Statistical properties of the time evolution of complex systems. I

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The time evolution of a bounded quantum system is considered in the framework of the orthogonal, unitary and symplectic circular ensembles of random matrix theory. For an N dimensional Hilbert space we prove that in the large N limit the return amplitude to the initial state and the transition amplitude to any other state of Hilbert space are Gaussian distributed. We further compute the exact first and second moments of the distributions. The return and transition probabilities turn out to be non self-averaging quantities with a Poisson distribution. Departures from this universal behaviour are also discussed.

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I. INTRODUCTION

During the last decades much theoretical progress has been made concerning the physical properties of complex quantum systems. In the 1950s and 1960s, following the pioneering work of Wigner, a statistical theory was developed[1] to interpret the basic features of highly excited nuclear states. Known today as the random matrix theory (RMT), it has had an impressive success in the description of a large class of systems, ranging from nuclei and atoms down to microwave chaotic cavities or transport properties of electrons in metallic or ballistic disordered quantum dots (for a review of the developments see for instance Refs. 2,4,5).

Most past studies focus on the spectral statistics of complex systems based on linear wave dynamics. Our purpose here is to concentrate on the time evolution of such systems. We are particularly interested in specific quantum (or wave) manifestations in the dynamics. If at $t = 0$ the system is prepared in a given state $|\psi(0)\rangle$, a quantity which has been extensively studied is the average of the return probability $|C(t)|^2 = |\langle\psi(0)|\psi(t)\rangle|^2$ (sometimes called the survival probability). This function, which is accessible from experimental measurements as the modulus squared of the Fourier transform of the spectrum, was studied by Leviandier et al[6] in their analysis of spectral correlations in molecular physics. Their basic finding is that the spectral rigidity is manifested as a "hole" in the average of $|C(t)|^2$ for times shorter than the Heisenberg time (proportional to the inverse of the mean level spacing), this test being more robust than other ways of testing the spectral correlations. The technique was also applied to the nuclear data ensemble[7] and the spectra of superconducting microwave cavities[8]. From the theoretical point of view, a rigorous interpretation of this phenomenon in the context of RMT was given in Refs. 9,10 (see also 11). The correlation hole was also obtained using supersymmetry techniques and interpreted as a quantum dynamical echo in the time evolution of a wave packet within a disordered mesoscopic sample[12], and in acoustics[13]. In Ref. 12 a direct dynamical experimental test of the correlation hole based on the transient currents through a disordered dot was also proposed, but such a direct test is still lacking.

Our purpose here is to describe the statistical properties of the return probability as well as those of more fundamental quantities such as the return amplitude $C(t)$ and the transition amplitude to a different state $|\chi\rangle$, $T(t) = \langle\chi|\psi(t)\rangle$. Our basic assumption is that the system can be described, from a statistical point of view, by a random matrix belonging to one of the three well known symmetry classes of RMT. In this respect, and due to the Hilbert space rotational invariance of these ensembles, the theory applies to the relaxation process of a closed system for which the rate of exploration of phase space is sufficiently high in order to avoid localization effects. We show in the following that in the large-N limit the real and imaginary part of $C(t)$ and $T(t)$ are Gaussian distributed for any time. It then follows that their modulus squared (the return and transition probabilities) are Poisson distributed and that moreover

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these are non self-averaging quantities (in the sense that their fluctuations are of the same order of magnitude as their mean value in the large-N limit). This provides incidentally a proof of the results of Ref. 15 where the Poisson distribution was conjectured. Finally we obtain an exact expression, valid for any N, for the form factor and thus for the average return probability. This generalizes the asymptotic results obtained in Refs. 9,10,12. An intuitive interpretation of the results obtained in terms of a correlated random walk process in the complex plane is also given.

