^2)$
flow around the origin ($\omega = \lambda $)					

Table 1: Classification of generic quadratic hamiltonians and normal forms in the neighborhood of an isolated fixed point \mathfrak{o} .

Together with equations (3.2.12), (3.2.13) and (3.2.14) it constitutes our first classification of normal forms. In the next subsections, we will refine it by including higher order terms in the hamiltonians and by considering one-parameter families of hamiltonians and normal forms.

3.3 Birkhoff normal form for the full hamiltonian: unstable and stable cases

If we write the equations of motion in complex coordinates

$$\begin{cases} z(p, q) = p + iq & (3.3.1a) \\ \bar{z}(p, q) = p - iq & (3.3.1b) \end{cases} \iff \begin{cases} p(z, \bar{z}) = \frac{1}{2}(\bar{z} + z) & (3.3.1c) \\ q(z, \bar{z}) = \frac{1}{2}i(\bar{z} - z), & (3.3.1d) \end{cases}$$

then for the stable case (3.2.14) we get the differential equations $\dot{z} = i\omega z$ and $\dot{\bar{z}} = -i\omega \bar{z}$ which correspond to the complexified hamiltonian $h^{(2)}(z, \bar{z}) = -i\omega z\bar{z}$ with (z, \bar{z}) playing the role of complexified canonical variables. Moreover, from a function $F(z', \bar{z})$ holomorphic with respect to its two complex variables in the neighborhood of the origin and satisfying $\partial_{z', \bar{z}} F(0, 0) \neq 0$, one can construct a *real* symplectic

transformation of the form (3.2.1) for the generating function $S(p', q) = \frac{1}{2}(p'q' + pq - \Im[F(z', \bar{z})])$, where (p, q) and (z, \bar{z}) are connected by the relations (3.3.1) and similarly for (p', q') , (z', \bar{z}') .

Then, we will study both unstable and stable cases from a hamiltonian of the form

$$H(u, v; t) = \lambda uv + \sum_{m \in \mathbb{Z}} \sum_{\substack{(\alpha, \beta) \in \mathbb{N}^2 \\ \alpha + \beta = k}} \mathfrak{h}_{m\alpha\beta} u^\alpha v^\beta e^{im\frac{2\pi}{\tau}t} + O_{u,v}(k+1) \quad (3.3.2)$$

where we made use of the τ -periodicity.

From what precedes, if we are dealing with an unstable case then $\lambda \in \mathbb{R}^+ \setminus \{0\}$, (u, v) are the real coordinates (p, q) and the coefficients $\mathfrak{h}_{m\alpha\beta}$ are complex numbers with a priori only one constraint, $\mathfrak{h}_{m\alpha\beta}^* = \mathfrak{h}_{-m\alpha\beta}$ ($H(p, q; t)$ must be real for real (p, q)). If the origin is stable, $\lambda \in -i\mathbb{R}^+ \setminus \{0\}$ and (u, v) are the complexified coordinates (z, \bar{z}) from which the real ones are obtained via equations (3.3.1). In this case, $\mathfrak{h}_{m\alpha\beta}$ are complex numbers such that $\mathfrak{h}_{m\alpha\beta}^* = \mathfrak{h}_{-m\beta\alpha}$ ($H(z, \bar{z}; t)$ must be real for $\bar{z} = z^*$). In both cases, k is the least integer greater or equal to three such that $\mathfrak{h}_{m\alpha\beta} \neq 0$ for some (α, β) with $\alpha + \beta = k$.

We now want to simplify the expression of H by eliminating as many terms as possible in the series (3.3.2). The symplectic change of coordinates is given by the generating function

$$S(u', v; t) = u'v + \sum_{m \in \mathbb{Z}} \sum_{\substack{(\alpha, \beta) \in \mathbb{N}^2 \\ \alpha + \beta = k}} \sigma_{m\alpha\beta} u'^\alpha v^\beta e^{im\frac{2\pi}{\tau}t} + O_{u',v}(k+1) \quad (3.3.3)$$

where the transformation of degree less or equal to $k-1$ corresponds to the identity since we do not want to modify the terms of the same order in H . $\sigma_{m\alpha\beta}$ are complex coefficients which we now fix. The symplectic transformation generated by the function (3.3.3) is

$$\left\{ \begin{array}{l} u(u', v'; t) = u' + \frac{1}{v'} \sum_{m \in \mathbb{Z}} \sum_{\substack{(\alpha, \beta) \in \mathbb{N}^2 \\ \alpha + \beta = k}} \beta \sigma_{m\alpha\beta} u'^\alpha v'^\beta e^{im\frac{2\pi}{\tau}t} + O_{u',v'}(k) \end{array} \right. \quad (3.3.4a)$$

$$\left\{ \begin{array}{l} v(u', v'; t) = v' - \frac{1}{u'} \sum_{m \in \mathbb{Z}} \sum_{\substack{(\alpha, \beta) \in \mathbb{N}^2 \\ \alpha + \beta = k}} \alpha \sigma_{m\alpha\beta} u'^\alpha v'^\beta e^{im\frac{2\pi}{\tau}t} + O_{u',v'}(k) \end{array} \right. \quad (3.3.4b)$$

and the hamiltonian $H'(u', v'; t) = H[u(u', v'; t), v(u', v'; t); t] + \frac{\partial S}{\partial t}[u', v(u', v'; t); t]$ takes the form

$$H'(u', v'; t) = \lambda u' v' + \sum_{m \in \mathbb{Z}} \sum_{\substack{(\alpha, \beta) \in \mathbb{N}^2 \\ \alpha + \beta = k}} \left[\mathfrak{h}_{m\alpha\beta} + \left(\lambda(\beta - \alpha) + im \frac{2\pi}{\tau} \right) \sigma_{m\alpha\beta} \right] u'^\alpha v'^\beta e^{im \frac{2\pi}{\tau} t} + O_{u', v'}(k + 1). \quad (3.3.5)$$

By taking $\sigma_{m\alpha\beta}$ so that the term between square brackets vanishes we can cancel all the terms in the double sum except the ones which correspond to m, α, β such that the so-called resonant condition is fulfilled

$$\lambda(\beta - \alpha) + im \frac{2\pi}{\tau} = 0. \quad (3.3.6)$$

Now the original hamiltonian may be simplified as follows. If we assume that all the non-resonant terms have been cancelled up to degree $k - 1$ in (p, q) , we can get rid of those of degree k with the help of the generating function (3.3.3) provided we choose

$$\begin{cases} \sigma_{m\alpha\beta} = -\frac{\mathfrak{h}_{m\alpha\beta}}{\lambda(\beta - \alpha) + im \frac{2\pi}{\tau}} & \text{for the non-resonant } m, \alpha, \beta, \\ \sigma_{m\alpha\beta} = 0 & \text{for the resonant } m, \alpha, \beta. \end{cases} \quad (3.3.7)$$

Since we can repeat this procedure order by order, we are left with a hamiltonian whose normal form is made up exclusively of resonant terms. This obstruction to the linearization of differential equations is well known from the works of Poincaré, Dulac and later on Sternberg, Grobman and Hartman [2] among others. It is remarkable that the resonant relations which prevent the reduction via diffeomorphism are the same resonant relations as those we obtained above, even though they lie in a more restricted class of coordinate transformations (the symplectic ones). The convergence of the normalization procedure of a dynamical system is an old and delicate problem because of the possibly small denominators in expression (3.3.7).

Let us now discuss the nature of the resonant terms depending on whether the origin is unstable or stable.

(i) Unstable case: Since $\lambda \in \mathbb{R}^+ \setminus \{0\}$ the resonant relation is fulfilled only if $m = 0$ and $\alpha = \beta$. Thus, for $n \in \mathbb{N} \setminus \{0, 1\}$ there always exists a real symplectic coordinate chart in which the hamiltonian

has the form

$$H(p, q; t) = \lambda pq + \sum_{k=2}^n \mathfrak{h}_k (pq)^k + O_{p,q}(2n+1) \quad (3.3.8)$$

for some $\mathfrak{h}_k \in \mathbb{R}$. This normal form has been derived for the first time by [8, § III.8] [25]. $n \geq 2$ is a cut off integer whose value fixes the accuracy with which we want to reproduce the full dynamics and $2n$ will be called the order of the normal form. The limit $n \rightarrow \infty$ has been shown to be locally convergent [39, §4] and hence the dynamics is equivalent to an integrable one.

(iii) Stable case: We have $\lambda = i\omega \in \frac{1}{\tau} i(\mathbb{R}^+ \setminus \pi\mathbb{Z})$.

$\rightarrow \frac{\omega\tau}{2\pi} \notin \mathbb{Q}$: The hamiltonian can also be reduced to a Birkhoff normal form, which in real coordinates, is

$$H(p, q; t) = \frac{1}{2} \omega (p^2 + q^2) + \sum_{k=2}^n \tilde{\mathfrak{a}}_k (p^2 + q^2)^k + O_{p,q}(2n+1). \quad (3.3.9)$$

for some $\tilde{\mathfrak{a}}_k \in \mathbb{R}$ and for every integer $n \geq 2$. In action-angle variables,

$$\begin{cases} I = \frac{1}{2} (p^2 + q^2) & (3.3.10a) \\ \theta = \text{arctg}\left(\frac{q}{p}\right) & (3.3.10b) \end{cases}$$

\iff

$$\begin{cases} p = \sqrt{2I} \cos \theta & (3.3.10c) \\ q = \sqrt{2I} \sin \theta & (3.3.10d) \end{cases}$$

and after having rescaled the coefficients, $\mathfrak{a}_k \stackrel{\text{def}}{=} 2^k \tilde{\mathfrak{a}}_k$, it takes the form,

$$H(I, \theta; t) = \omega I + \sum_{k=2}^n \mathfrak{a}_k I^k + O_I(2n+1). \quad (3.3.11)$$

The problem of small denominators appears now clearly. The non-convergence of the normalization procedure as $n \rightarrow \infty$ is due to the fact that the rational numbers \mathbb{Q} are dense in \mathbb{R} : $\sigma_{m\alpha\beta}$ in equation (3.3.7) can be arbitrarily large if we take $\beta - \alpha$ small enough.

$\rightarrow \frac{\omega\tau}{\pi} \in \mathbb{Q} \setminus \mathbb{N}$: Let $\ell \in \mathbb{N} \setminus \{0, 1, 2\}$ be the smallest integer such that it exists $r \in \mathbb{N} \setminus \{0\}$ with

$$\omega = \frac{r}{\ell} \frac{2\pi}{\tau}. \quad (3.3.12)$$

Then, the resonant terms correspond to $(m, \alpha, \beta) \in \mathbb{Z} \times \mathbb{N}^2$ obeying $(\beta - \alpha)r = -\ell m$. They are of degree $s \stackrel{\text{def}}{=} \alpha + \beta$ in (p, q) . We can get rid of the time-dependence of the normal form if we describe the dynamics in a frame uniformly rotating around the origin at angular speed ω . This corresponds

to the symplectic transformation

$$\begin{cases} z & \longmapsto z'(z, \bar{z}) = z e^{+i\omega t} \\ \bar{z} & \longmapsto \bar{z}'(z, \bar{z}) = \bar{z} e^{-i\omega t} \end{cases} \quad \text{generated by } S(z', \bar{z}) = z' \bar{z} e^{-i\omega t}, \quad (3.3.13)$$

leading to

$$H'(z', \bar{z}'; t) = H(z' e^{-i\omega t}, \bar{z}' e^{i\omega t}; t) + \frac{\partial S}{\partial t}(z', \bar{z}' e^{i\omega t}) \quad (3.3.14)$$

which has no longer any quadratic terms nor any time-dependence in all the resonant ones.

The latter can be classified in three sets R_- , R_0 and R_+ defined by

$$R_{\pm} \begin{cases} \beta = \alpha \pm \kappa \ell \\ s = 2\alpha \pm \kappa \ell \geq 3 \end{cases}, \quad R_0 \begin{cases} \beta = \alpha \\ s = 2\alpha \geq 3 \end{cases} \quad \text{for } \alpha = 0, 1, 2, \dots \text{ and } \kappa = 0, 1, 2, \dots \quad (3.3.15)$$

R_0 corresponds to the Birkhoff like terms. We obtain R_- from R_+ exchanging β with α .

These sets have a graphical interpretation in the (s, α) -plane (see figure 3 and Ref.[13]). The resonant terms are associated with the points of integer coordinates $s \geq \alpha \geq 0$ and $s \geq 3$ which lie on straight lines of slope two, whose intercept are strictly positive, zero and strictly negative for R_- , R_0 and R_+ respectively.

For $\ell \geq 3$, in addition of the Birkhoff like terms, the resonant terms of lowest degree in (p, q) are $(\alpha = 0, \beta = \ell, m = -r)$ and $(\alpha = \ell, \beta = 0, m = r)$. Hence, dropping the primes of the rotating coordinates, we have

$$H(z, \bar{z}; t) = \sum_{\substack{k \in \mathbb{N} \\ 2 \leq k \leq \ell/2}} \tilde{\mathbf{a}}_k (z \bar{z})^k + \mathbf{h}_{r\ell 0}^* \bar{z}^\ell + \mathbf{h}_{r\ell 0} z^\ell + O_{z, \bar{z}}(\ell + 1), \quad (3.3.16)$$

for some $\tilde{\mathbf{a}}_k \in \mathbb{R}$ and $\mathbf{h}_{r\ell 0} \in \mathbb{C}$. In real coordinates (p, q) , we get

$$H(p, q; t) = \sum_{\substack{k \in \mathbb{N} \\ 2 \leq k \leq \ell/2}} \tilde{\mathbf{a}}_k (p^2 + q^2)^k + |\mathbf{h}_{r\ell 0}| \Re[(p + iq)^\ell] + O_{p, q}(\ell + 1). \quad (3.3.17)$$

In action-angle variables

$$H(I, \theta; t) = \sum_{\substack{k \in \mathbb{N} \\ 2 \leq k \leq \ell/2}} \mathbf{a}_k I^k + \mathbf{b}_\ell I^{\ell/2} \cos(\ell \theta) + O_{\sqrt{I}}(\ell + 1) \quad (3.3.18)$$

where $\mathbf{a}_k \stackrel{\text{def}}{=} 2^k \tilde{\mathbf{a}}_k$, $\mathbf{b}_\ell \stackrel{\text{def}}{=} 2^{\ell/2} |\mathbf{h}_{r\ell 0}|$. In order to cancel the irrelevant phase $\theta_0 \stackrel{\text{def}}{=} \arg(\mathbf{h}_{r\ell 0})$, we have made a final rotation: $\theta \mapsto \theta - \theta_0$.

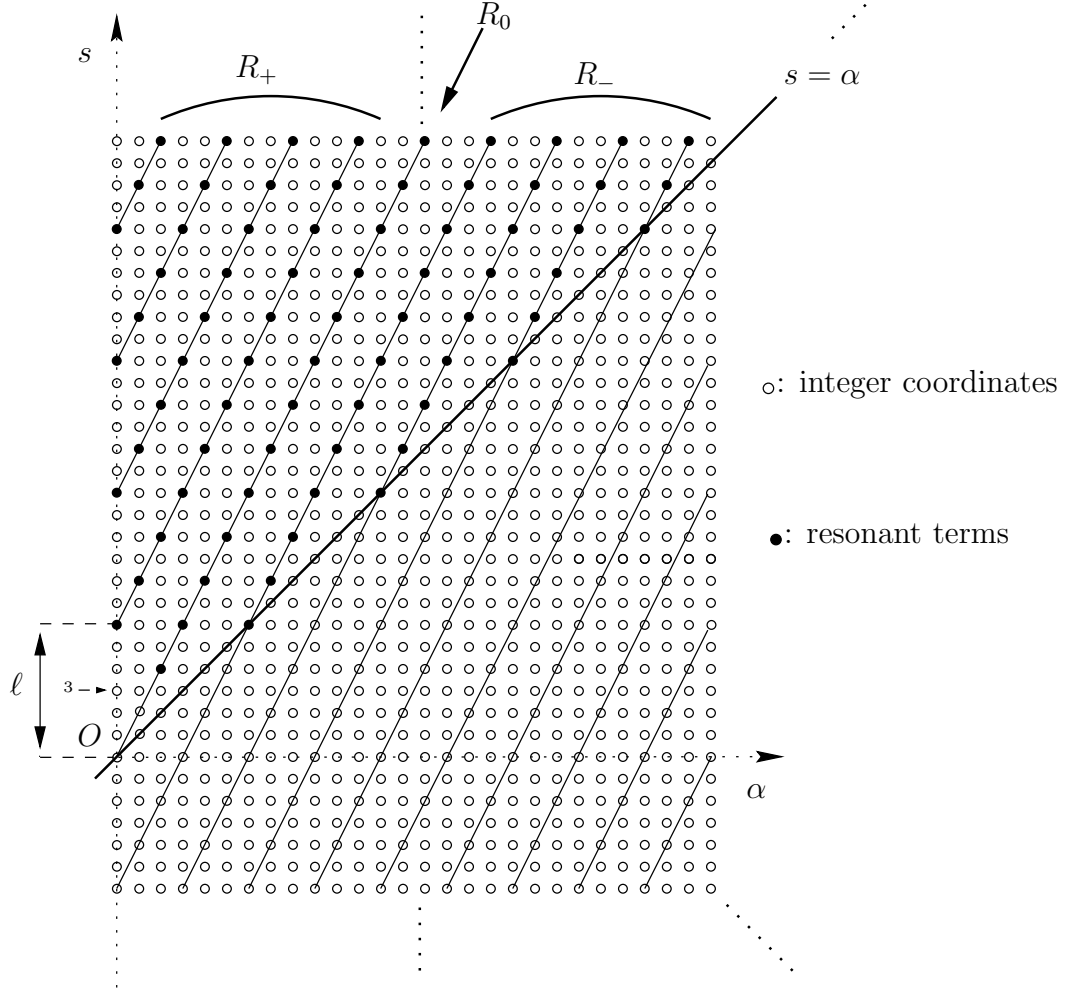


Figure 3: Graphical representation of the resonant terms in a hamiltonian in the neighborhood of a stable fixed point whose linearized motion is a rotation of rational angular speed.

