

On the limiting power of set of knots generated by $1 + 1-$ and $2 + 1-$ braids

R Bikbov[†] and S Nechaev^{‡†}

[†] *L D Landau Institute for Theoretical Physics,
117940, Moscow, Russia*

[‡] *Institut de Physique Nucléaire, Division de Physique Théorique*,
91406 Orsay Cedex, France*

Abstract

We estimate from above the set of knots, $\Omega(n, \mu)$, generated by closure of n -string $1 + 1-$ and $2 + 1-$ -dimensional braids of irreducible length μ ($\mu \gg 1$) in the limit $n \gg 1$.

Key words: standard and surface braid groups, graph of the group, primitive word, normal form

PACS:

* Unité de Recherche des Universités Paris XI et Paris VI associée au C.N.R.S.

1 Introduction

Besides the traditional fundamental topological issues concerning the construction of new topological invariants, investigation of homotopic classes and fibre bundles we mark a set of ajoint but much less studied problems. First of all, we mean the problem of so-called “knot entropy” calculation. Most generally it can be formulated as follows. Take the lattice \mathbb{Z}^3 embedded in the space \mathbb{R}^3 . Let Ω be the ensemble of all possible closed nonselfintersecting N -step paths with one common fixed point on \mathbb{Z}^3 ; by ω_N we denote the particular configuration of the trajectory. The main question is: what is the fraction P_N of the the trajectories $\omega_N \in \Omega$ belonging to some specific homotopic class characterized by the topological invarant Inv (we do not specify the way of defining the topological invariant). The distribution function $P\{Inv\}$ satisfies the obvious normalization condition $\sum_{\text{all } \omega_N \in \Omega} P_N\{Inv\} = 1$.

In the present paper we pay attention to the statistical problem concerning the estimation of the set $\Omega = \{\Omega^{(1)}, \Omega^{(2)}\}$ of knots generated by closure of braids embedded in $1 + 1$ - and $2 + 1$ - dimensions (see the definitions below).

The paper is organized as follows. Below we give the basic definitions of the standard $1 + 1$ -dimensional and $2 + 1$ -dimensional braid groups as well as formulate the basic results; the Section 2 is devoted to the estimations of the sets $\Omega^{(1)}$ and $\Omega^{(2)}$ using the concept of $1 + 1$ - and $2 + 1$ - locally-free groups; while in Conclusion we discuss in more details the corollaries following from our consideration.

1.1 The basic definitions

1. The $1 + 1$ -dimensional (“standard”) braid group $B_{n+1}^{(1)}$ of $n + 1$ strings has n generators $\{\sigma_1, \sigma_2, \dots, \sigma_n$ and their inverses} (see fig.1a) with the following relations:

$$\left\{ \begin{array}{ll} \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} & (1 \leq i < n) \\ \sigma_i \sigma_j = \sigma_j \sigma_i & (|i - j| \geq 2) \\ \sigma_i \sigma_i^{-1} = \sigma_i^{-1} \sigma_i = e & \end{array} \right. \quad (1)$$

2. The $2 + 1$ -dimensional (“surface”) braid group $B_{n+1}^{(2)}$ can be defined in the following way (see, for instance [1, 2]). Consider the two-dimensional lattice \mathbb{Z}^2 and take distinct points $P_1, P_2, \dots, P_{n+1} \in \mathbb{Z}^2$. A $2 + 1$ -braid of $n + 1$ strings on \mathbb{Z}^2 based at $\{P_1, P_2, \dots, P_{n+1}\}$ is an $n + 1$ -tuple $b = (b_1, \dots, b_{n+1})$ of paths, $b_i : [1, N] \rightarrow \mathbb{Z}^2$, such that

- (i) $b_i(1) = P_i$ and $b_1(1) \in \{P_1, P_2, \dots, P_{n+1}\} \forall i \in \{1, \dots, n + 1\}$;
- (ii) $b_i(t) \neq b_j(t) \forall \{i, j\} \in \{1, \dots, n + 1\}, i \neq j; t \in [1, N]$.

The braid group $B_{n+1}^{(2)}$ on \mathbb{Z}^2 based at $\{P_1, P_2, \dots, P_{n+1}\}$ is the group of homotopy classes of braids based at $\{P_1, P_2, \dots, P_{n+1}\}$. The group $B_{n+1}^{(2)}$ has $2(n \times n)$ generators $\{(\sigma_{11}^{(x)}, \sigma_{11}^{(y)}), \dots, (\sigma_{1n}^{(x)}, \sigma_{1n}^{(y)}); \dots; (\sigma_{n1}^{(x)}, \sigma_{n1}^{(y)}), \dots, (\sigma_{nn}^{(x)}, \sigma_{nn}^{(y)})$ and their inverses} (see fig.1b) with the following relations:

$$\left\{ \begin{array}{ll} \sigma_{i,j}^{(x)} \sigma_{i+1,j}^{(x)} \sigma_{i,j}^{(x)} = \sigma_{i+1,j}^{(x)} \sigma_{i,j}^{(x)} \sigma_{i+1,j}^{(x)} & (1 \leq \{i, j\} \leq n) \\ \sigma_{i,j}^{(y)} \sigma_{i,j+1}^{(y)} \sigma_{i,j}^{(y)} = \sigma_{i,j+1}^{(y)} \sigma_{i,j}^{(y)} \sigma_{i,j+1}^{(y)} & (1 \leq \{i, j\} \leq n) \\ \sigma_{i,j}^{(x)} \sigma_{i,j}^{(y)} \sigma_{i,j}^{(x)} = \sigma_{i,j}^{(y)} \sigma_{i,j}^{(x)} \sigma_{i,j}^{(y)} & (1 \leq \{i, j\} \leq n) \\ \sigma_{i_1, j_1}^{(x)} \sigma_{i_2, j_2}^{(x)} = \sigma_{i_2, j_2}^{(x)} \sigma_{i_1, j_1}^{(x)} & (|i_1 - i_2| > 1 \quad \text{or} \quad |j_1 - j_2| > 0) \\ \sigma_{i_1, j_1}^{(x)} \sigma_{i_2, j_2}^{(y)} = \sigma_{i_2, j_2}^{(y)} \sigma_{i_1, j_1}^{(x)} & (i_2 - i_1 \neq \{0, 1\} \quad \text{or} \quad j_1 - j_2 \neq \{0, 1\}) \\ \sigma_{i,j}^{(x)} (\sigma_{i,j}^{(x)})^{-1} = \sigma_{i,j}^{(y)} (\sigma_{i,j}^{(y)})^{-1} = e & \end{array} \right. \quad (2)$$

The braid groups $B_n^{(1)}$ and $B_n^{(2)}$ have the following general properties:

– Any arbitrary word written in terms of “letters”—generators of the groups $B_n^{(1)}$ or $B_n^{(2)}$ —gives a particular braid.

– The *length*, N , of the braid is the total number of used letters, while the *minimal irreducible length*, μ , hereafter referred to as the “primitive length” is the shortest noncontractible length of a particular braid which remains after applying of all possible group relations. Diagrammatically the braid can be represented as a set of crossed strings going from the top to the bottom appeared after subsequent gluing the braid generators.

– The closed braid appears after gluing the “upper” and the “lower” free ends of the braid on the cylinder.

1.2 The main results

Our basic results might be formulated in a geometrically clear way. Consider two sets of braids $\{B_n^{(1)}\}$ and $\{B_n^{(2)}\}$, embedded in $1+1-$ and $2+1-$ dimensions correspondingly. Let each particular braid has the primitive length μ and is represented by n strings.

Then:

- The set $\Omega^{(1)}(n, \mu)$ of knots which can be generated by the standard braids of given irreducible length μ ($\mu \gg 1$) from the set $\{B_n^{(1)}\}$ ($n = \text{const} \gg 1$) is restricted from above by the value

$$\Omega^{(1)}(n, \mu) < \frac{32\pi^2}{\ln^4 2} \frac{2^n}{n^3} 7^{\mu-1} \quad (3)$$

- The set $\Omega^{(2)}(n, \mu)$ of knots which can be generated by the surface braids of given irreducible length μ ($\mu \gg 1$) from the set $\{B_n^{(2)}\}$ ($n = \text{const} \gg 1$) is restricted from above by the value

$$\Omega^{(2)}(n, \mu) < \frac{32n^2}{\pi^2} \left(\frac{2n}{\ln n} \right)^{\mu-1} \quad (4)$$

(See the Conclusion for more detailed discussion of the results (3) and (4)).

2 Combinatorics of words

Any braid corresponds to some knot or link. The correspondence between braids and knots is not mutually single valued and each knot or link can be represented by infinite series of different braids. However, we can estimate from above the partition functions $\Omega^{(1)}(n, \mu)$ and $\Omega^{(2)}(n, \mu)$ of all possible knots generated by the ensemble of all $1 + 1-$ and $2 + 1-$ braids of primitive length μ using the following obvious fact. *The sets $\Omega^{(1)}(n, \mu)$ and $\Omega^{(2)}(n, \mu)$ are bounded from above by the number of all distinct words of the primitive length μ in $1 + 1-$ and $2 + 1-$ braid groups correspondingly.* Thus in what follows we are aimed in the estimation of the number of nonequivalent words in the standard and surface braid groups.

2.1 Definitions of $1 + 1-$ (“standard”) and $2 + 1-$ (“surface”) locally free groups

1. Following the ideas of A.M. Vershik concerning the notion of the “local groups” [3] and the papers [4], where the concept of a “locally free” group was proposed at first in the topological context, let us define the group, $\mathcal{LF}_{n+1}^{(1)}$, which has n generators $\{f_1, \dots, f_n$ and their inverses} with the relations:

$$\begin{cases} f_j f_k = f_k f_j & \text{for } |j - k| \geq 2 \\ f_i f_i^{-1} = e \end{cases} \quad (5)$$

We call the group with relations (5) the $1 + 1-$ -dimensional “locally free group”, because each pair of generators $(f_j, f_{j\pm 1})$ produces a free subgroup of the group $\mathcal{LF}_{n+1}^{(1)}$.