Deviations from the universal behaviour associated to a RMT description are expected to occur in particular for short times. These effects, which are not initially included in our description, are briefly discussed in the last part of the paper. Our results are moreover obtained for the circular ensembles of RMT. In the context of dynamical systems these ensembles are closely related to the quantized version of classically chaotic maps[14]. In fact, the time-dependent Schrödinger equation of these systems is precisely given by Eq.(1) below. We expect, however, that in the large-N limit the results reported here will be general, applicable to any fully chaotic system. This is so because the asymptotic behaviour of Circular and Gaussian ensembles coincide. As we will discuss here and in more detail elsewhere[16], some differences may however remain, in particular in connexion with the discrete nature of time in the Circular ensembles.

Following Dyson[17], we start by representing a system not by its Hamiltonian but by an $N \times N$ unitary matrix U which determines the time evolution of the system according to the equation

$$|\psi(n)\rangle = U^n |\psi(0)\rangle \tag{1}$$

where $n = 0, 1, 2, \ldots$ represents the time in units of an arbitrary interval of time which we fix to unity. The discrete nature of the time evolution encoded in this equation is not essential for the moment, and the time will be rescaled to a continuous variable later on. The eigenvalues of U are N complex numbers lying on the unit circle

$$U|\varphi_k\rangle = \exp(i\theta_k)|\varphi_k\rangle \qquad k = 1, \cdots, N \ . \tag{2}$$

Our main observables of interest are the amplitude of return

$$C(n) \equiv \langle \psi(0)|\psi(n)\rangle = \sum_{k=1}^{N} |a_k|^2 \mathrm{e}^{in\theta_k} \tag{3}$$

and the transition amplitude to a different state $|\chi\rangle$

$$T(n) \equiv \langle \chi|\psi(n)\rangle = \sum_{k=1}^{N} a_k \bar{b}_k \mathrm{e}^{in\theta_k} \ , \tag{4}$$

with $a_k = \langle \varphi_k|\psi(0)\rangle$ and $b_k = \langle \varphi_k|\chi\rangle$. Due to the discrete and finite nature of the spectrum these are quasiperiodic functions of time. However in the large N limit they will be highly oscillatory functions, and our purpose now is to study their statistical properties. A statistical analysis of these amplitudes may be introduced by considering $|\psi(0)\rangle$ as a random vector. However, this is not the way we proceed. More relevant to experiments is to fix the initial state vector and to consider the matrix U as a random matrix belonging to one of the circular ensembles of random matrices[17,18]: (a) symmetric and unitary for systems having time reversal invariance and rotational symmetry or having time reversal invariance and integral spin ($\beta = 1$); (b) unitary if there is no time reversal symmetry ($\beta = 2$); (c) self-dual unitary quaternion for half-integral spin systems with time reversal invariance ($\beta = 4$). $C(n)$ and $T(n)$ thus become random functions of the random variables a_k, b_k and θ_k whose distribution we wish to study. The state $|\chi\rangle$ will be considered to be statistically independent with respect to $|\psi(0)\rangle$, and thus the b_k's independent from the a_k's. For $\beta = 4$ each eigenvalue in Eqs.(3) and (4) is in fact doubly degenerate[17] (Kramers degeneracy) and $|a_k|^2$ represents in this case the sum of the square of the amplitudes in the degenerate subspace $|a_k^{(1)}|^2 + |a_k^{(2)}|^2$ (and analogously $a_k \bar{b}_k$ represents the sum $a_k^{(1)}\bar{b}_k^{(1)} + a_k^{(2)}\bar{b}_k^{(2)}$).