3.4 Normal form for the full hamiltonian: marginal case

We start from the real hamiltonian

$$H(p, q; t) = \frac{1}{2\tau} p^2 + \sum_{m \in \mathbb{Z}} \sum_{\substack{(\alpha, \beta) \in \mathbb{N}^2 \\ \alpha + \beta = k}} h_{m\alpha\beta} p^\alpha q^\beta e^{im\frac{2\pi}{\tau}t} + O_{p,q}(k+1), \quad (3.4.1)$$

k being a fixed integer greater or equal to three and $h_{m\alpha\beta}^* = h_{-m\alpha\beta}$. We are looking for a suitable real

generating function of the form

$$S(p', q; t) = p'q + \sum_{m \in \mathbb{Z}} \sum_{\substack{(\alpha, \beta) \in \mathbb{N}^2 \\ \alpha + \beta = k}} \sigma_{m\alpha\beta} p'^\alpha q^\beta e^{im\frac{2\pi}{\tau}t} + O_{p',q}(k+1). \quad (3.4.2)$$

Similarly to transformation (3.3.4) we get

$$\left\{ \begin{array}{l} p(p', q'; t) = p' + \sum_{m \in \mathbb{Z}} \sum_{\substack{(\alpha, \beta) \in \mathbb{N}^2 \\ \alpha + \beta = k}} \beta \sigma_{m\alpha\beta} p'^\alpha q'^{\beta-1} e^{im\frac{2\pi}{\tau}t} + O_{p',q'}(k) \\ q(p', q'; t) = q' - \sum_{m \in \mathbb{Z}} \sum_{\substack{(\alpha, \beta) \in \mathbb{N}^2 \\ \alpha + \beta = k}} \alpha \sigma_{m\alpha\beta} p'^{\alpha-1} q'^\beta e^{im\frac{2\pi}{\tau}t} + O_{p',q'}(k). \end{array} \right. \quad (3.4.3a)$$

$$\left\{ \begin{array}{l} p(p', q'; t) = p' + \sum_{m \in \mathbb{Z}} \sum_{\substack{(\alpha, \beta) \in \mathbb{N}^2 \\ \alpha + \beta = k}} \beta \sigma_{m\alpha\beta} p'^\alpha q'^{\beta-1} e^{im\frac{2\pi}{\tau}t} + O_{p',q'}(k) \\ q(p', q'; t) = q' - \sum_{m \in \mathbb{Z}} \sum_{\substack{(\alpha, \beta) \in \mathbb{N}^2 \\ \alpha + \beta = k}} \alpha \sigma_{m\alpha\beta} p'^{\alpha-1} q'^\beta e^{im\frac{2\pi}{\tau}t} + O_{p',q'}(k). \end{array} \right. \quad (3.4.3b)$$

For all integers m and (α, β) such that $\alpha + \beta = k \geq 3$, the coefficient of $p'^\alpha q'^\beta e^{im\frac{2\pi}{\tau}t}$ in the expansion of the hamiltonian $H'(p', q'; t) = H[p(p', q'; t), q(p', q'; t); t] + \frac{\partial S}{\partial t}[p', q(p', q'; t); t]$ is

$$\left\{ \begin{array}{l} \mathfrak{h}_{m\alpha\beta} + im \frac{2\pi}{\tau} \sigma_{m\alpha\beta} - 2(\alpha + 1) \sigma_{m(\alpha+1)(\beta-1)} \quad \forall \alpha \geq 0, \forall \beta > 1 \text{ such that } \alpha + \beta = k; \\ \mathfrak{h}_{mk0} + im \frac{2\pi}{\tau} \sigma_{mk0} \quad \text{for } \alpha = 0 \text{ and } \beta = k. \end{array} \right. \quad (3.4.4)$$

If we want to cancel them for fixed k and m , we must solve the matrix equation

$$\begin{bmatrix} im \frac{2\pi}{\tau} & -2 & 0 & \dots & 0 \\ 0 & im \frac{2\pi}{\tau} & -4 & 0 & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & im \frac{2\pi}{\tau} & -2(\alpha + 1) & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & im \frac{2\pi}{\tau} & -2k \\ 0 & \dots & 0 & 0 & im \frac{2\pi}{\tau} \end{bmatrix} \begin{bmatrix} \sigma_{m0k} \\ \sigma_{m1(k-1)} \\ \vdots \\ \sigma_{m\alpha(k-\alpha)} \\ \vdots \\ \sigma_{mk0} \end{bmatrix} = - \begin{bmatrix} \mathfrak{h}_{m0k} \\ \mathfrak{h}_{m1(k-1)} \\ \vdots \\ \mathfrak{h}_{m\alpha(k-\alpha)} \\ \vdots \\ \mathfrak{h}_{mk0} \end{bmatrix} \quad (3.4.6)$$

which we denote by

$$A_0 \sigma = \mathfrak{h}. \quad (3.4.7)$$

Since $\det A_0 = \left(im \frac{2\pi}{\tau}\right)^{k+1}$, the above equation has always a solution for every $m \neq 0$ and \mathbf{h} , thus all the time-dependent terms can be cancelled. When $m = 0$, the kernel of A_0 is one-dimensional and generated by $\mathbf{u}_1 \stackrel{\text{def}}{=} (1, 0, \dots, 0)$ and the image $\text{Im } A_0$ is a hyper plane of equation $\mathbf{u}_{k+1} \cdot \mathbf{h} = 0$ with $\mathbf{u}_{k+1} \stackrel{\text{def}}{=} (0, \dots, 0, 1)$ and where we have introduced the canonical scalar product. Let us now decompose the right hand side of (3.4.6) in the following manner

$$\mathbf{h} = \mathbf{h}_{\parallel} + \mathbf{h}_{\perp} \quad (3.4.8)$$

where $\mathbf{h}_{\parallel} \in \text{Im } A_0$ and $\mathbf{h}_{\perp} \cdot \text{Im } A_0 = 0$. We can choose $\sigma = A_0^{-1} \mathbf{h}_{\parallel}$, and the only coefficients which cannot be eliminated correspond to the elements orthogonal to $\text{Im } A_0$, that is those which are parallel to \mathbf{u}_{k+1}

$$\mathbf{h}_{00k} = \mathbf{h}_{01(k-1)} = \dots = \mathbf{h}_{0\alpha(k-\alpha)} = \dots = \mathbf{h}_{0(k-1)0} = 0 \quad \text{and} \quad \mathbf{h}_{0k0} \neq 0. \quad (3.4.9)$$

The latter are therefore the resonant terms and the form of the hamiltonian is, after relabeling the coordinates

$$H(p, q; t) = \frac{1}{2\tau} p^2 + \sum_{k=3}^n \mathbf{h}_k q^k + O_{p,q}(n+1), \quad (3.4.10)$$

for some $\mathbf{h}_k \in \mathbb{R}$ and for any integer $n \geq 3$.

4 Normal forms for one-dimensional parameter families of hamiltonians

Following what we have planned in section 2, we will now consider that the hamiltonian $H(p, q; t)$ is embedded in a smooth family depending on one real parameter, say ε . A typical example of the control parameter is the energy of a non-reduced two degrees of freedom system. Without loss of generality, we assume that the origin \mathfrak{o} is an isolated equilibrium point of $H(p, q; t; \varepsilon)$, obeying equations (2.0.4) at $\varepsilon = 0$. We can follow step by step the discussion of the previous section for $H(p, q; t; \varepsilon = 0)$ and classify its generic normal form around \mathfrak{o} . Now the question is to study what happens to the dynamics if we vary ε in a neighborhood of zero. We must first wonder about the behavior of the fixed point. By virtue of the implicit function theorem applied to the equations

$$\forall t \in \mathbb{R}, \quad \begin{cases} \frac{\partial H}{\partial p}(p, q; t; \varepsilon) = 0; & (4.0.11a) \\ \frac{\partial H}{\partial q}(p, q; t; \varepsilon) = 0 & (4.0.11b) \end{cases}$$

in a neighborhood of $(p, q; \varepsilon) = (0, 0; 0)$, we can follow as a function of ε the isolated solutions if

$$0 \neq \det(\text{Hess}[H(w; t; \varepsilon)])|_{w=(\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}), \varepsilon=0} = \det \begin{pmatrix} \partial_{p,p}^2 H(w; t; \varepsilon) & \partial_{p,q}^2 H(w; t; \varepsilon) \\ \partial_{q,p}^2 H(w; t; \varepsilon) & \partial_{q,q}^2 H(w; t; \varepsilon) \end{pmatrix}|_{w=(\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}), \varepsilon=0}. \quad (4.0.12)$$

An equivalent condition may be written for the monodromy matrix \mathbb{M} . If $w \mapsto \Phi(w; \varepsilon)$ is the (local) Poincaré map, the implicit function theorem applied to

$$\Phi(w; \varepsilon) = w \quad (4.0.13)$$

states that we can follow smoothly the fixed point if

$$0 \neq \det[\partial_w \Phi(w; \varepsilon) - \mathbb{1}]|_{w=(\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}), \varepsilon=0} = \det[\mathbb{M}(\varepsilon) - \mathbb{1}]|_{\varepsilon=0} = 2 - \text{tr}[\mathbb{M}(\varepsilon)]|_{\varepsilon=0}. \quad (4.0.14)$$

The last equality comes from property (3.2.6). From table 1 we see that the latter condition is fulfilled in both stable and unstable cases as well as in case $(ii)''$, because there exists a neighborhood \mathcal{S} with $0 \in \mathcal{S} \subset \mathbb{R}$ and a smooth family of points $\mathfrak{p}(\varepsilon)$ such that $\mathfrak{p}(0) = \mathfrak{o}$, and $\mathfrak{p}(\varepsilon)$ is a fixed point of $\Phi(\cdot; \varepsilon)$ for all ε in \mathcal{S} . If we now make an appropriate smooth ε -dependent change of coordinates in order to translate back $\mathfrak{p}(\varepsilon)$ to \mathfrak{o} , then we deal with a hamiltonian family $H(w; t; \varepsilon)$ which admits \mathfrak{o} as equilibrium point for all ε in \mathcal{S} . Elimination of the time dependence of the quadratic part of the hamiltonian around \mathfrak{o} can be done as described in subsection 3.1 in the whole \mathcal{S} . The τ or 2τ -periodic moving frame in which we follow the dynamics is smooth with respect to ε . At this stage, we must distinguish cases (i) and (iii) from case $(ii)''$. On the one hand, in the former two cases the stability of the fixed point will not change if \mathcal{S} is small enough because from a topological point of view property $\text{tr}[\mathbb{M}(\varepsilon)] \neq 2$ defines an open set for ε . On the other hand, case $(ii)''$ corresponds to $\text{tr}[\mathbb{M}(\varepsilon=0)] = -2$ which breaks down as soon as $\varepsilon \neq 0$. More precisely, generically $\text{tr}[\mathbb{M}(\varepsilon)]$ will cross transversally the value -2 and \mathfrak{o} will change from stable to unstable being marginal only at $\varepsilon = 0$. We will study first the stable and unstable unfoldings, then the marginal unfolding will be considered and we will finish with the study of case $(ii)'$ where even the presence of a fixed point cannot be guaranteed as soon as $\varepsilon \neq 0$. When embedded in a one-parameter unfolding, case $(ii)'$ and $(ii)''$ are respectively called *extremal* and *transitional* by [34, resp. definition 1.9 and definition 1.12].

4.1 Normal forms for unstable and stable unfoldings

From what precedes, we can deduce that if \mathfrak{o} is an unstable (resp. stable) fixed point of $H(w; t; \varepsilon = 0)$ it will remain a fixed point of $H(w; t; \varepsilon)$ for all ε in a small enough neighborhood \mathcal{I} of zero. Indeed, in both cases all the symplectic changes of coordinates we have made in subsection 3.2 and 3.3 vary smoothly with ε .

In the unstable case, we are led to the Birkhoff normal form (3.3.8) with now λ and $\{\mathfrak{h}_k\}_{k \in \{2, \dots, n\}}$ being smooth real functions of ε

$$H_u(p, q; t; \varepsilon) = \lambda(\varepsilon) pq + \sum_{k=2}^n \mathfrak{h}_k(\varepsilon) (pq)^k + O_{p,q}(2n+1). \quad (4.1.1)$$

In the stable case, we cannot prevent ω from crossing $\frac{\pi}{\tau} \mathbb{Q} \setminus \mathbb{N}$ infinitely many times. Let us then assume that

$$\omega(\varepsilon = 0) = \omega_0 \stackrel{\text{def}}{=} \frac{r}{\ell} \frac{2\pi}{\tau} \quad (4.1.2)$$

with $r \in \mathbb{N} \setminus \{0\}$ and $\ell \in \mathbb{N} \setminus \{0, 1, 2\}$ coprimes. Choose the neighborhood $I \supset 0$ so that it contains no rational $\frac{\tau}{\pi} \omega(\varepsilon)$ whose denominator is lower than ℓ . Moreover since generically $\frac{d\omega}{d\varepsilon}|_{\varepsilon=0} \neq 0$, we can always choose the control parameter in order to satisfy $\omega(\varepsilon) - \omega_0 = \varepsilon$. Thus, in a rotating frame defined by (3.3.13) with $\omega = \omega_0$ we get, from expressions (3.3.17) and (3.3.18),

$$H_\ell(p, q; t; \varepsilon) = \frac{1}{2} \varepsilon (p^2 + q^2) + \sum_{\substack{k \in \mathbb{N} \\ 2 \leq k \leq \ell/2}} \tilde{\mathfrak{a}}_k(\varepsilon) (p^2 + q^2)^k + |\mathfrak{h}_{\ell\mathfrak{o}\ell}(\varepsilon)| \Re[(p + iq)^\ell] + O_{p,q}(\ell+1) \quad (4.1.3a)$$

and

$$H_\ell(I, \theta; t; \varepsilon) = \varepsilon I + \sum_{\substack{k \in \mathbb{N} \\ 2 \leq k \leq \ell/2}} \mathfrak{a}_k(\varepsilon) I^k + \mathfrak{b}_\ell(\varepsilon) I^{\ell/2} \cos(\ell\theta) + O_{\sqrt{I}}(\ell+1) \quad (4.1.3b)$$

in momentum-position and action-angle variables, respectively.

4.2 Normal form for the transitional fixed point

Since the stability of the fixed point will change when crossing $\varepsilon = 0$, derivation of the normal form of the unfolding is not as straightforward as in the previous subsection. Let us then start from the

following hamiltonian

$$H(p, q; t; \varepsilon) = \underbrace{\frac{1}{2} c(\varepsilon) p^2 + d(\varepsilon) pq + \frac{1}{2} e(\varepsilon) q^2}_{\stackrel{\text{def}}{=} h^{(2)}(p, q; \varepsilon)} + O_{p,q}(3). \quad (4.2.1)$$

c , d and e are smooth functions of ε such that

$$c(0) = \frac{1}{\tau}, \quad d(0) = 0, \quad e(0) = 0. \quad (4.2.2)$$

In Eq.(4.2.1) there are no linear terms in (p, q) because we have already shown that the origin \mathfrak{o} is fixed in a suitable symplectic chart for all ε . Recall that the hamiltonian (4.2.1) corresponds to a 2τ -periodic stroboscopic view of the true dynamics.

We will proceed as in subsection 3.2. We look for a family of new symplectic charts obtained via a family of generating functions of the form

$$S(p', q; \varepsilon) = \frac{1}{2} \gamma(\varepsilon) p'^2 + \delta(\varepsilon) p'q + \frac{1}{2} \eta(\varepsilon) q^2 + O_{p',q}(3) \quad (4.2.3)$$

where (δ, γ, η) are now smooth real functions of ε such that δ does not vanish in \mathcal{I} . We are led to an expression identical to (3.2.4) for $h^{(2)}(p', q')$. The length of \mathcal{I} can possibly be reduced in order to have, $\forall \varepsilon \in I, c(\varepsilon) > 0$. Moreover since generically $\frac{d(d^2 - ce)}{d\varepsilon} \Big|_{\varepsilon=0} \neq 0$, we can reparametrize the unfolding by $\varepsilon' \stackrel{\text{def}}{=} -\tau(d^2 - ce)$. We can then choose

$$\forall \varepsilon \in I, \quad \gamma(\varepsilon) = 0; \quad \delta(\varepsilon) = 1/\sqrt{\tau c(\varepsilon)}; \quad \eta(\varepsilon) = -d/c, \quad (4.2.4)$$

which implies that $c' = 1/\tau$, $d' = 0$, and $e' = \varepsilon$.

Dropping all the primes, the normal form of the quadratic terms of the unfolding of a transitional point is

$$h^{(2)}(p, q; \varepsilon) = \frac{1}{2\tau} p^2 + \varepsilon q^2. \quad (4.2.5)$$

We can now reproduce step by step the reasoning of subsection 3.4. Starting from

$$H(p, q; t; \varepsilon) = \frac{1}{2\tau} p^2 + \varepsilon q^2 + \sum_{m \in \mathbb{Z}} \sum_{\substack{(\alpha, \beta) \in \mathbb{N}^2 \\ \alpha + \beta = k \geq 3}} h_{m\alpha\beta}(\varepsilon) p^\alpha q^\beta e^{im \frac{2\pi}{\tau} t} + O_{p,q}(k+1), \quad (4.2.6)$$

we construct a generating function (3.4.2) with the σ 's being smooth functions of ε . We are led to a matrix equation $A(\varepsilon) \sigma(\varepsilon) = \mathfrak{h}(\varepsilon)$ with $A(\varepsilon=0) = A_0$ [see (3.4.6)]. Possibly reducing the length

of \mathcal{I} , we can assume that $\forall \varepsilon \in \mathcal{I}, \det A(\varepsilon) \neq 0$ when $m \neq 0$, and then the time dependence can be cancelled uniformly in ε . In fact, we can also eliminate all the monomials of order greater than three except those terms corresponding to (3.4.9) since we cannot cancel them *uniformly* in ε . We thus obtain the full normal form for the unfolding in the transitional case

$$H(p, q; t; \varepsilon) = \frac{1}{2\tau} p^2 + \varepsilon q^2 + \sum_{k=3}^n \mathfrak{h}_k(\varepsilon) q^k + O_{p,q}(n+1), \quad (4.2.7)$$

where $\{\mathfrak{h}_k\}_{k \in \{3, \dots, n\}}$ are some real functions of ε .

There are certain constraints on the \mathfrak{h} 's due to the fact that the hamiltonian (4.2.7) describes the dynamics in a (-2τ) -periodic stroboscopic view. After one period τ , the monodromy matrix verifies

$$\mathbb{M}(\varepsilon=0) \doteq \begin{pmatrix} -1 & 0 \\ 1 & -1 \end{pmatrix}. \quad (4.2.8)$$

It can be shown using the Lie transformation [37, §VII. E. 5] that we must have $H(p, -q; t; \varepsilon = 0) = H(p, q; t; \varepsilon = 0) + O_{p,q}(n+1)$ which implies $\forall k \in \mathbb{N} \setminus \{0\}, \mathfrak{h}_{2k+1}(0) = 0$. We are going to prove the same result for the first terms of the normal form (up to $n = 6$) but without the help of the Lie transformation theory. We will show in this subsection that $\mathfrak{h}_3(0) = 0$. We will later on show in subsection 6.3 that $\frac{d\mathfrak{h}_3}{d\varepsilon}|_{\varepsilon=0} = 0$ and $\mathfrak{h}_5(0) = 0$ (see equations (6.3.10)).

