Classification of Conformal Field Theories Based on Coulomb gases. Application to Loop Models.

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ABSTRACT

We present a method for classifying conformal field theories based on Coulomb gases (bosonic free-field construction). Given a particular geometric configuration of the screening charges, we give necessary conditions for the existence of degenerate representations and for the closure of the vertex-operator algebra. The resulting classification contains, but is more general than, the standard one based on classical Lie algebras. We then apply the method to the Coulomb gas theory for the two-flavoured loop model of Jacobsen and Kondev. The purpose of the study is to clarify the relation between Coulomb gas models and conformal field theories with extended symmetries.

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1 Introduction

Conformal Field Theories (CFT) based on Coulomb gases arise in a variety of two-dimensional problems in statistical physics [1]. Furthermore, a great number of CFTs are known to possess a Coulomb gas formulation. Early examples, such as the Potts model or the critical and tricritical points in the O(n) model, were based on a single scalar field, and the physical operators could be interpreted as particles carrying scalar quantised electric and magnetic charges.

More recently, multicomponent Coulomb gases employing several bosonic free fields have appeared in the study of critical phases in the so-called fully-packed loop models [2]. A first step in their resolution consists in bijectively mapping configurations of oriented loops to those of a discretised surface. The basic idea is here to interpret the loops as contour lines of the surface height, but due to the fully-packing constraint it turns out that the height variables in general have to be vector valued. Based on symmetry and entropic considerations, an effective action of the Liouville type can then be written down for the continuum limit of this interfacial representation. However, this action contains a certain number of elastic constants whose exact values cannot be inferred directly from the discrete model.

Important technical progress was achieved with the use of the *loop ansatz*. It states that the most relevant vertex operators in a given model have to be exactly marginal and taken into the action as screenings, thus allowing all the elastic constants to be fixed [3]. This situation is truly remarkable: The discrete model precludes an a priori knowledge of the parameters defining its continuum limit, but it nevertheless fixes the geometry of the screening charges and thus permits an exact a posteriori determination of the very same parameters. As the end result one obtains a CFT in the form of a Coulomb gas (in general multicomponent, the number of components being the dimensionality of the height space) with a given background charge and screening operators. Physical operators are represented by vertex operators

$$V_{\vec{\beta}}(z,\bar{z}) =: e^{i\vec{\beta}\vec{\varphi}(z,\bar{z})} :, \tag{1.1}$$

with $\vec{\beta}$ taking values on a particular lattice, specific to the given model. The vector $\vec{\varphi}(z,\bar{z})$ contains a set of bosonic free fields $\vec{\varphi}(z,\bar{z}) = \{\varphi_1(z,\bar{z}), \varphi_2(z,\bar{z}), \ldots, \varphi_D(z,\bar{z})\}$, which can

be interpreted as the continuum limit of the components of the discrete interfacial height.

The question which appears naturally is whether there is a chiral algebra hidden behind this conformal theory, defined as a multicomponent Coulomb gas, and what it might be. In particular, given the fact that fully-packed loop models may possess a central charge $c > 1$ [2], one may wonder if the chiral algebra could be bigger than just the Virasoro one, ensured by the conformal invariance of the model. Put differently: Might a particular model, represented in the continuum limit as a Coulomb gas, have additional, or extended symmetries? By this we mean not just global symmetries, which are usually explicit, but symmetries which are extended on the level of the chiral algebra, i.e. extended infinitely. In case of a positive answer this would imply the existence of extra chiral operators, like $W(z)$ operators, which form an extended chiral algebra together with the stress-energy tensor $T(z)$.

Presence of such symmetries in the Coulomb gas would not automatically ensure that they persist in the original model defined on the lattice, and that the model would be integrable. Still, in case of a positive answer for the continuum theory (existence of extended symmetries) but negative for the lattice model, one could try to find an integrable version of it on the lattice.

On the other hand, in case of a negative answer in the continuum (no extra chiral operators in the multicomponent Coulomb gas model, in addition to $T(z)$) the prediction for the lattice model is likely to be definitive: Extended symmetries will not come about on the lattice, and the model will not be integrable. It is in this perspective, looking for the existence of extra symmetries or their absence, that we shall try to clarify the relation between Coulomb gas models and CFTs which are based on extended chiral algebras.