A basic property of circular (as well as Gaussian) ensembles of random matrices is their Hilbert space rotational invariance. There are two basic consequences of this invariance. The first one is the statistical independence of the eigenvalues with respect to the components of $|\psi(0)\rangle$: $P(\vec{a}, \vec{\theta}) = P_1(\vec{a})\, P_2(\vec{\theta})$, where $\vec{a} = (a_1, \cdots, a_N)$, $\vec{\theta} = (\theta_1, \cdots, \theta_N)$ and P represents the joint probability density for the corresponding ensemble. The second consequence is the isotropic form of P_1

$$P_1(\vec{a}) = \frac{2}{S_{\beta N}} \delta \left(\sum_{k=1}^{N} |a_k|^2 - 1 \right) \ . \tag{5}$$

The normalization constant $S_{\beta N}$ is the surface of a $(\beta N - 1)$ dimensional sphere. The coefficients a_k are real for $\beta = 1$ and complex for $\beta = 2$ and $\beta = 4$. We will not need the exact expression for $P_2(\vec{\theta})$ but simply note that P_2 is invariant under a global phase shift of the θ_k. Thus the distribution of $C(n)$ depends only on its modulus.

II. THE DISTRIBUTION OF THE RETURN AND TRANSITION AMPLITUDES

We now show that the probability distribution of the real part of $C(n) = C_1 + i\, C_2$ is, in the large N limit, Gaussian. Similar steps can be repeated for the imaginary part. From Eq.(3) the distribution is defined as

$$\mathcal{P}(C_1) =< \delta \left[C_1 - \sum_{k=1}^{N} |a_k|^2 \cos(n\theta_k) \right] >$$

where the brackets denote the ensemble average over the amplitudes and phases. Using the integral representation of the delta function, the average over the amplitudes reduces to

$$\prod_{k=1}^{N} \int d^2 a_k P_1(a_k) \exp\left[-i\xi |a_k|^2 \cos(n\theta_k)\right]$$

where $P_1(a_k)$ is the distribution of a single component. Because asymptotically the latter distribution is Gaussian, then

$$\mathcal{P}(C_1) \approx < \int \frac{d\xi}{2\pi} \mathrm{e}^{i\xi C_1} \exp\left[-\frac{1}{2} \sum_k \ln\left(1 + \frac{2\; i\; \xi \cos(n\theta_k)}{N}\right)\right] >$$

where the remaining ensemble average is over the phases. In the large N limit we can expand the logarithm and keep only the $1/N$ term. The latter equation then reduces to

$$\mathcal{P}(C_1) \approx < \int \frac{d\xi}{2\pi} \exp\left[i\; \xi \left(C_1 - \frac{1}{N} \sum_{k=1}^{N} \cos(n\theta_k) \right)\right] > ,$$

i.e. the distribution of the function $\tilde{C}_1(n) = \sum_k \cos(n\theta_k)/N$. This is a function that depends only on the phases (the powers of the traces of U). The distribution of the powers of the traces of unitary operators was discussed in Ref. 19, where it was shown to be Gaussian for low powers. More generally, it has been shown[20] that asymptotically the distribution of any "linear" function of the form $\sum_k f(\theta_k)$ is Gaussian. Since $\tilde{C}_1(n)$ is precisely of that form, this then proves that the real and imaginary part of the return amplitude are Gaussian distributed.

For the transition amplitude the proof is slightly different. For simplicity we limit here to the case $\beta = 1$ (real coefficients); the other symmetries can be treated in the same way. The distribution of the real part of $T(n) = T_1 + i\, T_2$ is, by definition (cf Eq.(4))

$$\mathcal{P}(T_1) =< \delta \left[T_1 - \sum_{k=1}^{N} a_k b_k \cos(n\theta_k) \right] > \;.$$

The use of the integral representation for the delta function and the computation of the integrals over the a's and b's now leads to

$$\mathcal{P}(T_1) \approx < \int \frac{d\xi}{2\pi} \mathrm{e}^{i\xi T_1} \exp\left[-\frac{1}{2} \sum_k \ln\left(1 + \frac{\xi^2 \cos^2(n\theta_k)}{N^2}\right)\right] >$$

where again the remaining average is over the phases. Keeping the leading order term in $1/N$ in the expansion of the logarithm it follows that

$$\mathcal{P}(T_1) \approx < \int \frac{d\xi}{2\pi} \exp\left[i\; \xi T_1 - \frac{\xi^2}{2N^2} \sum_{k=1}^{N} \cos^2(n\theta_k) \right] > \;.$$