Let us fix $\varepsilon = 0$, define $v_3(q) \stackrel{\text{def}}{=} \mathfrak{h}_3(0)q^3$ and anticipate a little bit on the calculus of Poincaré stroboscopic maps of section 6. We apply formula (6.0.2) in order to calculate the Poincaré map at time -2τ . The Poisson brackets are given by

$$\{q, H\} = \frac{1}{\tau} p + O_{p,q}(3); \quad (4.2.9)$$

$$\{p, H\} = -v_3'(q) + O_{p,q}(3); \quad (4.2.10)$$

$$\{\{p, H\}, H\} = -\frac{1}{\tau} p v_3''(q) + O_{p,q}(3); \quad (4.2.11)$$

$$\{\{\{p, H\}, H\}, H\} = -\frac{1}{\tau^2} p^2 v_3'''(q) + O_{p,q}(3); \quad (4.2.12)$$

$$\underbrace{\{\dots \{p, H\}, \dots, H\}}_{n \text{ times, } n \geq 4} \in O_{p,q}(3). \quad (4.2.13)$$

Then it is straightforward to get $w_2 \stackrel{\text{def}}{=} w(-2\tau)$ up to order two in $w_0 \stackrel{\text{def}}{=} w(0)$

$$\left\{ \begin{array}{l} p_2 = p_0 + 2\tau v_3'(q_0) - 2\tau p_0 v_3''(q_0) + \frac{4}{3}\tau p_0^2 v_3'''(q_0) + O_{p_0, q_0}(3) ; \\ q_2 = q_0 - 2p_0 - 2\tau v_3'(q_0) - \frac{4}{3}\tau p_0 v_3''(q_0) - \frac{2}{3}\tau p_0^2 v_3'''(q_0) + O_{p_0, q_0}(3) ; \end{array} \right. \quad (4.2.14a)$$

$$\left\{ \begin{array}{l} p_2 = p_0 + 2\tau v_3'(q_0) - 2\tau p_0 v_3''(q_0) + \frac{4}{3}\tau p_0^2 v_3'''(q_0) + O_{p_0, q_0}(3) ; \\ q_2 = q_0 - 2p_0 - 2\tau v_3'(q_0) - \frac{4}{3}\tau p_0 v_3''(q_0) - \frac{2}{3}\tau p_0^2 v_3'''(q_0) + O_{p_0, q_0}(3) ; \end{array} \right. \quad (4.2.14b)$$

Now we use the fact that this map is the square of another map whose linear part is known (see equation (4.2.8))

$$\left\{ \begin{array}{l} p_1 = -p_0 + f_2(p_0, q_0) + O_{p_0, q_0}(3) ; \\ q_1 = -q_0 + p_0 + g_2(p_0, q_0) + O_{p_0, q_0}(3) ; \end{array} \right. \quad (4.2.15a)$$

$$\left\{ \begin{array}{l} p_1 = -p_0 + f_2(p_0, q_0) + O_{p_0, q_0}(3) ; \\ q_1 = -q_0 + p_0 + g_2(p_0, q_0) + O_{p_0, q_0}(3) ; \end{array} \right. \quad (4.2.15b)$$

(f_2 and g_2 are two quadratic forms in w). After having iterated the last map and identified the result with (4.2.14) we get

$$2\tau v_3'(q) - 2\tau p v_3''(q) + \frac{4}{3}\tau p^2 v_3'''(q) = f_2(-p, p - q) - f_2(p, q) \quad (4.2.16)$$

for all (p, q) in a neighborhood of the origin. In particular on the q -axis relation (4.2.16) implies $h_3(0) = 0$. We have then shown that the unfolding of a transitional point has the form

$$H(p, q; t; \varepsilon) = \frac{1}{2\tau} p^2 + \varepsilon q^2 + \varepsilon \tilde{h}_3(\varepsilon) q^3 + \sum_{k=4}^n h_k(\varepsilon) q^k + O_{p, q}(n+1), \quad (4.2.17)$$

where $\tilde{h}_3(\varepsilon) \stackrel{\text{def}}{=} h_3(\varepsilon)/\varepsilon$ is a smooth real function.

4.3 Normal form for an extremal fixed point

In this case condition (4.0.14) is not fulfilled. Generically we cannot guaranty that we can follow smoothly the fixed point as soon as $\varepsilon \neq 0$ and then we cannot even make use of the arguments of subsection 3.1 to remove smoothly the time-dependence of the quadratic part of $H(p, q'; t; \varepsilon)$ uniformly with respect to ε in a neighborhood of zero. We must therefore reconsider our approach from its very beginning.

4.3.1 Uniform elimination of the time-dependence in the quadratic part of the unfolding

The most general form for a one-parameter linearized τ -periodic differential equation is

$$\dot{w}(t; \varepsilon) = \Lambda(t; \varepsilon) w(t; \varepsilon) + a(t; \varepsilon) \quad (4.3.1)$$

where $\mathbb{A}(t; \varepsilon)$ (resp. $a(t; \varepsilon)$) is a real one-parameter family of τ -periodic 2×2 matrices (resp. 2-vectors). If $\mathbb{U}(t; \varepsilon)$ is the operator defined by equations (3.1.6) and (3.1.7), then the extremal case corresponds precisely to $\mathbb{U}(t; \varepsilon=0) = \mathbb{M}(\varepsilon=0) = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$. From now on and until the end of this subsection we should keep in mind that all the quantities we are dealing with depend smoothly on ε but for simplicity we will not write down explicitly this dependence. Let \mathbb{L} (resp. b) be any smooth family of time-independent infinitesimal symplectic matrices (resp. 2-vectors). In the new symplectic coordinate system defined by $\tilde{w}(t) = e^{t\mathbb{L}}[\mathbb{U}(t)]^{-1} w(t) + b(t)$, equation (4.3.1) becomes

$$\dot{\tilde{w}}(t) = \mathbb{L} \tilde{w}(t) + \dot{b} - \mathbb{L} b + \underbrace{e^{t\mathbb{L}}[\mathbb{U}(t)]^{-1} a(t)}_{\stackrel{\text{def}}{=} \check{a}} . \quad (4.3.2)$$

There will be no explicit time-dependence if

$$\dot{b} - \mathbb{L} b + \check{a}(t) = c \quad (4.3.3)$$

where now c is a (family of) constant 2-vector(s). As we have seen in subsection 3.1, the change of variables has to be real and τ -periodic. Actually it is an extremal case if we choose a real τ -periodic b and $\mathbb{L} = \frac{1}{\tau} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$. Let us then expand $a(t)$ and $b(t)$ in Fourier series

$$\check{a}(t) = \sum_{n \in \mathbb{Z}} \check{a}_n e^{in \frac{2\pi}{\tau} t} \quad \text{and} \quad b(t) = \sum_{n \in \mathbb{Z}} b_n e^{in \frac{2\pi}{\tau} t} \quad \text{where} \quad \forall n \in \mathbb{Z}, (\check{a}_n, b_n) \in \mathbb{C}^2 . \quad (4.3.4)$$

Then equation (4.3.3) leads to

$$\forall n \in \mathbb{Z}, \quad \left(in \frac{2\pi}{\tau} \mathbb{1} - \mathbb{L} \right) b_n = \delta_{n,0}^{(K)} c - \check{a}_n \quad (4.3.5)$$

($\delta^{(K)}$ is the Kronecker symbol). Since the eigenvalues of \mathbb{L} are both small for ε near zero, the matrix equation (4.3.5) can be solved for every $n \in \mathbb{Z} \setminus \{0\}$. It only remains $\mathbb{L} b_0 = \check{a}_0 - c$ which can also be solved provided that $\check{a}_0 - c$ belongs to $\text{Im } \mathbb{L} = \mathbb{R} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + O_\varepsilon(1)$. If $\check{a}_0 = \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix}$ we can therefore eliminate the time dependence of (4.3.1) uniformly in ε by choosing $c \stackrel{\text{def}}{=}} \begin{pmatrix} 0 \\ \alpha_2 \end{pmatrix} + O_\varepsilon(1)$ and $b_0 \stackrel{\text{def}}{=} \mathbb{L}^{-1}(\check{a}_0 - c) + O_\varepsilon(1)$. Moreover, since \mathbb{L} is real we have $\forall n \in \mathbb{Z}, b_n^* = b_{-n}$ which ensures that $b(t)$ is real.

We have then shown that even in the one-parameter unfolding of an extremal point, we can choose a suitable local symplectic coordinate system smooth in ε in which the dynamics corresponds to a hamiltonian whose terms of degree one and two in (p, q) are constant.

4.3.2 Normal form for the quadratic unfolding

From what precedes, let us start from the most general form for a time-independent hamiltonian

$$H(p, q; t; \varepsilon) = \underbrace{\mathbf{a}(\varepsilon) p + \mathbf{b}(\varepsilon) q + \frac{1}{2} \mathbf{c}(\varepsilon) p^2 + \mathbf{d}(\varepsilon) pq + \frac{1}{2} \mathbf{e}(\varepsilon) q^2}_{\stackrel{\text{def}}{=} h^{(2)}(p, q; \varepsilon)} + O_{p,q}(3) \quad (4.3.6)$$

with $(\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}, \mathbf{e})$ being smooth functions of ε such that

$$\mathbf{a}(0) = 0; \quad \mathbf{b}(0) = 0; \quad \mathbf{c}(0) = \frac{1}{\tau}; \quad \mathbf{d}(0) = 0; \quad \mathbf{e}(0) = 0. \quad (4.3.7)$$

If \mathcal{I} is small enough we have $\forall \varepsilon \in \mathcal{I}, \mathbf{c} > 0$. Then, if we make the smooth translation $p' = p + \frac{\mathbf{a}}{\mathbf{c}}$ and if we relabel the coefficients of $h^{(2)}$, we can write

$$h^{(2)}(p, q; \varepsilon) = \mathbf{b}(\varepsilon) q + \frac{1}{2} \mathbf{c}(\varepsilon) p^2 + \mathbf{d}(\varepsilon) pq + \frac{1}{2} \mathbf{e}(\varepsilon) q^2. \quad (4.3.8)$$

As was done in subsection 4.2, the symplectic change of variables corresponding to the generating function (4.2.3) gives

$$\begin{aligned} h^{(2)}(p', q'; \varepsilon) &= \frac{1}{\delta} \mathbf{b} q' + \frac{1}{2} \left[\frac{\gamma^2}{\delta^2} (\mathbf{c} \eta^2 + 2\mathbf{d} \eta + \mathbf{e}) - 2\gamma(\mathbf{d} + \mathbf{c} \eta) + \mathbf{c} \delta^2 \right] p'^2 \\ &+ \left[-\frac{\gamma}{\delta^2} (\mathbf{c} \eta^2 + 2\mathbf{d} \eta + \mathbf{e}) + \mathbf{d} + \mathbf{c} \eta \right] p' q' + \frac{1}{2} \left[\frac{1}{\delta^2} (\mathbf{c} \eta^2 + 2\mathbf{d} \eta + \mathbf{e}) \right] q'^2 + O_{p',q'}(3). \end{aligned} \quad (4.3.9)$$

Let us make the same choice as in (4.2.4). Aside from Eq. (4.3.7), generically we have no additional relations between $(\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}, \mathbf{e})$, in particular $\frac{d\mathbf{b}\sqrt{\tau\mathbf{c}}}{d\varepsilon}|_{\varepsilon=0} \neq 0$ and without loss of generality we can reparametrize the unfolding to $\varepsilon' \stackrel{\text{def}}{=} \mathbf{b}\sqrt{\tau\mathbf{c}}$. Dropping all the primes, the normal form of the quadratic terms of the unfolding of an extremal point is

$$h^{(2)}(p, q; \varepsilon) = \frac{1}{2\tau} p^2 + \varepsilon q + \varepsilon \mathbf{h}_2(\varepsilon) q^2, \quad (4.3.10)$$

where \mathbf{h}_2 is a smooth generic real function of ε . Getting the normal form for higher orders can be done uniformly with respect to ε exactly in the same way as described in subsection 4.2. A full normal form for the unfolding in the extremal case can then be written

$$H(p, q; t; \varepsilon) = \frac{1}{2\tau} p^2 + \varepsilon q + \varepsilon \tilde{\mathbf{h}}_2(\varepsilon) q^2 + \sum_{k=3}^n \mathbf{h}_k(\varepsilon) q^k + O_{p,q}(n+1), \quad (4.3.11)$$

where $\{\mathbf{h}_k\}_{k \in \{3, \dots, n\}}$ are some generic real functions of ε . Through the smooth translation $q' = q + \frac{\varepsilon \tilde{\mathbf{h}}_2(\varepsilon)}{3\mathbf{h}_3(\varepsilon)}$ we can moreover set $\tilde{\mathbf{h}}_2(\varepsilon = 0) = 0$.

5 Geometrical structure of the trajectories

We have now all the necessary elements to describe the unfolding of the dynamics for each of the above mentioned hamiltonian normal forms. In particular, staying in a neighborhood of the origin \mathfrak{o} we search for a characterization, as a function of ε , of the typical phase-space scales of the classical structures involved in each bifurcation. We also want to explicitly locate the real as well as the complex fixed points involved. The following will be valid as long as ε remains small compared to all algebraic expressions that can be made from the coefficients appearing in the normal form and having the same dimensions.

Recall that the dynamics generated by the normal forms coincides with the true one if we look at the trajectories stroboscopically with a period that depends on the nature of the equilibrium point at $\varepsilon = 0$ (see table 1). We must also keep in mind that, in the stable case, the hamiltonian (4.1.3) describe the dynamics in a rotating frame (see equations (3.3.13)).

5.1 Unstable Point

The unfolding of an unstable point is given by hamiltonian (4.1.1). For each ε in \mathcal{I} , the origin \mathfrak{o} is the only fixed point and it remains unstable for \mathcal{I} small enough.

5.2 Extremal bifurcation

The normal form (4.3.11) can be rewritten as follows (recall that we can take $\tilde{\mathfrak{h}}_2(\varepsilon = 0) = 0$)

$$H_1(p, q; t; \varepsilon) = \frac{1}{2\tau} p^2 + \varepsilon q + \frac{1}{3} a q^3 + \frac{1}{4} b q^4 + O_{p, q, \sqrt{|\varepsilon|}}(5) \quad (5.2.1)$$

where a, b are generic real numbers and ε is such that $|\varepsilon| \ll |a^3/b^2|$. In figure 4 we have plotted the potential $v_{\text{extr}}(q) \stackrel{\text{def}}{=} \varepsilon q + \frac{1}{3} a q^3$ whose behavior governs the dynamics when $q \simeq 0$. The fixed points of $H_1(p, q; t; \varepsilon)$ are given by $p = 0$ and $q^2 = -\varepsilon/a$.

→ $\varepsilon/a < 0$ There are two fixed points which are real

$$\mathfrak{p}_{\pm} \stackrel{\text{def}}{=} \begin{pmatrix} p = 0 \\ q = q_{\pm} \stackrel{\text{def}}{=} \pm \sqrt{|\varepsilon/a|} \end{pmatrix} + O_{\sqrt{|\varepsilon|}}(2). \quad (5.2.2)$$

The equations of motion are

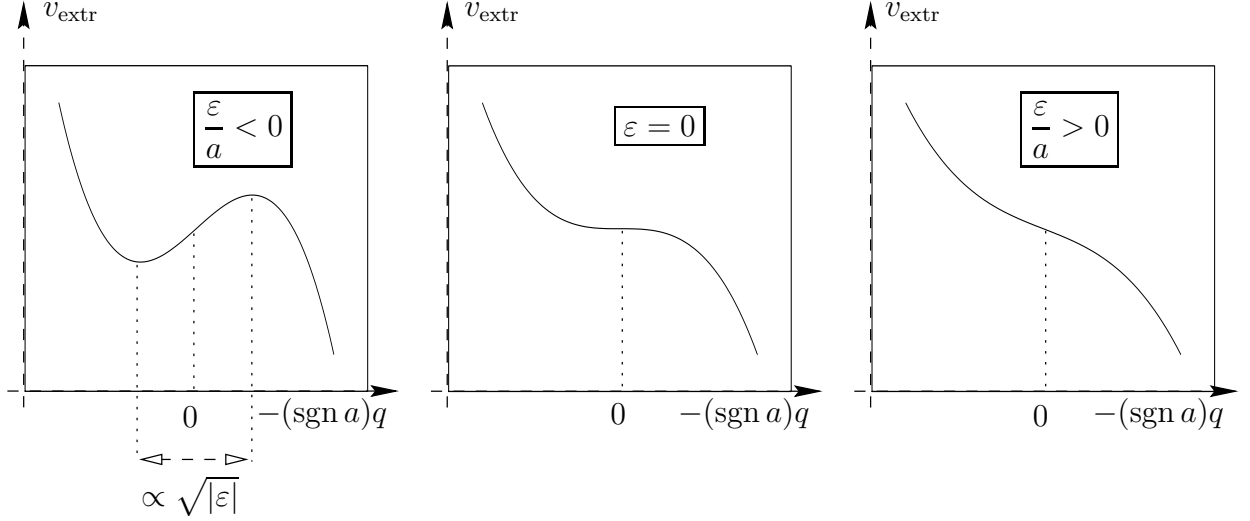


Figure 4: Potential $v_{\text{extr}}(q)$.

$$\begin{cases} \dot{p} = -\varepsilon - a q^2 + O_{p,q,\sqrt{|\varepsilon|}}(3) ; & (5.2.3a) \\ \dot{q} = \frac{1}{\tau} p + O_{p,q,\sqrt{|\varepsilon|}}(3) . & (5.2.3b) \end{cases}$$

Their linearization around \mathbf{p}_{\pm} gives

$$\begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix} = \begin{pmatrix} 0 & -2aq_{\pm} \\ 1/\tau & 0 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} \quad (5.2.4)$$

Applying the results of subsection 3.2 we deduce from the discussion of subsection 3.2 that $\mathbf{p}_{\text{sgn } a}$ is stable with a local angular speed given by $\omega = \sqrt{2/\tau} |\varepsilon a|^{1/4}$. $\mathbf{p}_{-\text{sgn } a}$ is an unstable point whose stable and unstable manifolds have respectively the following tangents: $u = -\sqrt{2/\tau} |\varepsilon a|^{1/4} v$ and $u = \sqrt{2/\tau} |\varepsilon a|^{1/4} v$. The equation of the separatrix is

$$\frac{1}{2\tau} p^2 + \varepsilon q + \frac{1}{3} a q^3 = H_1(0, q_{-\text{sgn } a}; t; \varepsilon) = -\frac{2}{3\sqrt{|a|}} |\varepsilon|^{3/2} . \quad (5.2.5)$$

It is the boundary of an area whose height and width vary with $|\varepsilon|$ as $|\varepsilon|^{1/2}$ and $|\varepsilon|^{3/4}$, respectively (see figure 5).

→ $\boxed{\varepsilon/a = 0}$ By construction, we have one and only one fixed point: the origin \mathbf{o} . The two fixed points \mathbf{p}_{\pm} which are present when $\varepsilon/a \neq 0$ now coalesce. The linearized flow has been studied in section 3.2 and looks like the marginal case in table 1. If we take into account the first non-linear

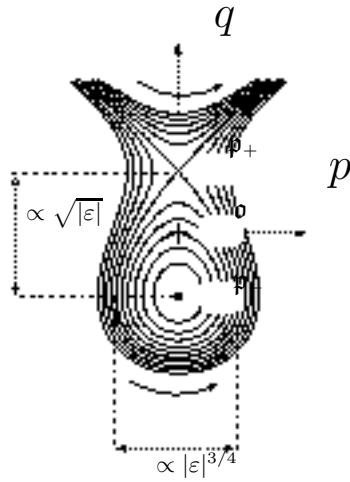


Figure 5: *Flow near the bifurcation of an extremal fixed point when $\varepsilon/a < 0$.*

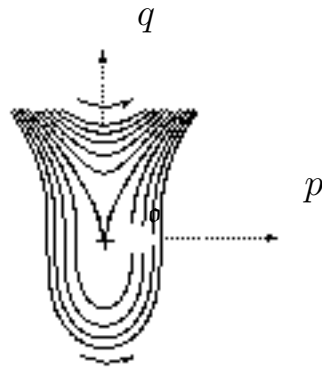


Figure 6: *Flow near the bifurcation of an extremal fixed point at $\varepsilon = 0$.*

terms we get two families of smooth trajectories separated by a curve given by

$$\frac{1}{2\tau} p^2 + \frac{1}{3} a q^3 = 0, \quad (5.2.6)$$

which has a vertical cusp at \mathfrak{o} where the stable and unstable manifolds meet (see figure 6).

→ $\boxed{\varepsilon/a > 0}$ There are two fixed points which are now complex

$$\mathbf{p}_{\pm} \stackrel{\text{def}}{=} \begin{pmatrix} p = 0 \\ q = iq_{\pm} = \pm i\sqrt{|\varepsilon/a|} \end{pmatrix} + O_{\sqrt{|\varepsilon|}}(2). \quad (5.2.7)$$

In the Poincaré surface of section, the flow has no fixed point and the linearized flow is given by equations (5.2.4) (see figure 7).