The group $\mathcal{LF}_{n+1}^{(1)}$ can be obtained from the braid group $B_{n+1}^{(1)}$ if we replace the braiding (“Yang-Baxter-type”) relations by the free ones. The geometrical interpretation of the generators of a group $\mathcal{LF}_{n+1}^{(1)}$ is shown in fig.2a.

Apparently, in mathematical literature the notion similar to our “locally free group” appeared firstly in the paper [5] devoted to the investigation of the combinatorial properties of rearrangements of sequences, known also as “partially commutative monoids” (see [6] and references therein).

2. The $2 + 1$ -dimensional (“surface”) locally free group $\mathcal{LF}_{n+1}^{(2)}$ has $2(n \times n)$ generators $\{(f_{11}^{(x)}, f_{11}^{(y)}), \dots, (f_{1n}^{(x)}, f_{1n}^{(y)}); \dots; (f_{n1}^{(x)}, f_{n1}^{(y)}), \dots, (f_{nn}^{(x)}, f_{nn}^{(y)})$ and their inverses} with the following relations:

$$\left\{ \begin{array}{ll} f_{i_1, j_1}^{(x)} f_{i_2, j_2}^{(x)} = f_{i_2, j_2}^{(x)} f_{i_1, j_1}^{(x)} & (|j_1 - j_2| > 0 \quad \text{or} \quad |i_1 - i_2| > 1) \\ f_{i_1, j_1}^{(x)} f_{i_2, j_2}^{(y)} = f_{i_2, j_2}^{(y)} f_{i_1, j_1}^{(x)} & (i_2 - i_1 \neq \{0, 1\} \quad \text{or} \quad j_1 - j_2 \neq \{0, 1\}) \\ f_{i, j}^{(x)} (f_{i, j}^{(x)})^{-1} = f_{i, j}^{(y)} (f_{i, j}^{(y)})^{-1} = e & \end{array} \right. \quad (6)$$

Thus, we can construct the $2 + 1$ -locally free group $\mathcal{LF}_{n+1}^{(2)}$ from the surface braid group $B_{n+1}^{(2)}$ if we replace the braiding relations of the neighbouring generators by the “full monodromy”, i.e. by the free group relations—see the fig.2b.

The following important properties of $1 + 1-$ and $2 + 1-$ locally free groups should be mentioned:

- (i) By definition the locally free groups $\mathcal{LF}_{n+1}^{(1)}$ and $\mathcal{LF}_{n+1}^{(2)}$ have less relations than the braid groups $B_{n+1}^{(1)}$ and $B_{n+1}^{(2)}$ correspondingly. *Thus, the number of distinct words of the primitive length μ in the $1 + 1-$ and $2 + 1-$ braid groups is bounded from above by the number of distinct words of the primitive length μ in the $1 + 1-$ and $2 + 1-$ locally free groups.*
- (ii) By construction (compare figures 1 and 2) the monodromy generators f_i ($i \in [1, n]$) of the group $\mathcal{LF}_{n+1}^{(1)}$ and $f_{i, j}^{(x, y)}$ ($\{i, j\} \in [1, n]$) of the group $\mathcal{LF}_{n+1}^{(2)}$ can be written as $f_i = (\sigma_i)^2$ ($i \in [1, n]$) and $f_{i, j}^{(x, y)} = (\sigma_{i, j}^{(x, y)})^2$ ($\{i, j\} \in [1, n]$), where σ_i and $\sigma_{i, j}^{(x, y)}$ are the generators of the groups $B_{n+1}^{(1)}$ and $B_{n+1}^{(2)}$ correspondingly. *Thus, the number of distinct words of the primitive length 2μ in the $1 + 1-$ and $2 + 1-$ braid groups is bounded from below by the number of distinct words of the primitive length μ in the $1 + 1-$ and $2 + 1-$ locally free groups.*

2.2 Computation of number of nonequivalent words in $1 + 1-$ and $2 + 1-$ locally free groups

We derive explicitly the expressions of the numbers $V^{(1)}(n, \mu)$ and $V^{(2)}(n, \mu)$ of all nonequivalent primitive words of length μ in the groups $\mathcal{LF}_{n+1}^{(1)}$ and $\mathcal{LF}_{n+1}^{(2)}$ respectively. Our computations are based on the so-called “normal order” representation of words proposed by A.M. Vershik in [7] (see also [4]).