Finally, computing the integral over ξ we arrive to

$$\mathcal{P}(T_1) \approx < \frac{1}{\sqrt{2\pi\sigma^2_{T_1}(\vec{\theta})}} \exp\left(-\frac{T^2}{2\sigma^2_{T_1}(\vec{\theta})} \right) > ,$$

where

$$\sigma^2_{T_1}(\vec{\theta}) = \frac{1}{N^2}\sum_{k=1}^{N}\cos^2(n\theta_k) \ .$$

Because in the large N limit the variance $\sigma^2_{T_1}(\vec{\theta})$ can be approximated by

$$\sigma^2_{T_1} \approx \frac{1}{2\pi N}\int_0^{2\pi}\cos^2(n\theta_k)\, d\theta = \frac{1}{2N}(1+\delta_{n,0}) \ ,$$

it then follows that the distribution of T_1 is Gaussian with a variance $\sigma^2_{T_1}$. Likewise, for the imaginary part of T we also get a Gaussian distribution with a variance

$$\sigma^2_{T_2} \approx \frac{1}{2\pi N}\int_0^{2\pi}\sin^2(n\theta_k) d\ \theta = \frac{1}{2N}(1-\delta_{n,0}) \ .$$

We thus recover, as it should, Eq.(9) below for the variance of T obtained from a direct computation of the average return probability.

III. THE RETURN AND TRANSITION PROBABILITIES

A. Their average value

Having established that the asymptotic distribution of the real and imaginary part of the return and transition amplitudes are Gaussian distributed, we now consider their first and second moment of those distributions. Of particular interest is the second moment of the return amplitude. But before that, let's consider the first moments.

Since the ensemble average of $|a_k|^2$ is given by $< |a_k|^2 >= \int |a_k|^2 P_1(\vec{a}) d\vec{a} = 1/N$, then the average of the return amplitude Eq.(3) is

$$< C(n) >= \frac{1}{N}\int_{-\pi}^{\pi} d\theta\, \mathrm{e}^{in\theta} R_1(\theta) = \delta_{n,0} \ ,$$

where $R_1(\theta) = N/2\pi$ is the level density. Therefore on average the vector $|\psi(n)\rangle$ decorrelates immediately from the initial state. This is obviously due to the randomization of the phase of $|\psi(n)\rangle$. For the transition amplitude we find $< T(n) >= 0, \ \forall n$.

The second moment of the distribution of the return amplitude is the probability of return

$$|C(n)|^2 = \sum_{k=1}^{N}|a_k|^4 + \sum_{j\neq k}^{N}|a_j|^2|a_k|^2 \mathrm{e}^{in(\theta_j-\theta_k)} \ . \qquad (6)$$

As mentioned before, this quantity was considered in earlier work by different authors. For the circular ensembles of RMT we obtain

$$< |C(n)|^2 >= \frac{N\delta_{n,0} + (\beta+2)/\beta - b_\beta^{(N)}(n)}{(N+2/\beta)} \ , \qquad (7)$$

which is an exact expression. The function $b_\beta^{(N)}(n) = \frac{4\pi}{N}\int_0^\pi d\theta \cos(n\theta) T_2(\theta)$ is the Fourier transform of the two-level cluster function, the so-called spectral form factor[18]. This function, which for $n = 0$ is equal to one while for large times tends to zero, is responsible for the "correlation hole" mentioned in the introduction. In fact, aside from the delta function at the origin, the numerator in Eq.(7) goes from $2/\beta$ for short times up to its large-time stationary value $(\beta+2)/\beta$. Due to the periodicity of the circular ensembles and the discreteness of time, this result differs from the one obtained for the Gaussian ensembles (no convolution present, cf Refs. 10,16).