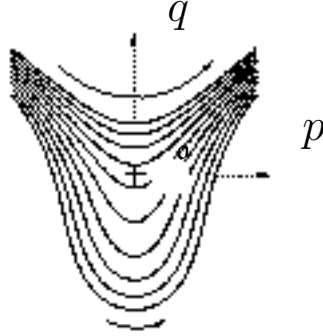


Figure 7: Flow near the bifurcation of an extremal fixed point when $\varepsilon/a > 0$.

5.3 Transitional point

The normal form (4.2.17) can be reordered as follows

$$H_2(p, q; t; \varepsilon) = \frac{1}{2\tau} p^2 + \varepsilon q^2 + \frac{1}{4} a q^4 + \frac{1}{3} \varepsilon b q^3 + \frac{1}{5} c q^5 + \mathbf{h}'_4(0)\varepsilon q^4 + \mathbf{h}_6(0)q^6 + O_{\sqrt{|p|}, q, \sqrt{|\varepsilon|}}(7) \quad (5.3.1)$$

where $a \stackrel{\text{def}}{=} 4\mathbf{h}_4(0)$, $b \stackrel{\text{def}}{=} 3\tilde{\mathbf{h}}_3(0) = 3\mathbf{h}'_3(0)$, and $c \stackrel{\text{def}}{=} 5\mathbf{h}_5(0)$. We are in a regime where $|\varepsilon| \ll |a/b^2|$ and $|\varepsilon| \ll |c/b^3|$. In figure 8 we have plotted the potential $v_{\text{tran}}(q) \stackrel{\text{def}}{=} \varepsilon q^2 + \frac{1}{4} a q^4$ whose behavior governs the dynamics.

The fixed points of $H(p, q; t; \varepsilon)$ are the origin \mathbf{o} and the two points defined by $p \simeq 0$, $q^2 \simeq -2\varepsilon/a$.

→ $\boxed{\varepsilon/a < 0}$ Aside from the origin, there are two other real fixed points

$$\mathbf{p}_{\pm} \stackrel{\text{def}}{=} \begin{pmatrix} p = 0 \\ q = q_{\pm} \stackrel{\text{def}}{=} \pm\sqrt{2|\varepsilon/a|} \end{pmatrix} + O_{\sqrt{|\varepsilon|}}(2). \quad (5.3.2)$$

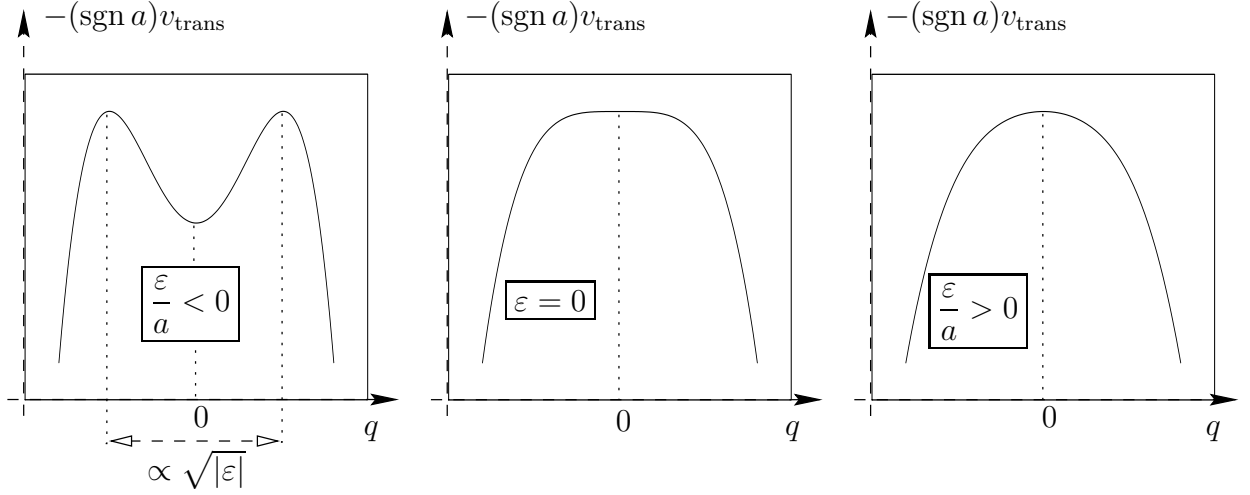


Figure 8: Potential $v_{\text{tran}}(q)$.

The equations of motion are

$$\begin{cases} \dot{p} = -2\varepsilon q - a q^3 + O_{\sqrt{p}, q, \sqrt{|\varepsilon|}}(4) ; & (5.3.3a) \\ \dot{q} = \frac{1}{\tau} p + O_{\sqrt{|p|}, q, \sqrt{|\varepsilon|}}(4) . & (5.3.3b) \end{cases}$$

Their linearization around \mathfrak{o} gives

$$\begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix} = \begin{pmatrix} 0 & -2\varepsilon \\ 1/\tau & 0 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} . \quad (5.3.4)$$

Therefore \mathfrak{o} is stable if $a < 0$ and the angular speed around it in this case is $\omega = 2\sqrt{2|\varepsilon|/\tau}$.

If $a > 0$, \mathfrak{o} is unstable and the stable and unstable manifolds have tangents $u = \mp\sqrt{2|\varepsilon|/\tau}v$.

The linearization of equations (5.3.3) about \mathfrak{p}_{\pm} gives

$$\begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix} = \begin{pmatrix} 0 & 4\varepsilon \\ 1/\tau & 0 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} . \quad (5.3.5)$$

Then \mathfrak{p}_{\pm} have a stability opposite to \mathfrak{o} : if $a < 0$, \mathfrak{p}_{\pm} are both unstable and in a chart centered on them, the stable and unstable manifolds have tangents $u = \pm 2\sqrt{|\varepsilon|\tau}v$. If $a > 0$, \mathfrak{p}_{\pm} are both stable and the angular speed around them is $\omega = 2\sqrt{|\varepsilon|\tau}$.

The equation of the separatrix is

$$\begin{cases} \frac{1}{2\tau} p^2 + \varepsilon q^2 + \frac{1}{4} a q^4 + O_{\sqrt{|\varepsilon|}}(5) = H_2(0, 0; t; \varepsilon) = 0 & \text{if } a > 0 \\ \frac{1}{2\tau} p^2 + \varepsilon q^2 + \frac{1}{4} a q^4 + O_{\sqrt{|\varepsilon|}}(5) = H_2(0, q_{\pm}; t; \varepsilon) = \frac{1}{a} \varepsilon^2 & \text{if } a < 0. \end{cases} \quad (5.3.6)$$

In both cases, the separatrix is the boundary of an area whose height and width varies with $|\varepsilon|$ as $|\varepsilon|^{1/2}$ and $|\varepsilon|$, respectively (see figure 9).

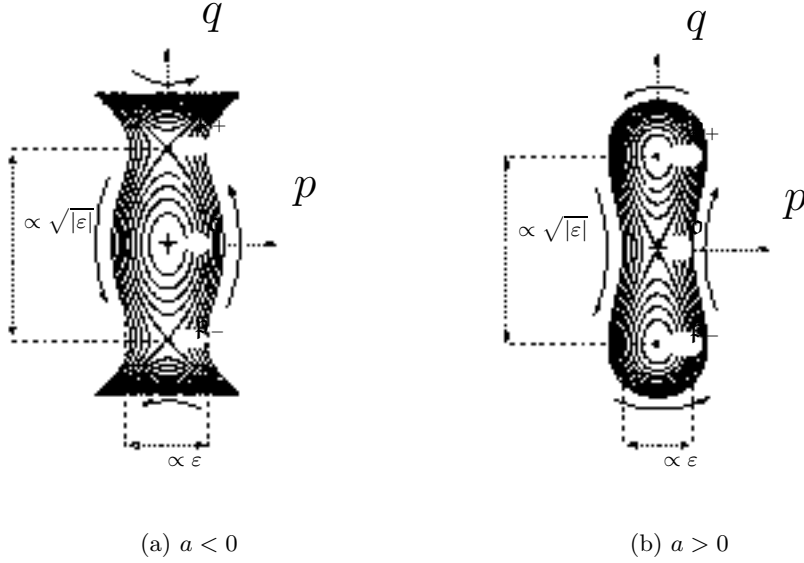


Figure 9: Flow near the bifurcation of a transitional fixed point when $\varepsilon/a < 0$.

→ $\boxed{\varepsilon/a = 0}$ By construction, the origin is the only fixed point. The two fixed points \mathbf{p}_{\pm} which are present when $\varepsilon/a \neq 0$ have merged at the origin. The linearized flow has been studied in section 3.2 and looks like the marginal case in table 1. If we take into account the non-linear terms we get families of smooth trajectories separated by the curve $\frac{1}{2\tau} p^2 + \frac{1}{4} a q^4 + O_{\sqrt{|\varepsilon|}}(5)$ represented by a couple of tangent parabolae if $a < 0$ and a single point if $a > 0$ (see figure 10).

→ $\boxed{\varepsilon/a > 0}$ \mathbf{o} is the only real fixed point and there are two fixed points in the complex plane

$$\mathbf{p}_{\pm} \stackrel{\text{def}}{=} \begin{pmatrix} p = 0 \\ q = iq_{\pm} = \pm i\sqrt{2|\varepsilon/a|} \end{pmatrix} + O_{\sqrt{|\varepsilon|}}(2). \quad (5.3.7)$$

The equations of motion and their linearization at \mathbf{o} are given by Eqs. (5.3.3) and (5.3.4), respectively. The origin changes its stability when ε crosses zero. \mathbf{o} is stable if $a > 0$ and the

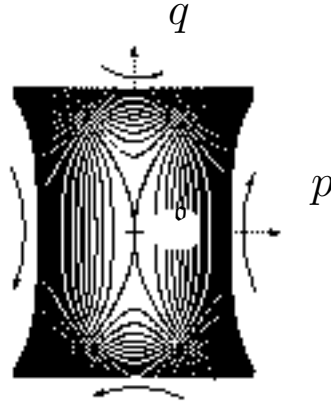
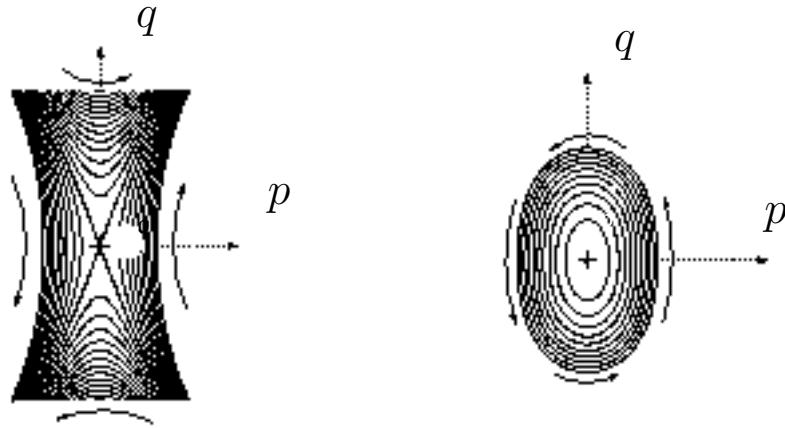


Figure 10: *Flow near the bifurcation of a transitional fixed point at $\varepsilon = 0$.*

angular speed around it is $\omega = \sqrt{2|\varepsilon|/\tau}$. If $a < 0$, \mathfrak{o} is unstable and the stable and unstable manifolds have tangents $p = \mp\sqrt{2|\varepsilon|/\tau} q$ (see figure 11).



(a) $a < 0$

(b) $a > 0$

Figure 11: *Flow near the bifurcation of a transitional fixed point when $\varepsilon/a < 0$.*

As we have said (see table 1), the -2τ -periodic stroboscopic view of these flows and the doubly iterated Poincaré map coincide. We will see, from the study of the generating functions of next section, that in order to describe the true dynamics after one-step iteration of the Poincaré map we must have a global symmetry with respect to \mathfrak{o} . That means that \mathfrak{p}_{\pm} are non-longer fixed points when $\varepsilon \neq 0$ but rather belong to a periodic orbit of period 2τ . In one period, they exchange each other and all the points in their neighborhood jump across \mathfrak{o} .

5.4 Stable point with $\omega_0 = 2r\pi/(\ell\tau)$ and $\ell \geq 3$

5.4.1 $\ell = 3$

For $\ell = 3$, the hamiltonian (4.1.3) takes the form

$$H_3(p, q; t; \varepsilon) = \frac{1}{2}\varepsilon(p^2 + q^2) + \frac{1}{3}b(p^3 - 3pq^2) + O_{p,q,\varepsilon}(4) \quad (5.4.1a)$$

and

$$H_3(I, \theta; t; \varepsilon) = \varepsilon I + \frac{1}{3}b(2I)^{3/2} \cos(3\theta) + O_{\sqrt{I},\varepsilon}(4) \quad (5.4.1b)$$

for some generic real number b . The fixed points of (5.4.1) are real for all ε . Aside from the origin we have

$$\mathbf{p}_0 \stackrel{\text{def}}{=} \begin{pmatrix} p = -\varepsilon/b \\ q = 0 \end{pmatrix} + O_\varepsilon(2) \quad \text{and} \quad \mathbf{p}_\pm \stackrel{\text{def}}{=} \begin{pmatrix} p = \varepsilon/(2b) \\ q_\pm \stackrel{\text{def}}{=} \pm\sqrt{3}\varepsilon/(2b) \end{pmatrix} + O_\varepsilon(2). \quad (5.4.2)$$

These points lie near the circle centered at \mathfrak{o} with radius $|\varepsilon/b|$. From expression (5.4.1b) we see that the flow is – to first order – invariant by rotation of $\frac{2\pi}{3}\mathbb{Z}$ around \mathfrak{o} . The equations of motion are

$$\begin{cases} \dot{p} = -\varepsilon q + 2bpq + O_{p,q,\varepsilon}(3); \\ \dot{q} = \varepsilon p + b(p^2 - q^2) + O_{p,q,\varepsilon}(3). \end{cases} \quad (5.4.3a)$$

$$\quad (5.4.3b)$$

By construction (see subsection 4.1) about \mathfrak{o} we have a rotation of angular speed ε . The linearization of (5.4.3) around \mathbf{p}_0 gives

$$\begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix} = \begin{pmatrix} 0 & -3\varepsilon \\ -\varepsilon & 0 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} \quad (5.4.4)$$

which shows that \mathbf{p}_0 and \mathbf{p}_\pm are unstable. The separatrix has the equation

$$0 = b \left[p - \frac{\varepsilon}{2b} \right] \left[q - \frac{1}{\sqrt{3}} \left(p + \frac{\varepsilon}{b} \right) \right] \left[q + \frac{1}{\sqrt{3}} \left(p + \frac{\varepsilon}{b} \right) \right] + O_{p,q,\varepsilon}(4). \quad (5.4.5)$$

which, to first order, is made by the three straight lines defining the equilateral triangle $(\mathbf{p}_0, \mathbf{p}_+, \mathbf{p}_-)$ (see figure 12-a). At $\varepsilon = 0$, all three points collapse at \mathfrak{o} (see figure 12-b).

The flows are seen in a frame rotating at angular speed $\omega_0 = 2\pi r/(3\tau)$ (cf (3.3.13)). Thus, in the fixed frame, $\{\mathbf{p}_0, \mathbf{p}_+, \mathbf{p}_-\}$ are no longer fixed points but rotate with angular speed ω_0 . They belong to a periodic orbit of period 3τ .

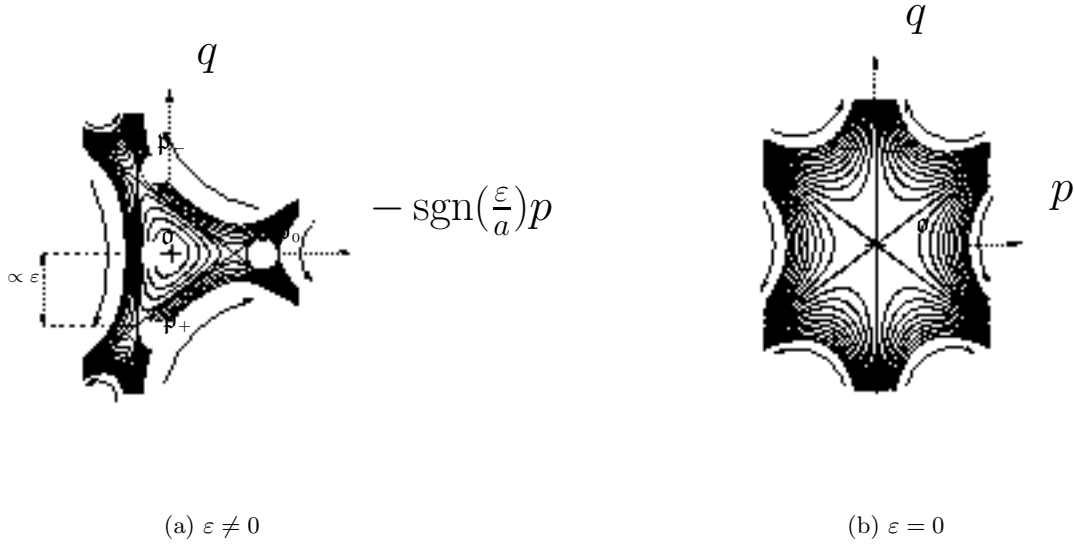


Figure 12: Flow near the bifurcation of a stable fixed point with $\ell = 3$

5.4.2 $\ell = 4$

For $\ell = 4$, the hamiltonian (4.1.3) takes the form

$$H_4(p, q; t; \varepsilon) = \frac{1}{2} \varepsilon (p^2 + q^2) + \frac{1}{4} a (p^4 + q^4) + \frac{1}{2} b p^2 q^2 + O_{p, q, \sqrt{|\varepsilon|}}(5) \quad (5.4.6a)$$

and

$$H_4(I, \theta; t; \varepsilon) = \varepsilon I + \frac{1}{4} (3a + b) I^2 + \frac{1}{4} (a - b) I^2 \cos(4\theta) + O_{\sqrt{I}, \sqrt{|\varepsilon|}}(5). \quad (5.4.6b)$$

for some generic real numbers a and b . Because we can change the global sign of H by inverting the arrow of time and the sign of ε , we can consider $a > 0$ without loss of generality. Moreover, by possibly make a global rotation of $\pi/4$, we can assume $a - b > 0$. From expression (5.4.6b), we see that the flow is – to first order – invariant by rotation of $\frac{\pi}{2}\mathbb{Z}$ around σ . The fixed points are solutions of the algebraic system

$$\begin{cases} \varepsilon p + a p^3 + b p q^2 + O_{p, q, \sqrt{|\varepsilon|}}(4) = 0 ; & (5.4.7a) \\ \varepsilon q + a q^3 + b p^2 q + O_{p, q, \sqrt{|\varepsilon|}}(4) = 0 & (5.4.7b) \end{cases}$$

or, in action-angle variables

$$\begin{cases} \sin(4\theta) + O_{\sqrt{I}, \sqrt{|\varepsilon|}}(3) = 0 ; & (5.4.8a) \end{cases}$$

$$\begin{cases} \varepsilon + \frac{1}{2}[3a + b + (a - b) \cos(4\theta)]I + O_{\sqrt{I}, \sqrt{|\varepsilon|}}(3) = 0 . & (5.4.8b) \end{cases}$$

Since generically $a \neq b$, we must distinguish the two cases $a + b \gtrless 0$.