THE GROUP $\mathcal{LF}_{n+1}^{(1)}$. Let us represent each word W_p of irreducible length μ in the group $\mathcal{LF}_{n+1}^{(1)}$ in the “standard” form

$$W_p = (f_{\alpha_1})^{m_1} (f_{\alpha_2})^{m_2} \dots (f_{\alpha_s})^{m_s} \quad (7)$$

where $\sum_{i=1}^s |m_i| = \mu$ ($m_i \neq 0 \forall i; 1 \leq s \leq \mu$) and the sequence of generators f_{α_i} in Eq.(7) for all distinct f_{α_i} satisfies the following local rules [4] ("normal order" representation):

- (i) If $f_{\alpha_i} = f_1$, then $f_{\alpha_{i+1}} = f_2$;
- (ii) If $f_{\alpha_i} = f_k$ ($2 \leq k \leq n-1$), then $f_{\alpha_{i+1}} \in \{f_1, \dots, f_{k-1}, f_{k+1}, \dots\}$;
- (iii) If $f_{\alpha_i} = f_n$, then $f_{\alpha_{i+1}} \in \{f_1, \dots, f_{n-1}\}$.

The rules (i)–(iii) give the prescription how to encode and enumerate all distinct primitive words in the group $\mathcal{LF}_{n+1}^{(1)}$. If the sequence of generators in the primitive word W_p does not satisfy the rules (i)–(iii), we commute the generators in the word W_p until the normal order is restored. Hence, the normal order representation enables one to give the unique coding of all nonequivalent primitive words in our group.

Let $\theta_n(m)$ be the number of all distinct sequences of $m+1$ generators, $1 \leq m \leq \mu-1$, satisfying the rules (i), (ii), (iii). The calculation of the number of distinct primitive words $V^{(1)}(n, \mu)$ of given primitive length μ is now straightforward:

$$V^{(1)}(n, \mu) = \sum_{m=1}^{\mu-1} 2^{m+1} \binom{\mu-1}{m} \theta_n(m). \quad (8)$$

The combinatorial factor $2^{m+1} \binom{\mu-1}{m}$ in Eq.(8) is the number of all primitive words of length μ written in a normal order form for the *fixed sequence* of $m+1$ generators.

Our approach to the computation of $\theta_n(m)$ is based on the consideration of a "correlation function" $\theta_n(x, x_0, m)$ which is defined as the number of all distinct sequences of $m+1$ generators satisfying the rules (i), (ii), (iii), beginning with the generator f_{x_0} and ending with the generator f_x . It is easy to write an evolution equation for $\theta(x, m) \equiv \theta_n(x, x_0, m)$ with the "time" m :

$$\theta(x, m+1) = \theta(x-1, m) + \sum_{y=x+1}^n \theta(y, m) \quad (9)$$

This equation should be completed by initial and boundary conditions

$$\begin{cases} \theta(x, 0) = \delta_{x, x_0} \\ \theta(0, m) = \theta(n+1, m) = 0 \end{cases} \quad (10)$$

We solve the boundary problem (9)–(10) in the limit $n \gg 1$ supposing the periodical boundary conditions on the segment $[0, n+1]$. Namely, we have:

$$\begin{cases} \theta(x+1, m+1) - \theta(x, m+1) = \theta(x, m) - \theta(x-1, m) - \theta(x+1, m) \\ \theta(x, 0) = \delta_{x_0, x} \\ \theta(0, m) = \theta(n+1, m) = 0. \end{cases} \quad (11)$$

The substitution

$$\theta(x, m) = \sum_{k=1}^n A_k \lambda_k^m \alpha_k(x) \quad (12)$$

enables us to pass to the following recursion relations:

$$\begin{cases} (\lambda_k + 1)\alpha_k(x + 1) - (\lambda_k + 1)\alpha_k(x) + \alpha_k(x - 1) = 0 \\ \alpha_k(0) = \alpha_k(n + 1) = 0 \end{cases} \quad (13)$$

One can readily find the eigen-values and eigen-functions of (13):

$$\begin{aligned} \lambda_k &= 4 \cos^2 \frac{\pi k}{n+1} - 1 \\ \alpha_k(x) &= \frac{\sin \frac{\pi k x}{n+1}}{\left(2 \cos \frac{\pi k}{n+1}\right)^x}, \quad k = 1, \dots, n \end{aligned} \quad (14)$$

As the function (13) is not symmetric, the set of eigen-functions is not orthogonal on the segment $[0, n + 1]$ and it is difficult to ensure the initial condition. It is convenient to pass from (13) to symmetric problem. Consider a generating function

$$Z(x, s) = \sum_{m=0}^{\infty} s^m \theta(x, m) \quad (15)$$

The equation for the function $Z(x, s)$ reads

$$\begin{cases} (s + 1)Z(x + 1, s) - (s + 1)Z(x, s) + sZ(x - 1, s) = \delta_{x, x_0-1} - \delta_{x, x_0} \\ Z(0, s) = Z(n + 1, s) = 0 \end{cases} \quad (16)$$

The last equation can be symmetrized via the substitution $Z(x, s) = A^x \varphi(x, s)$, where $A = \sqrt{\frac{s}{s+1}}$. Thus, we get

$$\begin{cases} \varphi(x + 1, s) - \frac{1}{A}\varphi(x, s) + \varphi(x - 1, s) = \frac{A^{-x}}{\sqrt{s(s+1)}}(\delta_{x, x_0-1} - \delta_{x, x_0}) \\ \varphi(0, s) = \varphi(n + 1, s) = 0. \end{cases} \quad (17)$$