The general method is presented in Section 2. We here study the necessary conditions for the existence of degenerate representations and for the closure of the vertex-operator algebra. These conditions are of course met by all W-type extended CFTs which are based on simply laced Lie algebras, but in fact they are less restrictive. In particular they are fulfilled by the two-flavour fully-packed loop model of Jacobsen and Kondev [2], whose continuum limit is reminiscent of, but not identical to, the classical WA_3 theory. To assess whether such theories actually contain extra chiral operators we work on a case-to-case basis, computing the exact form of such operators or proving that they do

not exist. In Section 3 this method is applied to a variety of two- and three-component Coulomb gases, in particular to those which arise in the solution of loop models. We finally present our conclusions in Section 4.

2 General method

2.1 Background charge and screenings

The primary information defining a Coulomb gas model can be summarised as follows:

1. Dimensionality of the Coulomb gas (number of free-field components), D.
2. The background charge, $\vec{\alpha}_0$.
3. A set of D screening operators,

$$\{: \mathrm{e}^{i\vec{\alpha}_a\vec{\varphi}(z,\bar{z})} : \mid a = 1, 2, \ldots, D\}. \tag{2.1}$$

We shall assume that, for a generic model, the number of screenings equals the dimensionality of the Coulomb gas. Namely, if the number of screenings were less than D then a subset of free fields would decouple, and the model could be represented as a direct product of a submodel with the background charge and screenings and a submodel of actual free fields, no background charge, and no screenings. If, alternatively, due to a special symmetry of the lattice to which the screenings belong, the number of screenings exceeds D, a subset of exactly D screenings ought to be sufficient to define correlation functions of physical operators. Different choices of a subset of D screenings should give equivalent results. At least this should be the case for a Coulomb gas with an underlying chiral algebra, which classifies all the operators in the model, and with respect to which the physical operators are primaries. This last point will be made more precise in the following. For the moment we do not have any reasons for assuming such special symmetries of the lattice to which the screenings should belong.

The primary information on a Coulomb gas, as stated above, is sketched in Fig. 1. For various lattice models, like Potts or O(n) models [1], or more general loop models [2], the

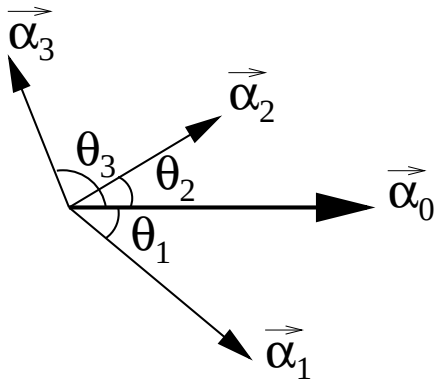

Figure 1: Background charge vector $\vec{\alpha}_0$ and the vectors $\{\vec{\alpha}_a\}$ defining the screening operators $V_{\vec{\alpha}_a}(z,\bar{z}) = :\mathrm{e}^{i\vec{\alpha}_a\vec{\varphi}(z,\bar{z})}:$ for $a = 1, 2, \ldots, D$. On the figure, $D = 3$. $\{\Theta_a\}$ are the angles which $\{\vec{\alpha}_a\}$ make with the background charge $\vec{\alpha}_0$.

geometry of $(\{\vec{\alpha}_a\}, \vec{\alpha}_0)$ is known explicitly. But we shall here derive general constraints on the possible geometries, by using consistency conditions of the corresponding CFT.

We take $\vec{\alpha}_0$ as given. The stress-energy tensor of D free fields $\{\varphi_a(z,\bar{z})\}$ will be taken in the form

$$T(z) = -\frac{1}{4} :\partial\vec{\varphi}(z)\partial\vec{\varphi}(z): + i\vec{\alpha}_0\partial^2\vec{\varphi}(z), \tag{2.2}$$

with the two-point correlation functions of the fields $\{\varphi_a(z,\bar{z})\}$ normalised as

$$\langle\varphi_a(z,\bar{z})\varphi_b(z',\bar{z}')\rangle = 2\delta_{a,b}\log\frac{1}{|z-z'|^2}. \tag{2.3}$$

With this normalisation, the conformal dimension (with respect to the stress-energy tensor (2.2)) of a vertex operator

$$V_{\vec{\alpha}}(z,\bar{z}) = :\mathrm{e}^{i\vec{\alpha}\vec{\varphi}(z,\bar{z})}: \tag{2.4}$$

will be equal to

$$\Delta_{\vec{\alpha}} = \vec{\alpha}^2 - 2\vec{\alpha}\vec{\alpha}_0. \tag{2.5}$$