$\alpha)$ $a + b > 0$

→ $\boxed{\varepsilon/a < 0}$ There are nine fixed points which are all real. Aside from the center we have

$$\mathbf{p}^{\pm} \stackrel{\text{def}}{=} \begin{pmatrix} p = \pm\sqrt{-\varepsilon/a} \\ q = 0 \end{pmatrix}, \quad \mathbf{p}_{\pm}^{\pm} \stackrel{\text{def}}{=} \begin{pmatrix} p = \pm\sqrt{-\varepsilon/(a+b)} \\ q = \pm\sqrt{-\varepsilon/(a+b)} \end{pmatrix} \text{ and } \mathbf{p}_{\pm} \stackrel{\text{def}}{=} \begin{pmatrix} p = 0 \\ q = \pm\sqrt{-\varepsilon/a} \end{pmatrix}, \quad (5.4.9)$$

modulo $O_{p, q, \sqrt{|\varepsilon|}}(2)$.

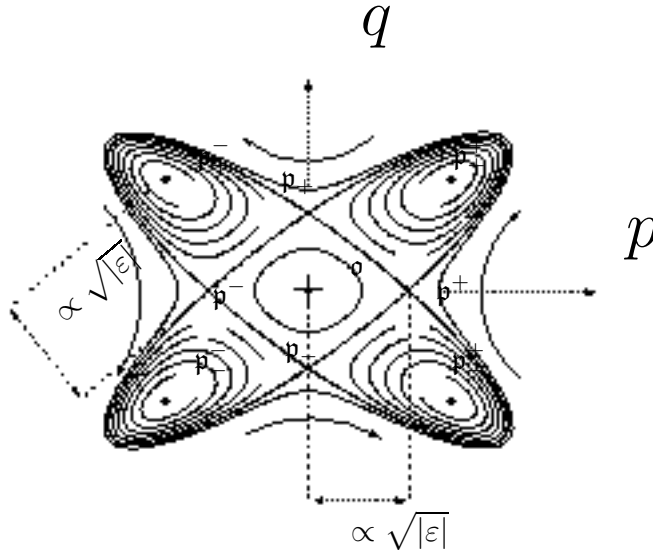


Figure 13: *Flow near the bifurcation of a stable fixed point of $\ell = 4$ when $a + b > 0$ and $\varepsilon/a < 0$.*

Linearizing the equations of motion we find that the four points $\{\mathbf{p}_{\pm}^{\pm}\}$ are stable while $\{\mathbf{p}_{\pm}, \mathbf{p}^{\pm}\}$

are unstable. The separatrix between them is given by

$$\left[a(p^2 + q^2) - \sqrt{2a(a-b)}pq + \varepsilon \right] \left[a(p^2 + q^2) - \sqrt{2a(a-b)}pq + \varepsilon \right] + O_{p,q,\sqrt{|\varepsilon|}}(3) = 0. \quad (5.4.10)$$

To first order the separatrix is made of two ellipses whose larger axes are oriented along $\theta = \pm\pi/4$. Their intersection defines four satellite islands of typical width proportional to $\sqrt{|\varepsilon|}$ (see figure 13).

→ $\boxed{\varepsilon = 0}$ \mathfrak{o} is the only fixed point and it is stable with a small positive angular rotation around it. The flow looks like the stable one in table 1.

→ $\boxed{\varepsilon/a > 0}$ \mathfrak{o} is the only real fixed point and it is stable with angular rotation ε around it. There are also eight fixed points with complex coordinates

$$\mathfrak{p}^\pm \stackrel{\text{def}}{=} \begin{pmatrix} p = \pm i\sqrt{\varepsilon/a} \\ q = 0 \end{pmatrix}, \quad \mathfrak{p}_\pm^\pm \stackrel{\text{def}}{=} \begin{pmatrix} p = \pm i\sqrt{\varepsilon/(a+b)} \\ q = \pm i\sqrt{\varepsilon/(a+b)} \end{pmatrix} \quad \text{and} \quad \mathfrak{p}_\pm \stackrel{\text{def}}{=} \begin{pmatrix} p = 0 \\ q = \pm i\sqrt{\varepsilon/a} \end{pmatrix}, \quad (5.4.11)$$

modulo $O_{p,q,\sqrt{|\varepsilon|}}(2)$. The flow looks like the stable one in table 1.

β) $a + b < 0$

→ $\boxed{\varepsilon/a < 0}$

There are five real fixed points

$$\mathfrak{o}, \quad \mathfrak{p}^\pm \stackrel{\text{def}}{=} \begin{pmatrix} p = \pm\sqrt{-\varepsilon/a} \\ q = 0 \end{pmatrix}, \quad \text{and} \quad \mathfrak{p}_\pm \stackrel{\text{def}}{=} \begin{pmatrix} p = 0 \\ q = \pm\sqrt{-\varepsilon/a} \end{pmatrix}, \quad (5.4.12)$$

modulo $O_{p,q,\sqrt{|\varepsilon|}}(2)$, and four fixed points with complex coordinates

$$\mathfrak{p}_\pm^\pm \stackrel{\text{def}}{=} \begin{pmatrix} p = \pm i\sqrt{-\varepsilon/(a+b)} \\ q = \pm i\sqrt{-\varepsilon/(a+b)} \end{pmatrix}, \quad (5.4.13)$$

modulo $O_{p,q,\sqrt{|\varepsilon|}}(2)$.

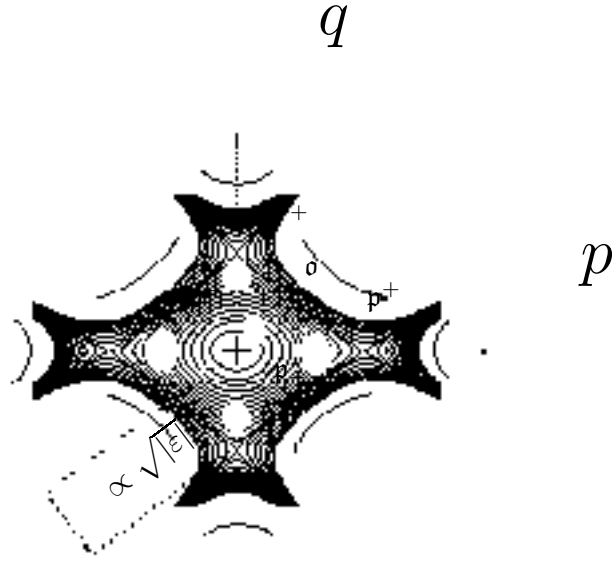


Figure 14: *Flow near the bifurcation of a stable fixed point when $a + b < 0$ and $\varepsilon/a < 0$.*

Linearizing the equations of motion we see that the four points $\{\mathbf{p}_{\pm}, \mathbf{p}^{\pm}, \}$ are all unstable. The separatrix joining them is given by

$$\left[a(p^2 + q^2) - \sqrt{2a(a-b)}pq - \varepsilon \right] \left[a(p^2 + q^2) + \sqrt{2a(a-b)}pq - \varepsilon \right] + O_{p,q,\sqrt{|\varepsilon|}}(3) = 0. \quad (5.4.14)$$

To first order, it is locally made of two hyperbolae of principal axes $\theta = \pm\pi/4$ (see figure 14).

→ $\boxed{\varepsilon = 0}$ The nine fixed points have collapsed into a single point \mathbf{o} which is unstable (see figure 15).

→ $\boxed{\varepsilon/a > 0}$

In addition to the stable fixed point at the origin, there are four real fixed points

$$\mathbf{p}_{\pm}^{\pm} \stackrel{\text{def}}{=} \begin{pmatrix} p = \pm\sqrt{\varepsilon/(a+b)} \\ q = \pm\sqrt{\varepsilon/(a+b)} \end{pmatrix}, \quad (5.4.15)$$

modulo $O_{p,q,\sqrt{|\varepsilon|}}(2)$. They are all unstable. To first order, the separatrix between them is given by two hyperbolae like in Eq. (5.4.14) but now the principal axes are oriented along $\theta = 0$ and $\theta = \pi/2$. Besides them, there are four fixed complex points

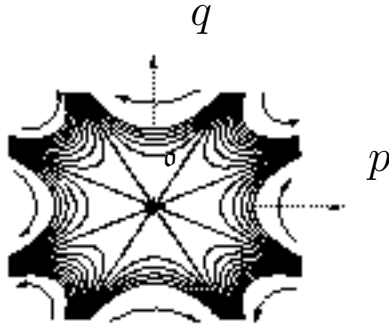


Figure 15: Flow near the bifurcation of a stable fixed point of $\ell = 4$ when $a + b < 0$ and $\varepsilon = 0$.

$$\mathfrak{o}, \mathfrak{p}^{\pm} \stackrel{\text{def}}{=} \begin{pmatrix} p = \pm i\sqrt{-\varepsilon/a} \\ q = 0 \end{pmatrix} \text{ and } \mathfrak{p}_{\pm} \stackrel{\text{def}}{=} \begin{pmatrix} p = 0 \\ q = \pm i\sqrt{-\varepsilon/a} \end{pmatrix}. \quad (5.4.16)$$

See figure 16.

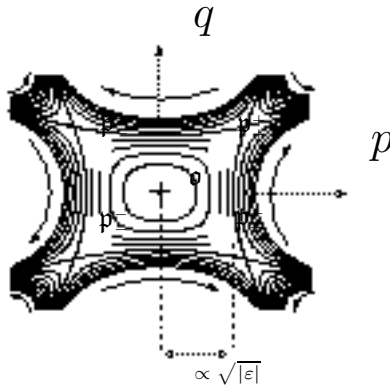


Figure 16: Flow near the bifurcation of a stable fixed point when $-b > a$ and $\varepsilon/a > 0$.

In all cases described in $\alpha)$ and $\beta)$ the reference frame rotates at angular speed $\omega_0 = \pi r / (2\tau)$ (cf Eq. 3.3.13). In the fixed frame, the points \mathfrak{p} rotate uniformly around \mathfrak{o} with speed ω_0 . $\{\mathfrak{p}_{\pm}^{\pm}\}$ belong to the same periodic orbit of period 4τ .

5.4.3 $\ell \geq 5$

We will work in action-angle variables with the normal form (4.1.3b). The origin remains stable across the bifurcation but the sense of rotation of the flow around it is inverted. The other fixed points are solutions of

$$\begin{cases} -\ell \mathbf{b}_\ell I^{\ell/2} \sin(\ell\theta) + O_{\sqrt{I}, \sqrt{|\varepsilon|}}(\ell+1) = 0; & (5.4.17a) \end{cases}$$

$$\begin{cases} \varepsilon + \sum_{\substack{k \in \mathbb{N} \\ 2 \leq k \leq \ell/2}} k \mathbf{a}_k(\varepsilon) I^{k-1} + \frac{1}{2} \ell \mathbf{b}_\ell(\varepsilon) I^{\ell/2-1} \cos(\ell\theta) + O_{\sqrt{I}, \sqrt{|\varepsilon|}}(\ell-1) = 0. & (5.4.17b) \end{cases}$$

To first order they correspond to: $\theta_k \stackrel{\text{def}}{=} k \frac{\pi}{\ell}$ for $k \in \{0, \dots, 2\ell-1\}$ and $I = -\frac{1}{2a_2} \varepsilon + O_{\sqrt{|\varepsilon|}}(3)$ where $a_2 \stackrel{\text{def}}{=} \mathbf{a}_2(\varepsilon=0)$.

→ $\boxed{\varepsilon/a_2 < 0}$ In addition to the origin, there are 2ℓ real fixed points

$$\mathbf{p}_k \stackrel{\text{def}}{=} \begin{pmatrix} I = -\varepsilon/(2a_2) \\ \theta_k \stackrel{\text{def}}{=} k\pi/\ell \end{pmatrix} + O_{\sqrt{|\varepsilon|}}(3) \quad \text{with} \quad k \in \{0, \dots, 2\ell-1\}. \quad (5.4.18)$$

To first order, they lie on the circle with center \mathfrak{o} and radius $\sqrt{-\varepsilon/a_2}$. The linearization of the

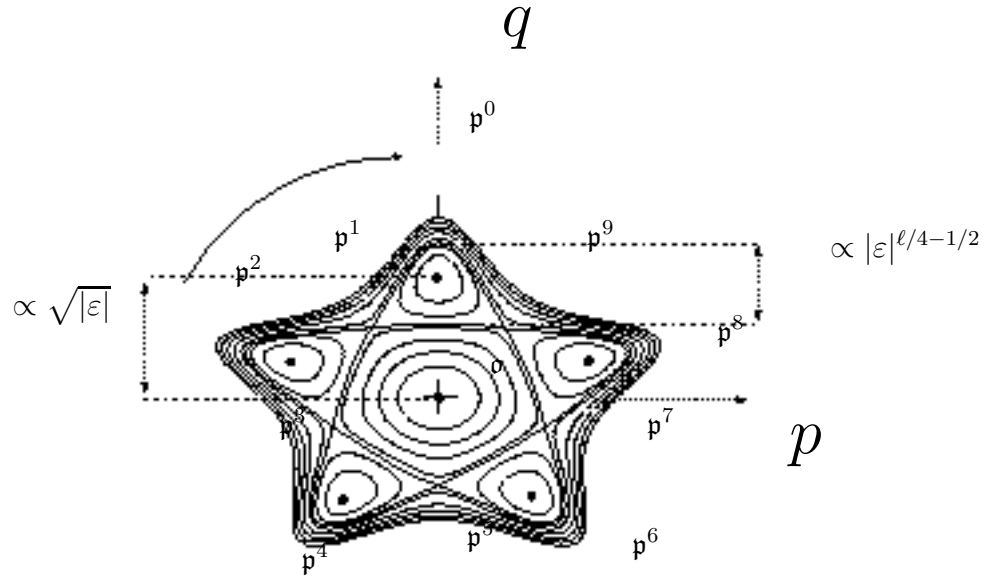


Figure 17: Flow near the bifurcation of a stable fixed point for $\ell = 5$ when $\varepsilon/a_2 < 0$.

equations of motion about \mathbf{p}_k is

$$\begin{pmatrix} \dot{I} \\ \dot{\theta} \end{pmatrix} = \begin{pmatrix} 0 & (-1)^k \ell^2 b |\varepsilon / 2a_2|^{\ell/2} \\ 2a_2 & 0 \end{pmatrix} \begin{pmatrix} I \\ \theta \end{pmatrix} \quad (5.4.19)$$

which shows that the points $\{\mathbf{p}_k\}_{k \in \{0, \dots, 2\ell-1\}}$ are alternately stable and unstable. Without loss of generality, we can assume that \mathbf{p}_0 is stable. The separatrix has the following equation

$$\varepsilon I + \sum_{\substack{k \in \mathbb{N} \\ 2 \leq k \leq \ell/2}} \mathbf{a}_k(\varepsilon) I^k + \mathbf{b}_\ell(\varepsilon) I^{\ell/2} \cos(\ell\theta) = H_\ell(\mathbf{p}_1; t; \varepsilon) + O_{\sqrt{I}, \varepsilon^*}(\ell + 1), \quad (5.4.20)$$

where

$$\varepsilon^* \stackrel{\text{def}}{=} \begin{cases} |\varepsilon| & \text{if } \ell = 3; \\ \sqrt{|\varepsilon|} & \text{if } \ell \neq 3. \end{cases} \quad (5.4.21)$$

It is a bounded set $\frac{2\pi}{\ell}$ -periodic in θ . To evaluate the typical transversal size δr of the island around the stable satellites we should solve to first order equation (5.4.20) for $\theta = 0$ with $I = I(\mathbf{p}_1) + \delta I$. We obtain $\delta I \propto |\varepsilon|^{\ell/4}$ and deduce that

$$\delta r \propto |\varepsilon|^{\frac{\ell}{4} - \frac{1}{2}}. \quad (5.4.22)$$

(see figure 17).

→ $\boxed{\varepsilon = 0}$ All the fixed points have collapsed at the origin around which the motion is stable with a positive rotating speed.

→ $\boxed{\varepsilon/a_2 > 0}$ \mathbf{o} is the only real fixed point. There are 2ℓ fixed points in the complex plane

$$\mathbf{p}_k \stackrel{\text{def}}{=} \begin{pmatrix} I = -\varepsilon/a_2 < 0 \\ \theta_k \stackrel{\text{def}}{=} k\pi/\ell \end{pmatrix} + O_{\sqrt{|\varepsilon|}}(3) = \begin{pmatrix} p_k \stackrel{\text{def}}{=} i\sqrt{\varepsilon/a_2} \cos(k\pi/\ell) \\ q_k \stackrel{\text{def}}{=} i\sqrt{\varepsilon/a_2} \sin(k\pi/\ell) \end{pmatrix} + O_{\sqrt{|\varepsilon|}}(2) \quad (5.4.23)$$

for $k \in \{0, \dots, 2\ell - 1\}$.

If we go back from the rotating frame to the fixed one, we add a global rotation of angular speed $\omega_0 = 2\pi r/(\ell\tau)$. This will make the points \mathbf{p} belong to two different periodic orbits of period $\ell\tau$.

5.5 Summary

Except the case $\ell = 3$ where there are none, in all other cases there are complex periodic orbits in the vicinity of the bifurcation. Each of them is connected continuously through the bifurcation with a real one. When $\ell = 4$ and $a + b < 0$, on each side of the bifurcation there is one real and one complex satellite orbit around the central point. As the external parameter goes through the bifurcation these two orbits interchange, the real one becoming complex and vice versa. In all other cases, when they exist, the complex orbits are present on one side of the bifurcation only.

The bifurcating (real and complex) orbits are typically separated in phase space by a scale ε^* given by Eq. (5.4.21). This quantity will play a crucial role in the semiclassical considerations of the following sections. In the particular case of stable bifurcating orbits, in addition to ε^* there exists another typical scale $\propto |\varepsilon|^{\ell/4-1/2}$ which gives the radial width of the stable satellite islands. For the extremal (resp. transitional) bifurcation the width of the stable island(s) is given by $|\varepsilon|^{3/4}$ (resp. $|\varepsilon|$). In all cases, the classical flow looks roughly the same in both sides of the bifurcation if one looks at it with a resolution of order ε^* or worst.

6 Normal form for one-dimensional unfolding of generating functions

So far, we have described the τ -periodic dynamics around an equilibrium point by simplifying the hamiltonian in its neighborhood. We have revisited the classification of the generic fixed points by describing, in each case, the normal form of a hamiltonian whose dynamics, viewed every -2τ , is the same as the original one. The one-freedom dynamics thus represents the local Poincaré reduction around a τ -periodic orbit. The associated local Poincaré map $\Phi(w; \varepsilon)$ is the stroboscopic view of the reduced dynamics taken every τ . In order to complete our study, in this section we deduce the normal forms of the generating function of $\Phi(w; \varepsilon)$. In some cases, like in semiclassical theory, the Poincaré map plays actually a more central role than the reduced hamiltonian itself.