Previous works have considered the large N limit of the average return probability by employing the well-known large-N behaviour of the form factor. In order to have an expression valid for any dimension, we have computed the form factor for any N, with the result

$$b_1^{(N)}(\tau) = 2\left(1-\tau-\frac{\tau}{2}\sum_{p=1}^{N-n}\frac{1}{(N+1)/2-p}\right)\Theta(1-\tau) - \frac{2}{N}\sum_{p=1/2}^{(N-1)/2}\left(\frac{p^2}{p^2-n^2}\right) ,$$
$$b_2^{(N)}(\tau) = (1-\tau)\Theta(1-\tau) , \tag{8}$$
$$b_4^{(N)}(\tau) = \left(1-\frac{\tau}{2}-\frac{\tau}{4}\sum_{p=1}^{2N-n}\frac{1}{N+1/2-p}\right)\Theta(2-\tau) ,$$

where we have rescaled the time in units of the Heisenberg time $\tau = n/t_H = n/N$, where $t_H = 2\pi/\Delta$ and $\Delta = 2\pi/N$ is the average level spacing (for $\beta = 4$ this is the nondegenerate average level spacing). The rescaled time tends to a continuous variable in the large N limit. ($\Theta(x)$ is the step function). In a different context, related results where obtained in Ref. 19. It is remarkable that for the unitary ensemble the exact and asymptotic form factors coincide. For the symplectic case, it is well known that the strong spectral correlations produces a strong "revival" at the Heisenberg time ($\tau = 1$). Asymptotically this revival is mathematically associated to a divergence of the form factor, since in the large-N limit it takes the form $b_4(\tau) = \lim_{N\to\infty} b_4^{(N)}(\tau) = [1-\tau/2+\tau\ln(|1-\tau|)/4]\Theta(2-\tau)$. However, this asymptotic expression does not allow to estimate the exact magnitude of the revival, i.e. if $<|C(\tau)|^2>$ is finite at $\tau = 1$ for $\beta = 4$ as N tends to infinity, because both the numerator and the denominator diverge in Eq.(7). From Eqs.(8) we obtain that at large but finite N the average return probability for $\beta = 4$ behaves like $<|C(\tau = 1)|^2> \approx \ln N/N$, i.e. it is large in a scale $1/N$ but remains however negligible (no "macroscopic" effect).

The second moment of the transition amplitude (i.e., the average transition probability $<|T(\tau)|^2>$) differs from the average return probability in that it is time independent (no transient regime). In fact, assuming that the state $|\chi\rangle$ is statistically independent from $|\psi(0)\rangle$ we obtain for $\beta = 1$ and $\beta = 2$

$$<|T(\tau)|^2> = <|\langle\chi|\psi(\tau)\rangle|^2> = 1/N ; \qquad \beta = 1, 2 . \tag{9}$$

Analogously we find of course that the average transition probability is $1/(2N)$ for $\beta = 4$.

Before closing this subsection, and generalizing the discussion of Ref. 19 concerning the powers of traces of unitary matrices, we would like to include a random walk interpretation of the solutions of the time-dependent Schrödinger equation, Eqs.(3-4), and in particular of the average return probability, Eq.(7).

If the phases $(\theta_1, \ldots, \theta_N)$ are considered as random independent variables, their multiples $n\theta_k$ are also random independent. Under this assumption concerning the phases, the complex numbers $C(n)$ and $T(n)$ in Eqs.(3-4) precisely define the position of a particle in the complex plane undergoing a random walk, the length of each of the N steps of the walk also being a random variable with a distribution defined by Eq.(5). In particular, the variance or typical square distance traveled by the particle after N steps will be, for uncorrelated phases

$$<|C_u(n)|^2> = \frac{N\delta_{n,0} + (\beta+2)/\beta}{(N+2/\beta)} , \tag{10}$$

(the fact that the variance doesn't grow like N as in the usual random walk process is related to the normalization in Eq.(5) which implies $<|a_k|^2> = 1/N$, i.e. an average step length which decreases with the number of steps. Putting $<|a_k|^2> = 1$ multiplies the last equation by N^2, thus recovering the usual behaviour). In eq.(10) we obtain a time-independent variance (since it doesn't depend on n aside the delta dependence at the origin) whose value coincides with the asymptotic value of Eq.(7).