The first condition on the screening operators (2.1) is that they have to be marginal, $\Delta_{\vec{\alpha}_a} = 1$ for $a = 1, 2, \ldots, D$. This condition ensures that contour integrals of the screenings

$$Q_a = \oint_C dz V_{\vec{\alpha}_a}(z), \quad a = 1, 2, ..., D \tag{2.6}$$

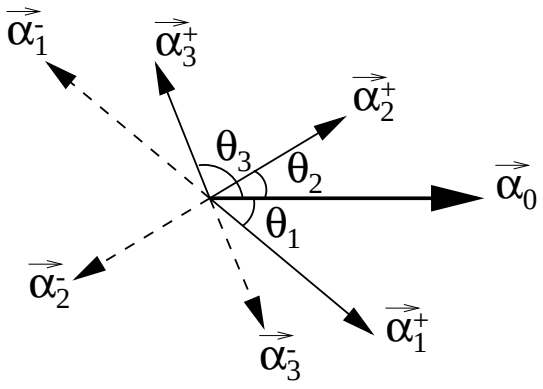

Figure 2: The complete set of screenings $\{\vec{\alpha}_a^+, \vec{\alpha}_a^-\}$, and the background charge $\vec{\alpha}_0$.

commute with the Virasoro algebra generated by $T(z)$. The operator $V_{\vec{\alpha}_a}(z)$ in (2.6) is assumed to be the holomorphic part of the screening operator $V_{\vec{\alpha}_a}(z, \bar{z})$ in (2.1), in the sense of holomorphic-antiholomorphic factorisation of correlation functions of vertex operators, or, more generally, in the sense of holomorphic-antiholomorphic factorisation of two-dimensional integrals of correlation functions of vertex operators with respect to the contour integrals [4, 5].

To make the screenings marginal, the vectors $\{\vec{\alpha}_a\}$ in (2.1) have to satisfy the condition

$$\Delta_{\vec{\alpha}_a} = \vec{\alpha}_a^2 - 2\vec{\alpha}_a\vec{\alpha}_0 = 1, \tag{2.7}$$

or

$$\alpha_a^2 - 2\alpha_a\alpha_0\cos\Theta_a = 1, \tag{2.8}$$

$$\alpha_a^\pm = \alpha_0\cos\Theta_a \pm \sqrt{1 + \alpha_0^2\cos^2\Theta_a}. \tag{2.9}$$

Here $\{\Theta_a\}$ are the angles in Fig. 1, and $\alpha_a^\pm$ are the "lengths" of the vectors $\vec{\alpha}_a$ (one positive, α_a^+, and one negative, α_a^-), which satisfy the condition (2.7).

In Fig. 1 we have indicated a particular geometry of the vectors $\vec{\alpha}_0$ and $\{\vec{\alpha}_a\}$. This choice of geometry is for the moment arbitrary. The only constraint, once the directions of $\vec{\alpha}_0$ and $\{\vec{\alpha}_a\}$ have been chosen, is on the lengths of the screenings, Eq. (2.9). There are two solutions, "+" and "−", for each direction, and we shall see in the following that the consistence of the corresponding CFT requires the use of them both. Since the "lengths" $\{\alpha_a^-\}$ are negative, the screening vectors $\{\vec{\alpha}_a^-\}$ are oriented in the opposite direction with respect to $\{\vec{\alpha}_a^+\}$. The set of screenings has thus been doubled, D screenings $\{\vec{\alpha}_a^+\}$ and D

screenings $\{\vec{\alpha}_a^-\}$, as indicated in Fig. 2. We also note the relations between α_a^+ and α_a^-

$$\alpha_a^+ + \alpha_a^- = 2\alpha_0 \cos\Theta_a \tag{2.10}$$

$$\alpha_a^+ \alpha_a^- = -1, \tag{2.11}$$

which follow from Eq. (2.8). These relations will be used in the following.

2.2 Necessary condition for degenerate representations. Kac formula

We shall be looking for a Coulomb gas theory based on a chiral algebra. The number of chiral operators forming this algebra has to be equal to the number of free fields, D. In addition to $T(z)$, this requires $D-1$ extra chiral operators to control all the degrees of freedom of the theory, i.e. to control D free fields.

We shall assume that these extra chiral operators belong to the module of the identity operator, just like $T(z)$, i.e. that they are made as linear combinations of products of derivatives of free fields. For a Coulomb gas theory based on D bosonic fields this last assumption appears to us to be natural. This is essentially because in this case the chiral operators will have zero Coulomb charge; their correlation functions will therefore not require the integrated screening operators, they will be "simple".

It should be noted that this argument is valid only in the case of $\vec{\alpha}_0 \neq 0$, where, in general, correlation functions of generic vertex operators *do* require screenings. Generic correlation functions are therefore not simple, but those made exclusively of chiral symmetry operators are. This argument implicitly supposes that an acceptable Coulomb gas theory will allow for a deformation of the trivial free-field point $\vec{\alpha}_0 = 0$, which has $c = D$.