The starting point for calculating the Poincaré map from a time-independent hamiltonian is the formula

$$\Phi(w; \varepsilon) = e^{\tau\{\cdot, H(w; \varepsilon)\}} w \tag{6.0.1}$$

$$= w + \tau\{w, H(w; \varepsilon)\} + \sum_{m=2}^{\infty} \frac{\tau^m}{m!} \underbrace{\{\dots \{w, H(w; \varepsilon)\}, \dots, H(w; \varepsilon)\}}_{m \text{ times}}, \tag{6.0.2}$$

where we define the Poisson bracket of two smooth functions on phase space \mathcal{P} by

$$\{A, B\} \stackrel{\text{def}}{=} \text{t}(\nabla_w A) \mathbb{J} \nabla_w B ; \quad (6.0.3)$$

$$\stackrel{\text{def}}{=} \frac{\partial A}{\partial q} \frac{\partial B}{\partial p} - \frac{\partial B}{\partial q} \frac{\partial A}{\partial p} . \quad (6.0.4)$$

Let $w_0 \in \mathcal{P}$ and construct the backward and forward sequence $\forall n \in \mathbb{Z}; w_n \stackrel{\text{def}}{=} \Phi^{(n)}(w_0, \varepsilon)$. Some relevant functions containing the same information as $\Phi^{(n)}(\cdot; \varepsilon)$ are

- The qq -generating function

$$F^{(n)}(w_n, w_0) \stackrel{\text{def}}{=} \int_0^{n\tau} \left(p(t) \dot{q}(t) - H[p(t), q(t); t; \varepsilon] \right) dt . \quad (6.0.5)$$

$w(t) \stackrel{\text{def}}{=} \begin{pmatrix} p(t) \\ q(t) \end{pmatrix}$ is the unique solution of (3.0.5) such that $w(0) = w_0$ and $w(n\tau) = w_n$. Locally $F^{(n)}$ depends only on two independent variables among (p_n, q_n, p_0, q_0) . These are often taken to be q_n and q_0 . The *values* of this function are canonically invariant, they do not depend on the choice of the symplectic coordinate chart – possibly time dependent – in which the integral is calculated. When H is constant we get

$$F^{(n)}(w_n, w_0) = -n\tau \underbrace{H(w_0; \varepsilon)}_{=H(w_n; \varepsilon)} + \int_0^{n\tau} p \{q, H\} dt . \quad (6.0.6)$$

- The mixed generating function

$$S^{(n)}(p_n, q_0; \varepsilon) \stackrel{\text{def}}{=} p_n q_n(p_n, q_0; \varepsilon) - F^{(n)}(p_n, q_0; \varepsilon) . \quad (6.0.7)$$

In all cases but the unstable one, we will not be able to complete the summation (6.0.2) exactly. Nevertheless following our general philosophy we will work at a finite precision. From section 5 we have determined for each bifurcation the typical scale in ε entering the dynamics. This scale is fixed by the order of truncation of the normal forms of the hamiltonian: the more terms are kept, the finer details can be described. Accordingly, we will now show that all the dynamical structures up to a fixed scale are actually contained in the first terms of the expansion (6.0.2). This allows to compute the generating functions for each type of bifurcation. Let us start with the unstable case which is special, as we said, in the sense that the sum can be computed explicitly to all orders. The physical reason for this is that in a generic unfolding of an unstable point there is no bifurcation and hence no local typical scale.

6.1 Unstable case

Since no structural change is induced when ε varies, we will leave the ε -dependence implicit up to the end of this subsection. The one-parameter unfolding is given by (4.1.1)

$$H_u(p, q; t) = \lambda pq + \underbrace{\sum_{k=2}^n \mathfrak{h}_k (pq)^k}_{\stackrel{\text{def}}{=} K_n(pq)} + O_{p,q}(2n+1) \quad (6.1.1)$$

We recall (see subsection 3.2) that the true dynamics is recovered by taking a stroboscopic view every $|\varsigma\tau|$, where $\varsigma=1$ for the hyperbolic case and $\varsigma=\pm 2$ for the inverse hyperbolic case. (In the following discussion we take, for the inverse hyperbolic case, the plus sign). Since $d(pq)/dt = \{pq, H_u(p, q; t)\} \in O_{p,q}(2n+1)$

pq is a constant of motion modulo $O_{p,q}(2n+1)$ and the solutions of the equations of motion are simply

$$\begin{cases} p(t) = p_0 e^{-tK'_n(p_0q_0)} + O_{p_0,q_0}(2n); & (6.1.2a) \\ q(t) = q_0 e^{+tK'_n(p_0q_0)} + O_{p_0,q_0}(2n). & (6.1.2b) \end{cases}$$

From (6.0.6) we get

$$F_u^{(\varsigma)}(w_1, w_0) = \varsigma\tau [p_0q_0 K'_n(p_0q_0) - K_n(p_0q_0)] + O_{p_0,q_0}(2n+1). \quad (6.1.3)$$

The mixed generating function is obtained from the definition (6.0.7) after inverting equations (6.1.2) up to first order

$$S_u^{(\varsigma)}(p_\varsigma, q_0) = p_0q_0 - \tau [K_n(p_0q_0) - p_0q_0 K'_n(p_0q_0)] \quad (6.1.4)$$

$$= e^{\varsigma\tau\lambda} p_\varsigma q_0 + \varsigma\tau \mathfrak{h}_2 e^{2\varsigma\tau\lambda} (p_\varsigma q_0)^2 + [\varsigma\tau \mathfrak{h}_3 + 2(\varsigma\tau \mathfrak{h}_2)^2] e^{3\varsigma\tau\lambda} (p_\varsigma q_0)^3 + O_{p_\varsigma,q_0}(8). \quad (6.1.5)$$

In the inverse hyperbolic case the hamiltonian (6.1.1) cannot alone give all the information about the dynamics at time τ . In particular, $w(\tau)$ given by (6.1.2) does not contain one transformation whose linear part is a symmetry with respect to the origin. If we instead propose as a guess the following one-step generating function

$$S_u^{(1)}(p_1, q_0) = \sigma_1 p_1 q_0 + \sigma_2 (p_1 q_0)^2 + \sigma_3 (p_1 q_0)^3 + O_{p_1,q_0}(8) \quad (6.1.6)$$

then the real coefficients σ_1 , σ_2 and σ_3 can be determined in the following way

- 1- Invert equations (3.2.1) modulo $O_{p_1, q_1}(7)$ to find out $w_1(w_0)$;
- 2- The monodromy matrix being $\mathbb{M} = \begin{pmatrix} -e^{-\tau\lambda} & 0 \\ 0 & -e^{\tau\lambda} \end{pmatrix}$, deduce that $\sigma_1 = -e^{\tau\lambda}$;
- 3- Iterate the mapping in order to get w_2 modulo $O_{p_1, q_1}(7)$ and identify the expression with $w(2\tau)$ obtained via equations (6.1.2);
- 4- Eventually find out σ_2 and σ_3 .

For the inverse hyperbolic case, we thus obtain

$$S_u^{(1)}(p_1, q_0) = -e^{\tau\lambda} p_1 q_0 - \tau h_2 e^{2\tau\lambda} (p_1 q_0)^2 + [\tau h_3 + \frac{1}{2}(\tau h_2)^2] e^{3\tau\lambda} (p_1 q_0)^3 + O_{p_1, q_0}(8). \quad (6.1.7)$$

Note that, in any case, the mixed generating functions have the same functional form as the hamiltonian but the relations between their respective coefficients are non-trivial.

6.2 Extremal case

From hamiltonian (5.2.1) the typical scales are $p \propto \varepsilon^{\frac{3}{4}}$ and $q \propto \sqrt{|\varepsilon|}$. We deduce the following Poisson brackets

$$\{q, H_1\} = \frac{1}{\tau} p + O_{p, q, \sqrt{|\varepsilon|}}(3); \quad (6.2.1)$$

$$\{p, H_1\} = -\varepsilon - a q^2 + O_{p, q, \sqrt{|\varepsilon|}}(3); \quad (6.2.2)$$

$$\{\{p, H_1\}, H_1\} = -\frac{2a}{\tau} p q + O_{p, q, \sqrt{|\varepsilon|}}(3); \quad (6.2.3)$$

$$\underbrace{\{\dots, \{p, H_1\}, \dots, H_1\}}_{n \text{ times, } n \geq 3} \in O_{p, q, \sqrt{|\varepsilon|}}(3). \quad (6.2.4)$$

From (6.0.2) we get the Poincaré map

$$\Phi_1^{(1)} \left\{ \begin{array}{l} p_1 = p_0 - \tau\varepsilon - \tau a q_0^2 - \frac{\tau}{3} a p_0^2 - \tau a p_0 q_0 + O_{p_0, q_0, \sqrt{|\varepsilon|}}(3); \\ q_1 = q_0 + p_0 - \frac{\tau}{2} \varepsilon - \frac{\tau}{2} a q_0^2 - \frac{\tau}{12} a p_0^2 - \frac{\tau}{3} a p_0 q_0 + O_{p_0, q_0, \sqrt{|\varepsilon|}}(3). \end{array} \right. \quad (6.2.5a)$$

$$\quad (6.2.5b)$$

To get the mixed generating function we have two methods

- The first is to write the general form of $S^{(1)}(p_1, q_0)$ modulo $O_{p_0, q_0, \sqrt{|\varepsilon|}}(4)$ and deduce the corresponding map. We get S by identifying the latter with (6.2.5);

- The second is to calculate $S^{(1)}$ via the formula (6.0.7).

We obtain

$$S_1^{(1)}(p_1, q_0) = \frac{1}{2} p_1^2 + \frac{\tau}{2} \varepsilon p_1 + \tau \varepsilon q_0 + p_1 q_0 + \frac{\tau}{12} a p_1^3 + \frac{\tau}{3} a p_1^2 q_0 + \frac{\tau}{2} a p_1 q_0^2 + \frac{\tau}{3} a q_0^3 + O_{p_1, q_0, \sqrt{|\varepsilon|}}(4), \quad (6.2.6)$$

and for the qq -generating function (6.0.5)

$$F_1^{(1)} = -\frac{1}{2} p_0^2 + \tau \varepsilon q_0 + \tau \varepsilon p_0 + \frac{\tau}{3} a q_0^3 + \frac{2\tau}{3} a p_0^2 q_0 + \tau a p_0 q_0^2 + \frac{\tau}{6} a p_0^3 + O_{p_0, q_0, \sqrt{|\varepsilon|}}(4). \quad (6.2.7)$$

Both functions have a functional form quite different from the one of H_1 .

6.3 Transitional case

From hamiltonian (5.3.1) the typical scales are $p \propto |\varepsilon|$ and $q \propto \sqrt{|\varepsilon|}$. Remember that this hamiltonian describes the true dynamics when we look at the dynamics *backwards* every 2τ . Moreover $\varsigma = -2$ in equation (3.1.10) and the monodromy matrix for one period is

$$\mathbb{M} = \begin{pmatrix} -1 & 0 \\ 1 & -1 \end{pmatrix}. \quad (6.3.1)$$

From the hamiltonian H_2 , we deduce the following Poisson brackets

$$\{q, H_2\} = \frac{1}{\tau} p + O_{\sqrt{|p|}, q, \sqrt{|\varepsilon|}}(5); \quad (6.3.2)$$

$$\{p, H_2\} = -2\varepsilon q - a q^3 - \varepsilon b q^2 + c q^4 + O_{\sqrt{|p|}, q, \sqrt{|\varepsilon|}}(5); \quad (6.3.3)$$

$$\{\{p, H_2\}, H_2\} = -\frac{1}{\tau} p(2\varepsilon + 3a q^2) + O_{\sqrt{|p|}, q, \sqrt{|\varepsilon|}}(5); \quad (6.3.4)$$

$$\{\dots \underbrace{\{p, H_2\}, \dots, H_2\}}_{n \text{ times, } n \geq 3}\} \in O_{\sqrt{|p|}, q, \sqrt{|\varepsilon|}}(5). \quad (6.3.5)$$

From (6.0.2) calculated with $\tau \mapsto -2\tau$, we get the Poincaré map

$$\Phi_2^{(2)} \begin{cases} p_2 = p_0 + 4\tau \varepsilon q_0 + 2\tau a q_0^3 - 4\tau \varepsilon p_0 + 2\tau \varepsilon b q_0^2 + 2\tau c q_0^4 - 6\tau c p_0 q_0^2 + O_{\sqrt{|p_0|}, q_0, \sqrt{|\varepsilon|}}(5); & (6.3.6a) \\ q_2 = q_0 - 2p_0 - 4\tau \varepsilon q_0 - 2\tau a q_0^3 + \frac{8}{3} \tau \varepsilon p_0 - 2\tau \varepsilon b q_0^2 - 2\tau c q_0^4 + 4\tau c p_0 q_0^2 + O_{\sqrt{|p_0|}, q_0, \sqrt{|\varepsilon|}}(5). & (6.3.6b) \end{cases}$$

We now compute the one-step mixed generating function. Since it is smooth with respect to ε , its general form is

$$S_2^{(1)}(p_1, q_0) = \frac{1}{2} \eta q_0^2 + \gamma p_1 q_0 + \frac{1}{3} \xi q_0^3 + \frac{1}{2} \delta p_1^2 + \kappa p_1 q_0^2 + \frac{1}{4} \theta_1 q_0^4 + \zeta_3 p_1^2 q_0 + \theta_2 p_1 q_0^3 + \frac{1}{5} \iota q_0^5 \\ + \frac{1}{3} \lambda_1 p_1^3 + \frac{1}{2} \lambda_2 p_1^2 q_0^2 + \lambda_3 p_1 q_0^3 + O_{\sqrt{|p_0|}, q_0, \sqrt{|\varepsilon|}}(5) . \quad (6.3.7)$$

where $(\theta_1, \zeta_3, \theta_2, \iota, \lambda_1, \lambda_2, \lambda_3) \in \mathbb{R}^6$ and $(\eta, \gamma, \xi, \delta, \kappa)$ are real functions of ε

$$\begin{aligned} \eta &= \eta_0 + \varepsilon \eta_1 ; & \gamma &= \gamma_0 + \varepsilon \gamma_1 ; & \xi &= \xi_0 + \varepsilon \xi_1 ; \\ \delta &= \delta_0 + \varepsilon \delta_1 ; & \kappa &= \kappa_0 + \varepsilon \kappa_1 ; \end{aligned} \quad (6.3.8)$$

with $(\eta_0, \eta_1, \gamma_0, \gamma_1, \xi_0, \xi_1, \delta_0, \delta_1, \kappa_0, \kappa_1) \in \mathbb{R}^6$ and $\gamma_0 \neq 0$.

Although they are of order $O_{\sqrt{|p_1|}, q_0, \sqrt{|\varepsilon|}}(6)$, the last three terms in $S_2^{(1)}$ must be taken into account. This is because when approximating the Poincaré map up to order four (see equations (4.2.15) and (3.2.1)) differentiation with respect to p_1 makes the order fall by two. From $p_0(p_1, q_0)$ and $q_1(p_1, q_0)$ modulo $O_{\sqrt{|p_1|}, q_0, \sqrt{|\varepsilon|}}(5)$, it is straightforward to invert them to obtain $p_1(p_0, q_0)$ and $q_1(p_0, q_0)$ modulo $O_{\sqrt{|p_0|}, q_0, \sqrt{|\varepsilon|}}(5)$. Condition (6.3.1) imposes

$$\gamma_0 = \delta_0 = -1 . \quad (6.3.9)$$

From this we can calculate $p_2(p_0, q_0)$ and $q_2(p_0, q_0)$ modulo $O_{\sqrt{|p_0|}, q_0, \sqrt{|\varepsilon|}}(5)$. Identifying these two latter equations with the Poincaré map $\Phi_2^{(2)}$ given by (4.2.15) we get a set of equations with a unique solution if and only if

$$b = 3 \frac{d\mathfrak{h}_3}{d\varepsilon} \Big|_{\varepsilon=0} = 0 \quad \text{and} \quad c = 5 \mathfrak{h}_5(0) = 0 . \quad (6.3.10)$$

(cf the discussion of subsection 4.2). We finally find

$$S_2^{(1)}(p_1, q_0) = -p_1 q_0 - \tau \varepsilon q_0^2 - \frac{1}{2} p_1^2 - \frac{1}{4} \tau a q_0^4 - \tau \varepsilon p_1 q_0 - \frac{1}{2} \tau a p_1 q_0^3 + O_{\sqrt{|p_1|}, q_0, \sqrt{|\varepsilon|}}(5) , \quad (6.3.11)$$

as well as the Poincaré map for one period

$$\Phi_2^{(1)} \left\{ \begin{aligned} p_1 &= -p_0 - 2\tau \varepsilon q_0 - \tau a q_0^3 + \tau \varepsilon p_0 + \frac{3}{2} \tau \varepsilon a p_0 q_0^2 + O_{\sqrt{|p_0|}, q_0, \sqrt{|\varepsilon|}}(5); \\ q_1 &= -q_0 + p_0 + \tau \varepsilon q_0 + \frac{1}{2} \tau a q_0^3 - \frac{1}{3} \tau \varepsilon p_0 + \varepsilon \kappa_1 q_0^2 \\ &\quad + \lambda_1 p_0^2 - \frac{1}{2} a \tau p_0 q_0^2 + (\lambda_3 - 2\zeta_3 \tau a) q_0^4 + O_{\sqrt{|p_0|}, q_0, \sqrt{|\varepsilon|}}(5). \end{aligned} \right. \quad (6.3.12a)$$

$$(6.3.12b)$$

We should have worked to a higher order to determine $(\kappa_1, \lambda_1, \lambda_3)$. To complete the study, let us give the expression of the two-step mixed generating function

$$S_2^{(2)}(p_2, q_0) \stackrel{\text{def}}{=} p_2 q_2 - 2\tau H_2 - \int_0^{-2\tau} p(t) \dot{q}(t) dt, \quad (6.3.13)$$

where $p(t)$ and $q(t)$ are calculated via (6.0.2). We obtain

$$S_2^{(2)}(p_2, q_0) = p_2 q_0 - p_2^2 - 2\tau \varepsilon q^2 - \frac{1}{2} \tau a q^4 + O_{\sqrt{|p_2|}, q_0, \sqrt{|\varepsilon|}}(5), \quad (6.3.14)$$

as well as the qq -generating function

$$F_2^{(2)} = S_2^{(2)}(p_2, q_0) - p_2 q_2 = 2\tau H_2(p_2, q_2; \tau; \varepsilon) - 2p_2^2 + O_{\sqrt{|p_2|}, q_2, \sqrt{|\varepsilon|}}(5); \quad (6.3.15)$$

$$= -p_2^2 + 2\tau \varepsilon q^2 + \frac{1}{2} a q^4 + O_{\sqrt{|p_2|}, q_2, \sqrt{|\varepsilon|}}(5). \quad (6.3.16)$$

6.4 Stable cases

We will treat all stable cases for $\ell \geq 3$ simultaneously by regrouping hamiltonians (5.4.1b), (5.4.6b) and (4.1.3b) in the unified form

$$H_\ell(I, \theta; t; \varepsilon) = \varepsilon I + \sum_{\substack{k \in \mathbb{N} \\ 2 \leq k \leq \ell/2}} a_k(\varepsilon) I^k + b_\ell(\varepsilon) I^{\ell/2} \cos(\ell\theta) + O_{\sqrt{I}, \varepsilon^*}(\ell + 1). \quad (6.4.1)$$

We have seen that both p and q scale as ε^* given by (5.4.21). Recall (cf Eq. (3.3.13)) that this normal form is obtained looking at the dynamics in a rotating frame of angular speed ω_0 given by Eq. (4.1.2).