The solutions of the time-dependent Schrödinger equation obtained in the context of RMT may also be thought of as a random walk in the complex plane, with the important difference that now there exist *correlations* between the angles of the different steps of the walk. The comparison of Eqs.(7) and (10) indicate that for times shorter than the Heisenberg time the variance of this correlated motion is smaller than the variance of the uncorrelated process. This is easy to understand intuitively. Indeed, it is well known, and this information is of course contained in the form factor, that there exist long range correlations between the θ's[18]. For example, the two-point correlation function for $\beta = 2$ exhibits for sufficiently large separations between phases an average repulsion which goes like the inverse of the square of the distance. In the random walk process, there exist therefore an average repulsion between the direction θ_k of the kth step and all the other ones. For $n = 1$, the orientation in the complex plane of one step increases the probability for the subsequent steps to go in the *opposite* direction. This mechanism of course diminishes the variance or typical square distance traveled by the particle.

As n increases, the multiplication of the phases by n and their periodicity contributes to progressively destroy this effect, since the repulsion may be transform into attraction by the multiplication and periodization process (depending on the relative position of the two phases in the unit circle). At the Heisenberg time, two phases separated by a mean level spacing coincide when multiplied by $n = N$. Then, for times bigger or comparable to the Heisenberg time the product $n\theta_k$ start to behave as a random phase, uncorrelated with respect to the other ones, and the variance of an uncorrelated process should be recovered. This is exactly what is observed in Eq.(7) for $\beta = 1$ and $\beta = 2$, since the function $b_\beta^{(N)}(n)$ monotonically reaches the value zero at $\tau \approx 1$. For $\beta = 4$, however, the level repulsion is so strong that it produces a very rigid structure for short distances: the probability to observe equidistant phases

$$\theta_k = 2\pi k/N \tag{11}$$

for $k = 1, 2, 3, \ldots$ is considerable enhanced. This strong local order, which is practically absent for $\beta = 1$ and $\beta = 2$, decreases with k and becomes irrelevant for $k = 5 - 10$ (see for instance Fig.6 in Ref. 2). In the random walk process, all steps which approximately satisfied Eq.(11) are in phase since $n\theta_k = N\theta_k = 2\pi k$. This produces the observed behaviour of the variance for $\beta = 4$ at the Heisenberg time, which grows like $\ln N/N$.

Finally, we comment on the absence of transient regime in the average transition probability (Eq.(9)). The difference between Eqs.(3) and (4) is that in the former $|a_k|^2$ is a real random number, while $a_k \bar{b}_k$ is a complex one (or real with a random sign for $\beta = 1$). This means that the prefactor in Eq.(4) contains a random phase on it, which of course suffices to destroy all the correlations between the θ's. The transition amplitude in quantum mechanics may then be thought of as a standard random walk process with uncorrelated phases. The variance obtained (Eq.(9)) coincides, aside from the normalization, with the usual result (multiplying by N^2 to recover the usual normalization we get a variance which grows linearly with the number of steps).

B. Their distribution

The distribution of the return and transition probabilities is obtained straightforwardly from that of the amplitudes. For example, introducing the rescaled variable $\eta = |C(\tau)|^2/(2 < |C(\tau)|^2 >)$, it follows from the Gaussian distribution of $C(\tau)$ that the distribution of the variable η is given by