Making use of the sin-Fourier transform, $f(k, s) = \sum_{x=1}^n \varphi(x, s) \sin \frac{\pi k x}{n+1}$, let us rewrite (17) in the form

$$\left(2 \cos \frac{\pi k}{n+1} - \frac{1}{A}\right) f(k, s) = \frac{A^{-x_0}}{\sqrt{s(s+1)}} \left(A \sin \frac{\pi k(x_0 - 1)}{n+1} - \sin \frac{\pi k x_0}{n+1}\right).$$

The final explicit expression of the function $Z(x, s)$ reads as follows

$$Z(x, s) = \frac{2}{(n+1)(s+1)} \left(\frac{s}{s+1}\right)^{\frac{x-x_0}{2}} \sum_{k=1}^n \frac{\sqrt{\frac{s+1}{s}} \sin \frac{\pi k x_0}{n+1} - \sin \frac{\pi k(x_0-1)}{n+1}}{\sqrt{\frac{s+1}{s}} - 2 \cos \frac{\pi k}{n+1}} \sin \frac{\pi k x}{n+1}.$$

Now we can restore the function $\theta(x, m)$ via contour integration

$$\theta(x, m) = \frac{1}{2\pi i} \oint_C \frac{Z(x, s)}{s^{m+1}} ds,$$

where the contour C surrounds the point $s = 0$ and is displaced in the regularity area of the function $Z(s) \equiv Z(x, s)$. Hence

$$\theta(x, m) = - \sum_{s_k} \text{Res} \left(\frac{Z(x, s_k)}{s_k^{m+1}} \right),$$

where s_k are the poles out of the regularity area:

$$s_k = \frac{1}{4 \cos^2 \frac{\pi k}{n+1} - 1}$$

(compare to (14)). We are interested only in the asymptotic behavior $m \gg 1$ of the function $\theta_n(x, x_0, m)$ which is determined by the poles nearest to the origin, $s_1 = s_{n-1} = \frac{1}{3}$ for $n \gg 1$. So we get

$$\theta_n(x, x_0, m) = \frac{4}{n+1} \sin \frac{\pi(x_0+1)}{n+1} \sin \frac{\pi x}{n+1} 2^{x_0-x} 3^m \quad (18)$$

To find the function $\theta_n(m)$ we should sum up $\theta(x, x_0, m)$ over all x and x_0 : $\theta_n(m) = \sum_{x, x_0=1}^n \theta_n(x, x_0, m)$. We obtain in the limit for $n = \text{const} \gg 1$ the following expression:

$$\theta_n(m) = \frac{16\pi^2}{\ln^4 2} \frac{2^n}{n^3} 3^m \quad (19)$$

The whole number of nonequivalent words follows from Eq.(8) in the limits $n = \text{const} \gg 1$, $\mu \gg 1$:

$$V^{(1)}(n, \mu) = \frac{32\pi^2}{\ln^4 2} \frac{2^n}{n^3} 7^{\mu-1} \quad (20)$$

THE GROUP $\mathcal{LF}_{n+1}^{(2)}$. It is convenient to enumerate the generators $f_{ij}^{(\alpha)}$, $i, j = 1, \dots, n^2$, $\alpha = x, y$, ordering them in a sequence: $(f_{11}^{(x)}, f_{11}^{(y)}), \dots, (f_{1n}^{(x)}, f_{1n}^{(y)}), \dots, (f_{n1}^{(x)}, f_{n1}^{(y)}), \dots, (f_{nn}^{(x)}, f_{nn}^{(y)})$. For any such sequence we define the "normal order" according to the prescriptions (i)-(iii). Let z be the serial number of the pair $(f_{ij}^{(x)}, f_{ij}^{(y)})$. Consider the functions $a(z, m)$ and $b(z, m)$ defined as numbers of all distinct sequences of $m+1$ generators satisfying the rules (i)-(iii) and ending with $f_{ij}^{(x)}$ and $f_{ij}^{(y)}$ respectively. One can readily write the evolution equations for $a(z, m)$ and $b(z, m)$ similar to (9):

$$\left\{ \begin{array}{l} a(z, m+1) = a(z-1, m) + b(z-n, m) + b(z-n+1, m) + b(z, m) + \\ \quad \sum_{z'=z+1}^{n^2} (a(z', m) + b(z', m)) \\ b(z, m+1) = b(z-n, m) + a(z-1, m) + a(z, m) + \sum_{z'=z+1}^{n^2} (a(z', m) + b(z', m)) \end{array} \right. \quad (21)$$

$k = 1$. (Compare to the case of the group $\mathcal{LF}_{n+1}^{(1)}$ —Eq.(14)). The Eq.(25) at $k = 1$ and $n \gg 1$ reads:

$$\begin{cases} p_1^{n+1} + 2p_1^n - p_1^{n-1} - n(p_1 - 1)^2 = 0 \\ \lambda_1 = p_1^{-2} - 1 + np_1^{-n-1} - (n-1)p_1^{-n}. \end{cases} \quad (26)$$