As in the two previous subsections, let us start computing the Poisson brackets in (6.0.2)

$$\{I, H_\ell\} = \ell b_\ell(\varepsilon) I^{\ell/2} \sin(\ell\theta) + O_{\sqrt{I}, \varepsilon^*}(\ell + 1); \quad (6.4.2)$$

$$\{\theta, H_\ell\} = \varepsilon + \sum_{\substack{k \in \mathbb{N} \\ 2 \leq k \leq \ell/2}} k a_k(\varepsilon) I^{k-1} + \frac{1}{2} \ell b_\ell(\varepsilon) I^{\ell/2-1} \cos(\ell\theta) + O_{\sqrt{I}, \varepsilon^*}(\ell - 1); \quad (6.4.3)$$

$$\{\dots \underbrace{\{I, H_\ell\}, \dots, H_\ell}_{n \text{ times}, n \geq 2}\} \in O_{\sqrt{I}, \varepsilon^*}(\ell + 1); \quad (6.4.4)$$

$$\{\dots \underbrace{\{\theta, H_\ell\}, \dots, H_\ell}_{n \text{ times}, n \geq 2}\} \in O_{\sqrt{I}, \varepsilon^*}(\ell + 1). \quad (6.4.5)$$

Then, in the rotating frame we obtain $I(t)$ and $\theta(t)$ modulo $O_{\sqrt{I}, \varepsilon^*}(\ell + 1)$ and $O_{\sqrt{I}, \varepsilon^*}(\ell - 1)$ respectively. Going back to the non-rotating frame by adding $\omega_0 t$ to $\theta(t)$ we have

$$I(t) = I_0 + \ell t b_\ell(\varepsilon) I_0^{\ell/2} \sin[\ell(\theta_0 + \omega_0 t)] + O_{\sqrt{I_0}, \sqrt[3]{|\varepsilon|}}(\ell + 1); \quad (6.4.6a)$$

$$\begin{aligned} \theta(t) = \theta_0 + (\omega_0 + \varepsilon)t + t \sum_{\substack{k \in \mathbb{N} \\ 2 \leq k \leq \ell/2}} k a_k(\varepsilon) I_0^{k-1} + \frac{1}{2} \ell b_\ell(\varepsilon) I_0^{\ell/2-1} \cos[\ell(\theta_0 + \omega_0 t)] \\ + O_{\sqrt{I_0}, \sqrt[3]{|\varepsilon|}}(\ell - 1). \end{aligned} \quad (6.4.6b)$$

For every integer n , it is then straightforward to express the first orders of $\theta_n \stackrel{\text{def}}{=} \theta(n\tau)$ and I_0 in terms of $I_n \stackrel{\text{def}}{=} I(n\tau)$ and θ_0 . The mixed generating function for time n follows immediately

$$\begin{aligned} S_\ell^{(n)}(I_n, \theta_0) = I_n \theta_0 + n(\omega_0 + \varepsilon)\tau I_n + n\tau \sum_{\substack{k \in \mathbb{N} \\ 2 \leq k \leq \ell/2}} a_k(\varepsilon) I_n^k + n\tau b_\ell(\varepsilon) I_n^{\ell/2} \cos(\ell\theta_0) \\ + O_{\sqrt{I_n}, \sqrt[3]{|\varepsilon|}}(\ell + 1) \end{aligned} \quad (6.4.7)$$

$$= I_n \theta_0 + n\omega_0 \tau I_n + n\tau H_\ell(I_n, \theta_0). \quad (6.4.8)$$

We finally calculate the qq -generating function from, for instance, its definition (6.0.5)

$$F_\ell^{(n)} = n\tau \sum_{\substack{k \in \mathbb{N} \\ 2 \leq k \leq \ell/2}} (k-1) a_k(\varepsilon) I_0^k + n\tau \left(\frac{\ell}{2} - 1 \right) b_\ell(\varepsilon) I_0^{\ell/2} \cos(\ell\theta_0) + O_{\sqrt{I_0}, \sqrt[3]{|\varepsilon|}}(\ell + 1). \quad (6.4.9)$$

7 Asymptotic expansion of oscillating integrals

In the previous sections, we have exclusively dealt with classical mechanics. The semiclassical analogue of these geometrical studies is highly non-trivial for essentially two reasons. The first is connected to the choice of the quantum hamiltonian one must start with. Infinitely many quantum hamiltonians differing from each other by \hbar -dependent terms correspond to the same classical dynamics. The second is that the semiclassical analogue of the Poincaré reduction (section 2) is very difficult to extend beyond the leading order (see [10, 43]). Nevertheless, we will assume in the following that one has already solved these problems for a given system.

Let us then begin with a given evolution operator $\hat{U}_\gamma(\hat{p}, \hat{q})$ depending on one external parameter γ . The Hilbert space on which the canonical operators \hat{p} and \hat{q} are defined can be finite or infinite. The associated classical dynamics corresponds to a one-step iteration Poincaré map $\Phi_\gamma^{(1)}(p, q)$.

Our aim now is to use Meyer's classification to write down semiclassical expansions of physical quantities related to the exact quantum evolution operator \hat{U}_γ . For instance, we may compute semi-

classically the trace of correlation operators of the form

$$\hat{C}(\gamma) \stackrel{\text{def}}{=} \prod_{i=0}^M a_i(\hat{p}, \hat{q}) \hat{U}_\gamma^{n_i} \quad (7.0.10)$$

where M is a strictly positive integer, $\{a_i\}_{i \in \{0, \dots, M\}}$ are some smooth functions which do not depend on \hbar and $\{n_i\}_{i \in \{0, \dots, M\}}$ are some integers whose sum is $\ell \neq 0$. Without loss of generality we can assume $\ell > 0$ by possibly changing the arrow of time. One particular case of such operators are of course the trace of the ℓ th power of \hat{U}_γ , which play a central role for calculating the spectral function of the quantum system [48].

Formally we have

$$Q_\ell(\hbar; \gamma) \stackrel{\text{def}}{=} \text{tr}[\hat{C}(\gamma)] = \sum_{k \geq 0} \hbar^k \iint_{\mathfrak{s}} F_{\ell, k}(p', q; \gamma) e^{\frac{i}{\hbar} N^{(\ell)}(p', q; \gamma)} dp' dq, \quad (7.0.11)$$

where the F 's are some smooth functions. The integration domain is the whole bidimensional phase space and the argument of the exponential is

$$N^{(\ell)}(p', q; \gamma) = N^{(\ell)}(0, 0; \gamma) + p'q - S^{(\ell)}(p', q; \gamma) \quad (7.0.12)$$

where $S^{(\ell)}$ is the mixed generating function of the ℓ th iterated Poincaré classical map $\Phi_\gamma^{(\ell)} \stackrel{\text{def}}{=} [\Phi_\gamma^{(1)}]^\ell$. Without loss of generality we can take $N^{(\ell)}(0, 0; \gamma) = 0$.

The basic steps leading to Eq. (7.0.11) are the following. First, the fact that $\Phi_\gamma^{(1)}(p, q)$ is the classical dynamics associated to $\hat{U}_\gamma(\hat{p}, \hat{q})$ means that we have

$$\frac{\langle p' | \hat{U}_\gamma | q \rangle}{\langle p' | q \rangle} \underset{\hbar \rightarrow 0}{\sim} e^{\frac{i}{\hbar} N^{(1)}(p', q; \gamma)} \sum_{k \geq 0} u^{(k)}(p', q) \hbar^k \quad (7.0.13)$$

where the u 's are some smooth functions except, perhaps, for a zero-measure set. For \hat{U}_γ to be unitary, u_0 has to be the square root of $\partial^2 N^{(1)}/\partial p' \partial q$ [38, Appendix A].

Second, remark that for every smooth $a(p, q)$ and for every integer n , we have the formal asymptotic series

$$\frac{\langle p | \hat{U}_\gamma^{-n} a(\hat{p}, \hat{q}) \hat{U}_\gamma^n | q \rangle}{\langle p | q \rangle} \underset{\hbar \rightarrow 0}{\sim} a[\Phi_\gamma^{(n)}(p, q)] + \sum_{k \geq 1} a^{(k)}(p, q) \hbar^k \quad (7.0.14)$$

for some functions $a^{(k)}$'s which are smooth everywhere except for a zero-measure set. This last formula can be understood if we write the formal expansion of $a(p, q)$ in powers of (p, q) , then use

the commutation relation between \hat{p} and \hat{q} and eventually make a stationary phase expansion at all orders in \hbar after having inserted the appropriate resolution of identity made with the projectors $|p\rangle\langle p|$ or $|q\rangle\langle q|$. Except for a zero-measure set of (p, q) 's the stationary points of the oscillating integral are non-degenerated.

Third, write $Q_\ell(\hbar; \gamma)$ in the following manner

$$Q_\ell(\hbar; \gamma) = \text{tr}[\hat{U}_\gamma^\ell a_1 \hat{U}_\gamma^{n_1} a_2 \hat{U}_\gamma^{-n_1} \hat{U}_\gamma^{n_1+n_2} a_3 \hat{U}_\gamma^{-n_1-n_2} \dots \hat{U}_\gamma^{\ell-n_M} a_M \hat{U}_\gamma^{-\ell+n_M}] , \quad (7.0.15)$$

and insert alternatively the resolution of identity made with the projectors $|p\rangle\langle p|$ or $|q\rangle\langle q|$. We are therefore led to a many-dimensional oscillating integral. A stationary phase expansion can be made formally to all orders in \hbar in the Morse (quadratic) directions of the critical points. Eq. (7.0.11) is thus recovered, where the integral in the quasi-degenerate directions remains to be done³.

If we make a stationary phase approximation of the integrals appearing in (7.0.11), we see that the domains of phase space which contribute are the neighborhoods of the fixed points of $\Phi^{(\ell)}$, which are the periodic orbits of $\Phi^{(1)}$ whose period divides ℓ . When we want to calculate the contribution to Q_ℓ of the k repetitions of an $m\tau$ -periodic orbit ($mk = \ell$) we should have specified the normal form of the generating functions corresponding to the mapping $\Phi_m^{(\ell)}$ (in section 6 it has been done only when $\ell = m$ for extremal and transitional cases). But it is clear, from the series (6.0.2), that changing $m\tau$ to $mk\tau$ will only modify the coefficients of the polynomials in (p, q, ε) but not their form (in the stable case we get a simple relation of proportionality: for $F_m^{(\ell)}$ the τ -dependence is given by $\ell\tau$, see Eq. (6.4.9)). Since we are mainly interested in the \hbar -dependence of the terms in the semiclassical expansion we are allowed to consider only primitive periodic orbits. Therefore we will take for $N^{(\ell)}$ the generating functions calculated in the previous section.

At this stage, we must recall the hierarchy of structures which are present in a generic mixed classical dynamics. In a compact domain of phase space there exists a finite number of periodic orbits of a given period ℓ . Let us consider one of these orbits. If it is far from its bifurcation point, then complex periodic orbits are neglected and the real ones — which are isolated — contribute in the usual way [26]. On the contrary, near its bifurcation one can always identify a typical distance ε^* which was introduced and calculated for each type of bifurcation in 5. Moreover there exists a typical phase-space quantum scale $\lambda \propto \sqrt{\hbar}$. If $\lambda \gtrsim \varepsilon^*$, then a uniform approximation is needed in order to compute the semiclassical contribution of the orbits to Q_ℓ [40, 49, 46, 47, 44]. Finally, if $\lambda < \varepsilon^*$, then one can still consider the satellites orbits are isolated but now the complex ones are no longer negligible. This

is the typical situation we will deal with in the remaining of this section. More precisely, we will evaluate the semiclassical contribution of the real as well as the complex periodic orbits for each type of bifurcation described previously.

After having constructed a suitable symplectic chart, we are therefore led to oscillating integrals of the form

$$\mathcal{I}_\ell \stackrel{\text{def}}{=} \iint F(p', q) e^{\frac{i}{\hbar} N_\ell(p', q; \varepsilon)} dp' dq, \quad (7.0.16)$$

where F is a smooth generic function. Without loss of generality we can suppose $F(0, 0) = 1$. N_ℓ , defined by (7.0.12), is obtained from the normal forms of the mixed generating function $S_\ell^{(\ell)}$ of section 5. The integration domain is a neighborhood of the fixed island whose central point is at the origin \mathfrak{o} . It is assumed not to overlap with other islands. ε must now be considered to be a fixed quantity small compared to all other classical quantities. We will control semiclassical errors by varying \hbar only.

7.1 Extremal case

From (6.2.6) we have

$$-N_1(p, q; \varepsilon) = \frac{\tau}{2} \varepsilon p + \tau \varepsilon q + \frac{1}{2} p^2 + \frac{\tau}{12} a p^3 + \frac{\tau}{3} a p^2 q + \frac{\tau}{2} a p q^2 + \frac{\tau}{3} a q^3 + O_{p, q, \sqrt{|\varepsilon|}}(4). \quad (7.1.1)$$

Near the fixed points we have $F(p, q) = 1 + O_{p, q}(1)$ and therefore the uniform approximation of (7.0.16) is an Airy function. The two fixed points are isolated if

$$\hbar \ll \tau \varepsilon^2. \quad (7.1.2)$$

They both give an oscillating contribution if they are real ($\varepsilon/a < 0$)

$$\mathcal{I}_1 = \frac{2\pi i \hbar}{\sqrt{2\tau|\varepsilon a|^{1/2}}} \left(e^{-i\frac{\pi}{2}} e^{-2i\tau|\varepsilon|^{3/2}/(3\hbar|a|^{1/2})} [1 + O_{\sqrt{\hbar}}(3)] + e^{2i\tau|\varepsilon|^{3/2}/(3\hbar|a|^{1/2})} [1 + O_{\sqrt{\hbar}}(3)] \right). \quad (7.1.3)$$

This is the usual Gutzwiller contributions where we have made explicit the Maslov index for each orbit (see figure 18 a)).

If $\varepsilon/a > 0$, the integration contour can be deformed to pass through only one complex saddle point, giving an exponentially small contribution (see figure 18 b))

$$\mathcal{I}_1 = \frac{2\pi i \hbar}{\sqrt{2\tau|\varepsilon a|^{1/2}}} e^{-i\frac{\pi}{4}} e^{-2\tau|\varepsilon|^{3/2}/(3\hbar|a|^{1/2})} (1 + O_{\hbar}(2)). \quad (7.1.4)$$

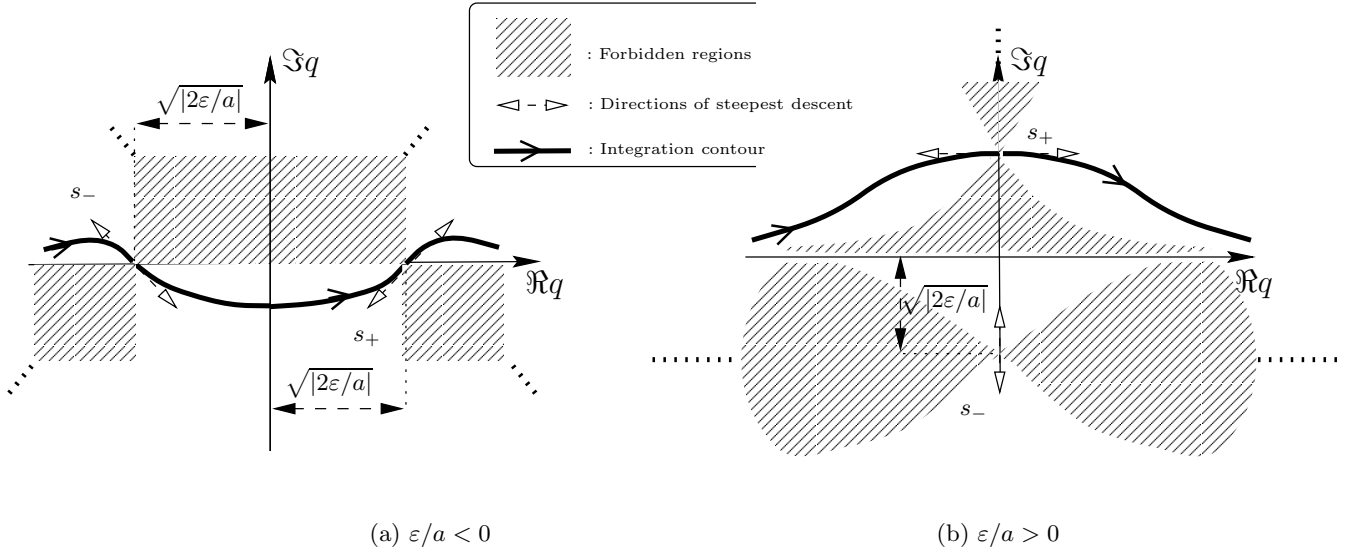


Figure 18: Deformations of the contour through the saddle points for the oscillating integrals \mathcal{I}_1 .

7.2 Transitional case

From (6.3.14) we have

$$N_2(p, q) = p^2 + 2\tau\varepsilon q^2 + \frac{1}{2}\tau a q^4 + O_{\sqrt{|p|}, q, \sqrt{|\varepsilon|}}(5). \quad (7.2.1)$$

The satellite points can be considered to be isolated from the origin if

$$\hbar \ll \tau \frac{2\varepsilon^2}{|a|}. \quad (7.2.2)$$

The contribution of the central fixed point is

$$\frac{\pi i \hbar}{\sqrt{2\tau|\varepsilon|}} e^{-i\frac{\pi}{2}\nu[\varepsilon]}, \quad (7.2.3)$$

where the Maslov index is given by

$$\nu[x] \stackrel{\text{def}}{=} \begin{cases} 0 & \text{if } x > 0; \\ 1 & \text{if } x < 0. \end{cases} \quad (7.2.4)$$

If $\varepsilon/a < 0$ the satellite points are real and contribute by

$$2\sqrt{\frac{\pi i \hbar}{8\tau|\varepsilon|}} e^{-i\frac{\pi}{2}\nu[-\varepsilon]} e^{-2i\tau\varepsilon^2/(a\hbar)}. \quad (7.2.5)$$

If $\varepsilon/a > 0$ we must study the relative orientation of the saddle points. Indeed after having done one stationary integration in the p direction, we have

$$\mathcal{I}_2 = \sqrt{\pi i \hbar} \int_{-\infty}^{+\infty} e^{\frac{i\varepsilon}{\hbar}(2\varepsilon q^2 + aq^4)} dq [1 + O_{\sqrt{\hbar}}(2)] . \quad (7.2.6)$$

The saddle points ($s_0 = 0, s_{\pm 1} = iq_{\pm}$) are represented in figure 19.

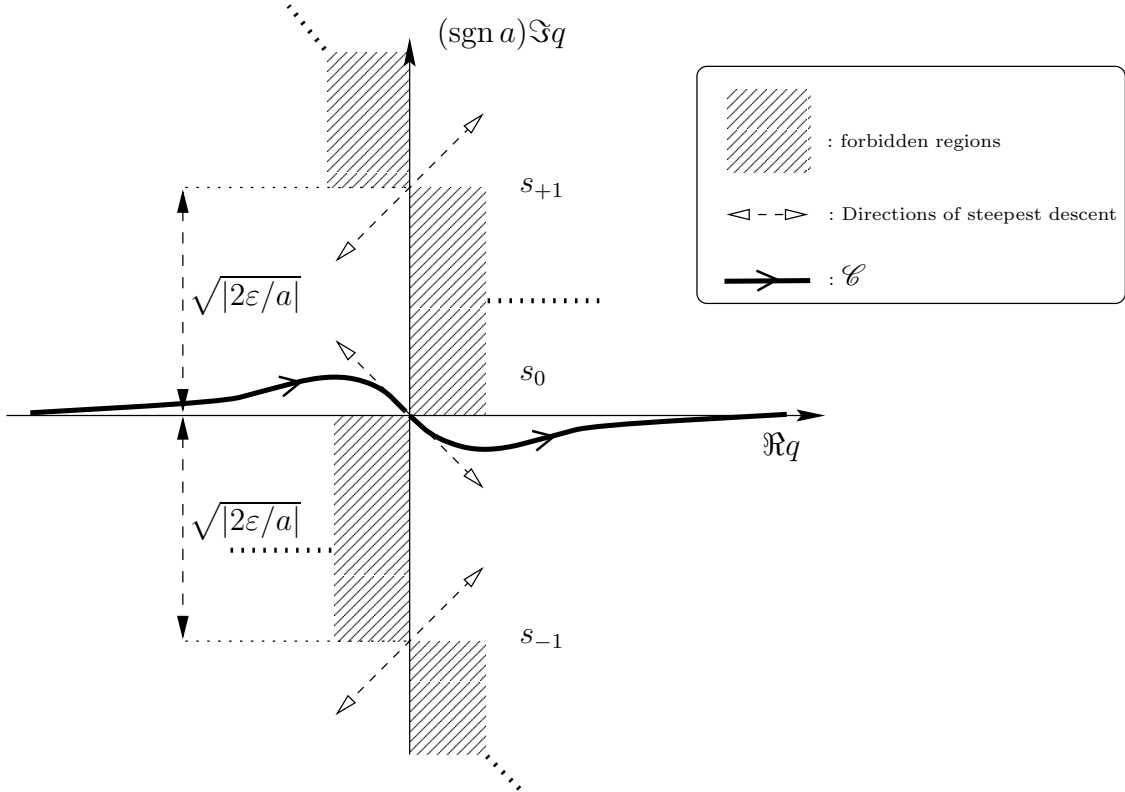


Figure 19: Saddle points for \mathcal{I}_2 when $\varepsilon/a > 0$.