$$\mathcal{P}(\eta) = \exp(-\eta) \ .$$

An analogous equation is valid for the transition probability. This distribution was in fact conjectured by Prange[15] in his study of the statistical properties of the form factor. As was discussed there, the Poissonian distribution of η implies that $|C(\tau)|^2$ and $|T(\tau)|^2$ are non self-averaging quantities, since the mean and the variance of η are both equal to one and therefore the fluctuations compare to the mean do not tend to zero in the large-N limit. In order to observe this quantities, some additional smoothing (over time or over the parameters of the system) is necessary. In fact, Eq.(6) represents an interferent sum of oscillating terms. The shortest period in the sum corresponds to an eigenphase difference of order π, giving $T_{min} = O(1) \sim t_{erg}$ (see the concluding remarks for a discussion concerning t_{erg}). On the contrary, the longest period is associated to differences in the eigenphases of the order of the mean level spacing, $T_{max} \sim 2\pi/\Delta = t_H$. In order to keep the time-dependent structure of the curve, a reasonable smoothing is over an interval δt such that $t_{erg} \ll \delta t \ll t_H$, thus reducing the fluctuations by a factor of order $1/\sqrt{\delta t}$. Averages over the initial state or parameters of the system may also be considered, like it was done in the experimental studies of $< |C(\tau)|^2 >$[6–8].

IV. CONCLUDING REMARKS

In Eq.(1), one iteration of the unitary matrix corresponds to the time it takes to the system to explore (without filling) the available phase space. This is because the state $U|\psi(0)\rangle$ can roughly be at any point of Hilbert space (U is an arbitrary rotation in that space). This time, called the ergodic time t_{erg}, can be relatively large in a particular physical system as compared to other time scales involved in the problem, meaning that the system is not able to immediately explore the available phase space. For example, in disordered metallic systems a diffusive motion takes place before the particle reaches the boundary of the sample. If the typical sample size is L and the diffusion constant D, then $t_{erg} \propto L^2/D$ (Thouless time) which may be much larger than the elastic mean free time. The diffusive motion is known to produce non-universal deviations from the RMT in the form factor for $t \lesssim t_{erg}$[21]. These non-universalities can also manifest themselves as a system-dependent transient in Eq.(9). Non-universal features will appear in $b_\beta^{(N)}(\tau)$ for short times in general for any dynamical system[22], where t_{erg} can roughly be identified with the period of the shortest periodic orbit. As extensively studied[23], in a scale of the order of t_{erg} the function $b_\beta^{(N)}(\tau)$ may show a series of peaks at multiples of the shortest periodic orbits whose amplitude decreases as $\exp(-\lambda t)$, where λ is the classical Lyapunov exponent. For longer times, it follows from the Gaussian distribution of the return amplitude with a variance of order $1/N$ that statistically the probability of revivals (i.e. a return probability of order one) in chaotic systems is exponentially small.

Although a natural choice (relevant for experiments) of $|\psi(0)\rangle$ is a wave packet, it is important to note that the relaxation process described here is not restricted to such a class of initial states. Our results are general, valid for any initial state (except of course the eigenstates). In particular, $|\psi(0)\rangle$ could be a state completely delocalized in the basis we are using to study the motion.

In conclusion, we have studied the relaxation process in the time evolution of complex quantum systems. Aside from the short time non-universal features already mentioned, the relaxation process described here is expected to be universal, applicable to a large class of physical systems including those whose classical limit is chaotic, disordered metallic mesoscopic samples, high lying excitations of many body systems, and equivalent problems in optics or acoustics. The $1/N$ scaling factor of the fluctuations amplitude is a purely normalization factor, proportional to the typical component of the initial state $|\psi(0)\rangle$ on the chaotic eigenstates of the system. For example, for maps defined on a compact phase space, N is the dimension of the Hilbert space, while for chaotic billiards N is the volume of the cavity. Rescaling the time by the Heisenberg time we obtain a universal description, expected to be independent of any specific feature of the system like for example the distribution or density of impurities or the Lyapunov exponents. In Ref. 24 numerical results for the transition probability for several chaotic maps were reported. These were found to be independent of the system considered and they all nicely agree with the present results.

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