One can easily check that the smallest positive root corresponding to the largest value of $\lambda_1^{(i)}$ is

$$p_1 = 1 - \frac{\ln n}{n} + o\left(\frac{\ln n}{n}\right), \quad n = \text{const} \gg 1$$

and

$$\lambda_1 = \frac{n}{\ln n} + o\left(\frac{n}{\ln n}\right), \quad n = \text{const} \gg 1 \quad (27)$$

In the $1+1$ -dimensional case the same value of λ_k was given by the right edge of the spectrum, but in $2+1$ -dimensional case one can prove that Eq.(25) for any $i > 1$ has no solutions $\lambda_n^{(i)}$ growing as fast as λ_1 . The coefficients B_k should be found from the initial condition

$$\sum_{k=1}^{n^2} B_k \beta_k(z) = 1.$$

As $p_1 \rightarrow 1$ ($n \gg 1$) we can expect the set $\beta_k(z)$ to be orthogonal in the vicinity of the left edge of the spectrum, so B_1 is determined basically by the expression

$$B_1 = \frac{\int_0^{n^2+1} \beta_1(z) dz}{\int_0^{n^2+1} \beta_1^2(z) dz}, \quad n \gg 1$$

Now we have the following equation for the function for $b(z, m)$ in the limits $n = \text{const} \gg 1$ and $m \gg 1$

$$b(z, m) = \frac{4}{\pi} \sin \frac{\pi z}{n^2 + 1} \left(\frac{n}{\ln n}\right)^m \quad (28)$$

If we suppose the equality $a(z, m) = b(z, m)$, where $b(z, m)$ is given by (28), it is easy to check that $a(z, m)$ and $b(z, m)$ really satisfy the equations (22) in the limits $n \gg 1$, $m \gg 1$. This fact proves our assumption about the behaviors of $a(z, m)$ and $b(z, m)$ in a selfconsistent way.

The limiting expression of the function $\tilde{\theta}_n(m)$ reads

$$\tilde{\theta}_n(m) = \sum_{z=1}^{n^2} a(z, m) + b(z, m) = \frac{16n^2}{\pi^2} \left(\frac{n}{\ln n}\right)^m.$$

Thus, the asymptotics of the number of nonequivalent words of given irreducible length, μ in the limit $n = \text{const} \gg 1$, $\mu \gg 1$ is

$$V^{(2)}(n, \mu) = \sum_{m=1}^{\mu-1} 2^{m+1} \binom{\mu-1}{m} \tilde{\theta}_n(m) = \frac{32n^2}{\pi^2} \left(\frac{2n}{\ln n}\right)^{\mu-1} \quad (29)$$

(compare to Eq.(20)).

3 Conclusion

The principal difference between the limiting behavior of the partition functions $V^{(1)}(n, \mu)$ and $V^{(2)}(n, \mu)$ (and, hence, between the upper boundaries of the sets $\Omega^{(1)}(n, \mu)$ and $\Omega^{(2)}(n, \mu)$) becomes at most illuminating in the limit $n = \text{const} \gg 1$ and $\mu \rightarrow \infty$ if we consider the following limit

$$f_{1,2} = \left[\lim_{\mu \rightarrow \infty} \frac{\ln V^{(1,2)}(n, \mu)}{\mu} \right]_{n=\text{const} \gg 1} \quad (30)$$

Using the equations (20) and (29), we get

$$\begin{cases} f_1 = \ln 7 \\ f_2 = \frac{2n}{\ln n} \end{cases}$$

Thus, we can conclude, that with the exponential accuracy in the limit $n = \text{const} \gg 1$ and $\mu \rightarrow \infty$ the set $\Omega^{(1)}(n, \mu)$ is bounded from above by the n -independent estimate, i.e. $\Omega^{(1)}(n, \mu)$ is "representation-independent"; while the set $\Omega^{(2)}(n, \mu)$ with the same accuracy and in the same limit depends strongly on the braid representation (i.e. on the number of strings, n).

The equations (20) and (29) enable us to make some conclusions about the structure of the graphs corresponding to the groups $\mathcal{LF}_n^{(1)}$ and $\mathcal{LF}_n^{(2)}$. These graphs can be viewed as follows. Take the *free* $1 + 1$ - or $2 + 1$ - groups, where all generators do not commute at all. The graphs of these groups have structures of $2n$ - and $4n^2$ - branching Cayley trees, where the number of distinct words of length μ is equal to

$$\begin{cases} V_{\text{free}}^{(1)}(n, \mu) = 2n(2n - 1)^{\mu-1} & \text{for } 1 + 1\text{- free group} \\ V_{\text{free}}^{(2)}(n, \mu) = 4n^2(4n^2 - 1)^{\mu-1} & \text{for } 2 + 1\text{- free group} \end{cases}$$

The graphs corresponding to the groups $\mathcal{LF}_n^{(1,2)}$ can be constructed from the graphs of the free groups in accordance with the following recursion procedure:

- (i) Take the root vertex of the free group graph and consider all vertices on the distance $\mu = 2$. Identify those vertices which correspond to the equivalent words in groups $\mathcal{LF}_n^{(1,2)}$;
- (ii) Repeat this procedure taking all vertices on the distance $\mu = (1, 2, \dots)$ and "gluing" them on the distance $\mu + 2$ according to the definition of the locally free groups.