The hatched regions represent the forbidden regions where $\Im[N(z) - N(s)] < 0$. It is clear from the figure that it is impossible to deform the contour \mathcal{C} in order to cross s_1 or s_{-1} and to come back to the real axis. The complex saddles will therefore contribute to semiclassical expansions only via power laws in \hbar when one tries to go beyond the dominant order of the semiclassical expansion at s_0 . This power law expansion will contain some information about the presence of $s_{\pm 1}$. Let us specify the corresponding power of \hbar

$$\begin{aligned}
\mathcal{I}_2 &= \iint_{\mathbb{R}^2} \left(1 + \frac{1}{2} p^2 \partial_{p,p} F|_{(0,0)} + \frac{1}{2} q^2 \partial_{q,q} F|_{(0,0)} + \frac{i\tau}{2\hbar} a q^4 \right) e^{\frac{i}{\hbar}(2\varepsilon\tau q^2 + p^2)} dp dq [1 + O_{\sqrt{\hbar}}(3)] \\
&= \frac{\pi i \hbar}{\sqrt{2\tau|\varepsilon|}} e^{-i\frac{\pi}{2}\nu[\varepsilon]} \left[1 + \frac{1}{4} i \hbar \left(\partial_{p,p} F|_{(0,0)} + \frac{1}{2\tau\varepsilon} \partial_{q,q} F|_{(0,0)} - \frac{3a}{8\tau\varepsilon^2} \right) + O_{\sqrt{\hbar}}(3) \right]. \quad (7.2.7)
\end{aligned}$$

It is the third term of the parenthesis which is the explicit leading contribution of the complex satellite points. It corresponds to a term \hbar smaller than the leading Gutzwiller contribution of the real one. Even if a is not known its sign can be determined by looking at the stability of the central orbit (see figure 11).

The fact that the complex saddle points are indeed encoded in the asymptotic series of \mathcal{I}_2 is well known: When $F \equiv 1$, one can re-sum the complete asymptotic expansion of \mathcal{I}_2 and give to it a sense in the whole complex \hbar -plane except over one half line. The discontinuity of \mathcal{I}_2 between the two sides of this cut can be interpreted in terms of an integral whose contour goes actually trough the saddles s_1 or s_{-1} . See for example [11, §2.1 a] or [29, §9.4 a].

7.3 Stable cases

From (6.4.7) we have in action-angle variables

$$-N_\ell(I, \theta; \varepsilon) = \ell\varepsilon\tau I + \ell\tau \sum_{\substack{k \in \mathbb{N} \\ 2 \leq k \leq \ell/2}} a_k(\varepsilon) I^k + \ell\tau b_\ell(\varepsilon) I^{\ell/2} \cos(\ell\theta) + O_{\sqrt{I}, \varepsilon^*}(\ell + 1). \quad (7.3.1)$$

We are here systematically ignoring problems related to the quantization procedure. In expressions like (7.0.16) the passage to action-angle variables or to any other symplectic coordinate system of jacobian unity can be done. If one *starts* from the classical problem expressed in action-angle coordinates, we may face however some difficulties to quantize it [14] because the phase space has a cylindrical topology and therefore cannot be described by a unique symplectic chart. In Feynman path integrals or their discretized analogues, this implies to take care of the homotopy of the paths. The interfering contribution of each of them is weighted by the character of its homotopy group [45]. For the cylinder, it corresponds to a infinite sum over the winding numbers. Semiclassically this will select exactly the central orbit whose winding number is $r = \ell\tau\omega/(2\pi)$, the others being at a distance much larger than ε and a fortiori than \hbar ; we are therefore led to (7.3.1). For a concrete illustration see [46, §2].

The contribution to the central fixed point $I = 0$ is given by

$$\begin{aligned} \mathcal{I}_\ell^0 = & \int_\theta^{2\pi} \int_{I=0}^\infty F(I, \theta) \left[1 - \frac{i\ell\tau}{\hbar} \left(\sum_{\substack{k \in \mathbb{N} \\ 2 \leq k \leq \ell/2}} a_k(\varepsilon) I^k + \ell\tau b_\ell(\varepsilon) I^{\ell/2} \cos(\ell\theta) \right) \right. \\ & \left. + \frac{1}{2} \left(-\frac{i\ell\tau}{\hbar} \right)^2 \left(\sum_{\substack{k \in \mathbb{N} \\ 2 \leq k \leq \ell/2}} a_k(\varepsilon) I^k + \ell\tau b_\ell(\varepsilon) I^{\ell/2} \cos(\ell\theta) \right)^2 + \dots \right] e^{-i\ell\tau\varepsilon I/\hbar} dI d\theta. \end{aligned} \quad (7.3.2)$$

The dominant term is $-\frac{i\hbar}{\ell\tau\varepsilon}$. Many powers of \hbar follow among which we want to determine the first which depends on b_ℓ . If we expand F in Fourier series we obtain

$$F(I, \theta) = \sum_{k=0}^{\infty} [F_k(I) \cos(k\theta) + \tilde{F}_k(I) \sin(k\theta)]. \quad (7.3.3)$$

Moreover, we know from the smooth behavior of F in a neighborhood of the origin that we have $F(I, \theta) = F(\sqrt{2I} \cos \theta, \sqrt{2I} \sin \theta)$ and thus that, $\forall k \in \mathbb{N}$, $F_k(I) = A_k I^{k/2}$ for some complex constant A_k . In (7.3.2) we get a term of the form

$$-\frac{i\ell\tau}{\hbar} \pi b_\ell \left(\frac{\hbar}{\ell\tau\varepsilon} \right)^{\ell/2+1} \int_0^\infty x^{\ell/2} F_\ell \left(\frac{\hbar x}{\ell\tau\varepsilon} \right) e^{-x} dx = -i\pi A_\ell b_\ell \left(\frac{\hbar}{\ell\tau} \right)^\ell \frac{\ell!}{\varepsilon^{\ell+1}} + O_{\sqrt{\hbar}}(2\ell + 1). \quad (7.3.4)$$

But this is not the largest term depending on b_ℓ since there is a term proportional to $\hbar^{\ell-1}$

$$-\left(\frac{\ell\tau}{\hbar} \right)^2 b_\ell^2 \int_{\theta=0}^{2\pi} \int_{I=0}^\infty I^\ell \cos^2(\ell\theta) e^{-i\ell\tau\varepsilon I/\hbar} dI d\theta = -\pi b_\ell^2 \left(\frac{\hbar}{\ell\tau} \right)^{\ell-1} \frac{\ell!}{\varepsilon^{\ell+1}}. \quad (7.3.5)$$

When the satellite points are real, then their contribution to semiclassical expansion corresponds to the standard Gutzwiller oscillating terms. To leading order one finds

— For $\ell = 3$

$$\mathcal{I}_3 = -\frac{i\hbar}{3\tau\varepsilon} [1 + O_{\sqrt{\hbar}}(3)] + \frac{2\pi i\hbar}{\sqrt{3}\tau|\varepsilon|} e^{-i(\nu[b_3] + \nu[\varepsilon])\pi/2} e^{-4i\tau\varepsilon^3/(9\hbar b_3^2)} [1 + O_{\sqrt{\hbar}}(3)]. \quad (7.3.6)$$

— For $\ell = 4$ and with the notations of subsection 5.4.2, we can have one or two real families of fixed points corresponding to $\cos(4\theta) = \pm 1$. When they are real their contribution is

$$4 \frac{2\pi i\hbar}{\left| \frac{a-b}{a+c} \right|^{1/2} \tau|\varepsilon|} e^{-i(\nu[-a-c] + \nu[a-b])\pi/2} e^{-2i\tau\varepsilon^2(\frac{1}{a} + \frac{1}{a+c})/\hbar} [1 + O_{\sqrt{\hbar}}(3)]. \quad (7.3.7)$$

— For $\ell \geq 5$ both stable and unstable satellite orbits give the same contribution when $\varepsilon/a_2 < 0$,

$$\frac{2\pi i \hbar}{|b_\ell|^{1/2} \tau} \left| \frac{2a_2}{\varepsilon} \right|^{\ell/4} e^{-i(\nu[\varepsilon] + \nu[\pm b_\ell])\pi/2} e^{i\ell\tau\varepsilon^2/(4a_2\hbar)} [1 + O_{\sqrt{\hbar}}(3)]. \quad (7.3.8)$$

Martin Sieber showed [46] that if one wants to compute the difference between the contribution of the stable and unstable satellite orbits as a function of ε one needs to work with normal forms up to $O_{\sqrt{I}, \sqrt[4]{|\varepsilon|}}(2\ell - 4)$. The expression of this normal form is given by equation (14) of Ref.[46]

When the satellite orbits are complex, they actually correspond to real negative action I . The integration contour in I goes from zero to $+\infty$ and therefore cannot cross the saddle points on the negative real axis. The contribution of the complex orbits comes entirely from the saddle expansion at the origin. It thus contains information about the complex solution through a term of order $\hbar^{\ell-1}$ given by Eq. (7.3.5).

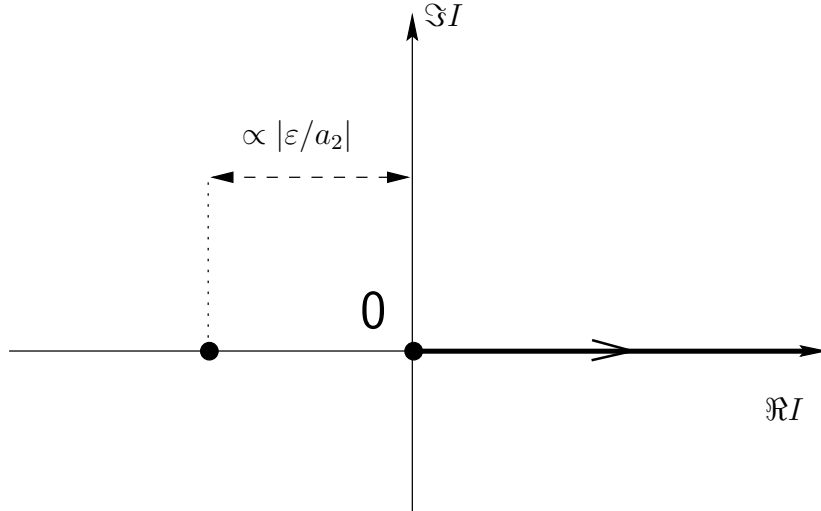


Figure 20: Saddle points for \mathcal{I}_ℓ when $\varepsilon/a_2 > 0$ and $\ell \geq 4$.

8 Conclusion

In the first part of this paper we have revisited Meyer’s classification of bifurcations in two-freedom systems by systematically including and locating the complex periodic orbits which are present on each side of the bifurcation (subsection 5.5 presents a summary of the main features concerning this point). Thereafter we have explicitly studied the contribution of complex periodic orbits to

semiclassical expansions (1.0.1) when the classical regime is a mixture of regular and irregular motion. Our starting hypothesis was that the ghost orbits which may generically contribute are those which are not too deeply located in the complex phase space and therefore can be associated to a bifurcation. We have shown that the only case where they do not give a power law contribution in \hbar is the extremal bifurcation where the monodromy matrix has a determinant equal to one. Their contribution is exponentially small in \hbar (see (7.1.4)), whereas the argument of the exponential is proportional to the distance from the bifurcation to the power $3/2$. Because of this large power and because no real orbits come to shadow them, these ghost orbits can be easily observed when one compares exact and semiclassical quantities [30, 31]. In the latter reference it has moreover been shown that these orbits may also play an important role in the semiclassical description of tunneling effects. In all the other types of bifurcations the complex orbits do not create new peaks in the Fourier transform of the density of states but rather modify the amplitude of the peak associated to the central real orbit.

We have systematically ignored in our discussion the consequences a symmetry may have on bifurcations. Because discrete as well as continuous symmetries are, from a technical point of view, equivalent to the introduction of another continuous parameter in the unfolding in addition to ε , their presence increases drastically the number of the possible types of bifurcations. One should for example consider cases where the linearized motion corresponds to $\mathbb{L} = \mathbb{0}$ (or $\mathbb{M} = \mathbb{1}$), and should therefore classify first all the possible cubic hamiltonians. To our knowledge there is no systematic classification of the bifurcations of hamiltonians of codimension two or higher. Nevertheless some normal forms for certain types of bifurcations can be found in the literature [36]. The simplest cases are of course the normal forms given in section 8 with some of the higher order coefficients put to zero. Then we get some additional phase portraits like the one obtained in Ref. [21, table 1 case c]. One can easily go in these cases through a semiclassical analysis similar to the one presented here.

A.M. would like to express his gratitude to Eugene Bogomolny, Stephen Creagh, Martin Sieber and Daniel Rouben for many fruitful discussions. He also thanks le Service de Prestations Informatiques du Département de Mathématiques et d'Informatique de l'École Normale Supérieure (Paris) for assistance.

Notes

¹ In the literature [3, App. 27], one can find *parabolic* fixed point.

² In the literature [3, App. 27] and [5, § 7], one can find *elliptic* fixed point.

³ Since we are working with a one-dimensional unfolding there is only one direction in the integration domain where a critical point of a smooth function can be non-quadratic and this result is independent of the number of arguments of the function [42, chap 4, § 5: the splitting lemma]. In (7.0.11) we nevertheless keep two directions of integration for stressing the physical interpretation of the critical points in term of the periodic orbits of the dynamics but it can be checked in the following that in all cases (extremal, transitional and stable) there is only one direction in which the integration is not a gaussian one.

Appendix: Summary tables

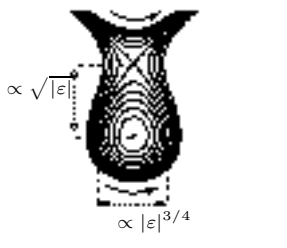
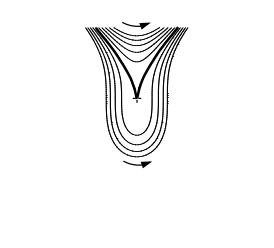

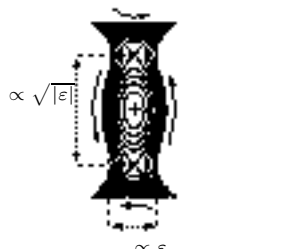




M: monodromy matrix	$\varepsilon/a < 0$	$\varepsilon = 0$	$\varepsilon/a > 0$
$\text{sp M} _{\varepsilon=0} = \{1, 1\}$	<i>Extremal case</i>		
			
Hamiltonian normal form:			
$H_1(p, q; t; \varepsilon) = \frac{1}{2\tau} p^2 + \varepsilon q + \frac{1}{3} a q^3 + \frac{1}{4} b q^4 + O_{p, q, \sqrt{ \varepsilon }}(5)$			
$\text{sp M} _{\varepsilon=0} = \{-1, -1\}$	<i>Transitional case</i>		
		<div style="border: 1px solid black; padding: 2px; width: fit-content; margin: 0 auto;">$a < 0$</div>  <div style="border: 1px solid black; padding: 2px; width: fit-content; margin: 10px auto;">$a > 0$</div> 	 
Hamiltonian normal form:			
$H_2(p, q; t; \varepsilon) = \frac{1}{2\tau} p^2 + \varepsilon q^2 + \frac{1}{4} a q^4 + b\varepsilon q^4 + cq^6 + O_{\sqrt{p}, q, \sqrt{ \varepsilon }}(7)$			

Table 2: Generic bifurcations of extremal and transitional points






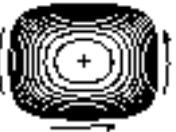



<i>Stable case: $\text{sp } \mathbb{M} _{\varepsilon=0} = \{e^{\pm i \frac{2r\pi}{\ell}}\}$</i>			
	$\varepsilon/a < 0$	$\varepsilon = 0$	$\varepsilon/a > 0$
	$\ell = 3$		
			
	<p>Hamiltonian normal forms:</p> $H_3(p, q; t; \varepsilon) = \frac{1}{2} \varepsilon (p^2 + q^2) + \frac{1}{3} b (p^3 - 3pq^2) + O_{p, q, \varepsilon}(4)$ $H_3(I, \theta; t; \varepsilon) = \varepsilon I + \frac{1}{3} b (2I)^{3/2} \cos(3\theta) + O_{\sqrt{I}, \varepsilon}(4)$		
	$\ell = 4$		
$\alpha) a + b > 0$			
$\beta) a + b < 0$			
	<p>Hamiltonian normal forms:</p> $H_4(p, q; t; \varepsilon) = \frac{1}{2} \varepsilon (p^2 + q^2) + \frac{1}{4} a (p^4 + q^4) + \frac{1}{2} b p^2 q^2 + O_{p, q, \sqrt{ \varepsilon }}(5)$ $H_4(I, \theta; t; \varepsilon) = \varepsilon I + \frac{1}{4} (3a + b) I^2 + \frac{1}{4} (a - b) I^2 \cos(4\theta) + O_{\sqrt{I}, \sqrt{ \varepsilon }}(5)$		

Table 3: *Generic bifurcations in stable cases for $\ell \in \{3, 4\}$.*

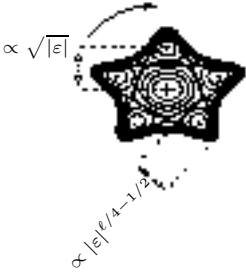


<i>Stable case: $\text{sp } \mathbb{M} _{\varepsilon=0} = \{e^{\pm i \frac{2r\pi}{\ell}}\}$</i>		
$\varepsilon/\alpha_2 < 0$	$\varepsilon = 0$	$\varepsilon/\alpha_2 > 0$
$\ell \geq 5$		
		
Hamiltonian normal forms:		
$H_\ell(p, q; t; \varepsilon) = \frac{1}{2} \varepsilon (p^2 + q^2) + \sum_{\substack{k \in \mathbb{N} \\ 2 \leq k \leq \ell/2}} \tilde{\mathbf{a}}_k(\varepsilon) (p^2 + q^2)^k + \mathbf{h}_{\ell 0 \ell}(\varepsilon) \Re[(p + iq)^\ell] + O_{p,q}(\ell + 1)$		
$H_\ell(I, \theta; t; \varepsilon) = \varepsilon I + \sum_{\substack{k \in \mathbb{N} \\ 2 \leq k \leq \ell/2}} \mathbf{a}_k(\varepsilon) I^k + \mathbf{b}_\ell(\varepsilon) I^{\ell/2} \cos(\ell\theta) + O_{\sqrt{I}}(\ell + 1)$		

Table 4: *Generic bifurcations in stable cases for $\ell \geq 5$.*

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