By means of this procedure we raise a graph which in average has $z_{\text{eff}}^{(1,2)} - 1$ distinct branches leading from the level μ to the level $\mu + 1$. We may easily find the expressions of $z_{\text{eff}}^{(1,2)}$ using the Eqs. (20) and (29). We have in the limit $n = \text{const} \gg 1$ and $\mu \rightarrow \infty$:

$$z_{\text{eff}}^{(1,2)} = \frac{V^{(1,2)}(n, \mu + 1)}{V^{(1,2)}(n, \mu)} + 1 = \begin{cases} 8 & \text{for } 1 + 1\text{- locally free group} \\ \frac{2n}{\ln n} + 1 & \text{for } 2 + 1\text{- locally free group} \end{cases}$$

We see that the graph of the group $\mathcal{LF}_n^{(1)}$ coincides (in average) with $(z_{\text{eff}} = 8)$ -branching Cayley tree for any $n \gg 1$, while the effective coordinational number of the graph of the group $\mathcal{LF}_n^{(2)}$ depends on n and does not "saturate" for $n \gg 1$.

Acknowledgments

We are very grateful to A. Comtet, J. Desbois and A.M. Vershik for fruitful discussions and useful suggestions.

References

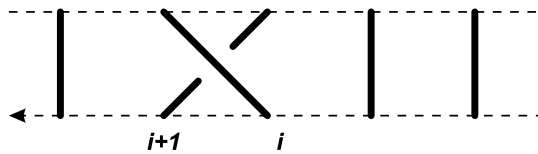
- [1] L. Paris, D. Rolfsen, *Geometric subgroups of surface braid groups*, Preprint No. 115 (1997) (Université de Bourgogne, Laboratoire de Topologie)
- [2] J. Birman, *Mapping class groups of surfaces*, *Contemp. Math.*, 78 (1988), 13
- [3] A.M. Vershik, S.V. Kerov, *Sov. Ac. Sci. Doklady*, 301 (1988), 777; A.M. Vershik, *Topics in Algebra*, 26 (1990), pt. 2, 467; *Proc. Am. Math. Soc.* 148 (1991), 1
- [4] S.K. Nechaev, A.Yu. Grosberg, A.M. Vershik, *J. Phys. A: Math. Gen.*, 29 (1996), 2411; J. Desbois, S. Nechaev, *J. Stat. Phys.*, 88 (1997), 201-223; J. Desbois, S. Nechaev, *J. Phys. (A): Math. Gen.*, 31 (1998), 2767-2784; A. Comtet, S. Nechaev, *J. Phys. (A): Math. Gen.* 31 (1998),
- [5] P. Cartier, D. Foata, *Lect. Notes in Math.*, 85 (1969)
- [6] G.X. Viennot, *Lect. Notes in Math.*, 1234 (1986), 321
- [7] A.M. Vershik, private communication

Figure Captions

Fig.1. Representation of: (a) 1+1-braid group generator, σ_i ; (b) 2+1-braid group generators $\sigma_{i,j}^x$ and $\sigma_{i,j}^y$

Fig.2. Representation of: (a) 1+1-locally free group generator, f_i ; (b) 2+1-locally free group generators $f_{i,j}^x$ and $f_{i,j}^y$

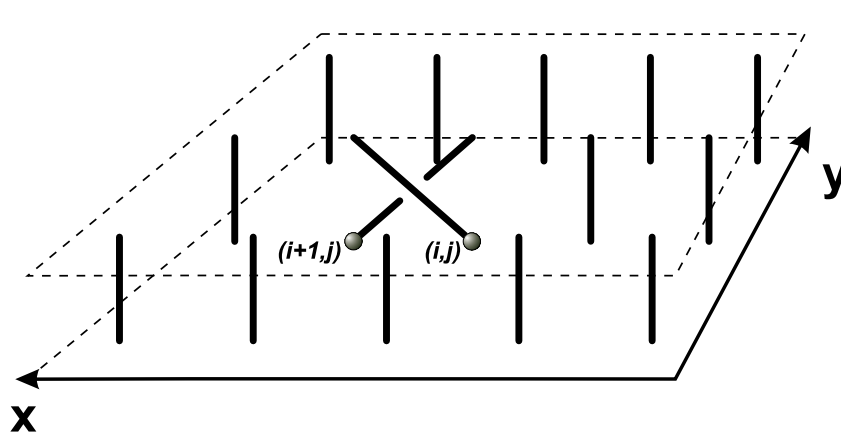
$B_n^{(1+1)}:$



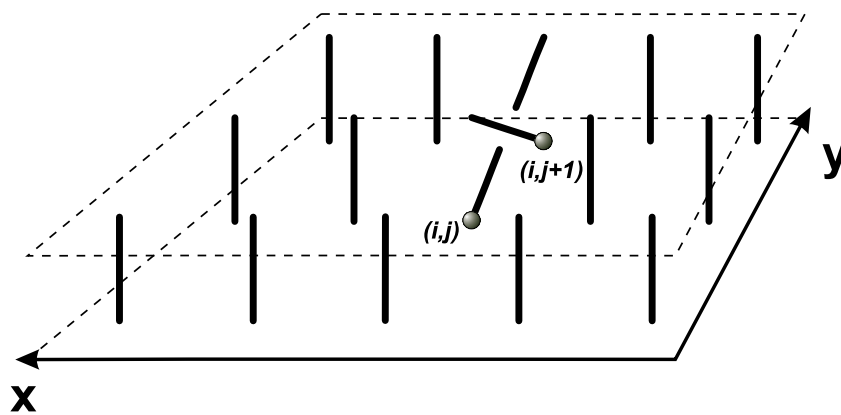
$= \sigma_i$

(a)

$B_n^{(2+1)}:$



$= \sigma_{ij}^{(x)}$

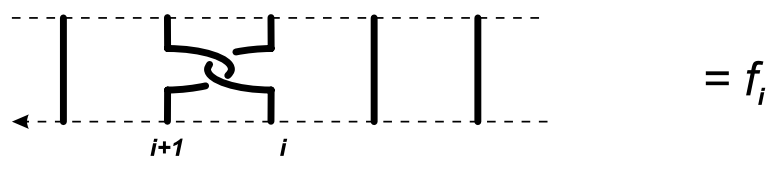


$= \sigma_{ij}^{(y)}$

(b)

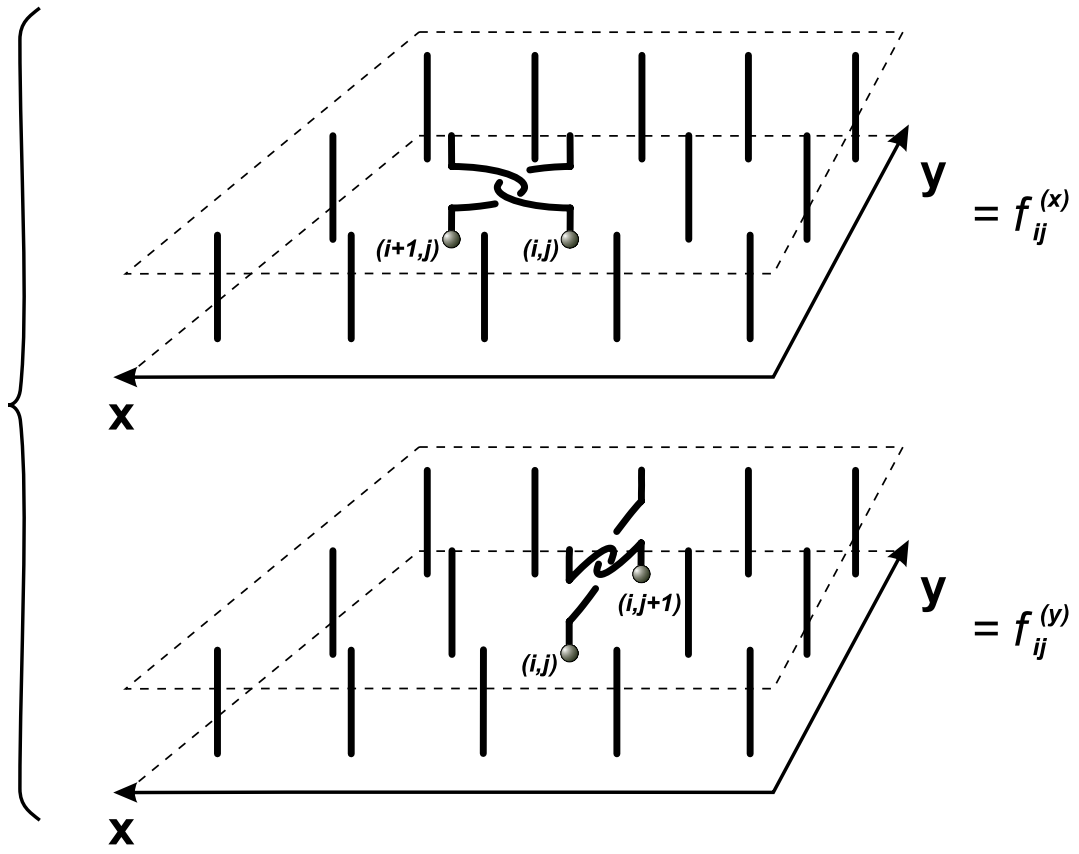
Fig. 1.

$\mathcal{LF}_n^{(1+1)}:$



(a)

$\mathcal{LF}_n^{(2+1)}:$



(b)

Fig. 2.