Assuming therefore that the extra chiral operators belong to the module of the identity, the possible chiral algebras for Coulomb gas theories should be made of operators with integer conformal dimensions, $\Delta \in \mathbb{N}$. Furthermore, since a dimension-one current is known to generate a continuous rather than a discrete symmetry, we require $\Delta \geq 2$. This means that the chiral algebras should be of the W type. If this algebra exists, the screening operators (integrated along the contours) have to commute with it, in order to respect the extended symmetry of the given theory.

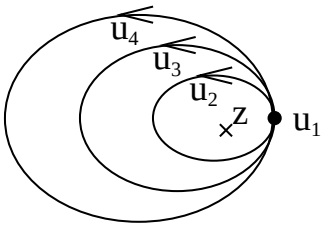

Figure 3: Having integrated first all the screenings except for one type (whose choice is not important), one finally integrates the last screening, at the point u_1, around z.

For chiral operators in the form of linear combinations of products of derivatives of free fields, the operators which are primary with respect to them are the vertex operators

$$V_{\vec{\beta}}(z,\bar{z}) = :\mathrm{e}^{i\vec{\beta}\vec{\varphi}(z,\bar{z})}:, \tag{2.12}$$

where $\vec{\beta}$ is for the moment arbitrary. In fact, an operator $\Phi(z,\bar{z})$ being primary with respect to a particular chiral operator $W(z)$ means that the operator product expansion (OPE) of $W(z)$ with $\Phi(z,\bar{z})$ takes the form

$$W(z)\Phi(z',\bar{z}') = \frac{A}{(z-z')^{\Delta_W}}\Phi(z',\bar{z}') + \ldots, \tag{2.13}$$

where A is a (structure) constant. Loosely speaking, the most singular term in the expansion in powers of $(z-z')$ produces again the operator $\Phi(z,\bar{z})$. Evidently, when $W(z)$ is a linear combination of products of derivatives of free fields, each term involving a total of Δ_W derivatives, and when $\Phi(z,\bar{z}) = V_{\vec{\beta}}(z,\bar{z})$ is an exponential of free fields, cf. Eq.(2.12), the expansion $W(z)V_{\vec{\beta}}(z',\bar{z}')$ will be precisely of the form (2.13).

Until now $\vec{\beta}$ has been an arbitrary vector. For special values of $\vec{\beta}$ it may however happen that primary operators appear in the module of $V_{\vec{\beta}}(z,\bar{z})$. They are called singular states, and their presence means degeneracy of the module. Since the integrated screenings commute with the operators of the chiral algebra, the singular state in the module of $V_{\vec{\beta}}(z,\bar{z})$ may be obtained by mapping, with the use of screenings, another vertex operator $V_{\vec{\beta}'}(z,\bar{z})$ into the module of $V_{\vec{\beta}}(z,\bar{z})$. This is brought about in the following way:

$$S(V_{\vec{\beta}}(z,\bar{z})) = \prod_{a=1}^{D}\left(\oint_{C_a} du_a V_{\vec{\alpha}_a^+}(u_a)\right)^{n_a} V_{\vec{\beta}'}(z,\bar{z}), \tag{2.14}$$

where $n_a \in \mathbb{N}_0$ for $a = 1,2,\ldots,D$. Here $S(V_{\vec{\beta}})$ is a singular state in the module of $V_{\vec{\beta}}$,

and $\vec{\beta}'$ should be related to $\vec{\beta}$ by

$$\vec{\beta}' = \vec{\beta} - \sum_a n_a \vec{\alpha}_a^+. \tag{2.15}$$

The configuration of contours $\{C_a\}$ in Eq. (2.14) is shown in Fig. 3. The set of contours in the figure is only schematic, since there are in fact n_a distinct contours corresponding to the screenings of type $a = 1, 2, \ldots, D$.

This prescription for realising closed contour integrations of screenings around a fixed vertex operator, on which they act to induce the mapping (2.14), was first used in Ref. [6].

More precisely, the mapping of Eq. (2.14) is defined, and the singular state $S(V_{\vec{\beta}})$ exists, only if

$$\Delta_{\vec{\beta}'} - \Delta_{\vec{\beta}} = N, \tag{2.16}$$

where N is a positive integer. In fact, as the screenings commute with $T(z)$, the conformal dimension of $S(V_{\vec{\beta}})$ is equal to that of $V_{\vec{\beta}'}$. Being a descendent state, the conformal dimension of $S(V_{\vec{\beta}})$ should differ from $\Delta_{\vec{\beta}}$ by an integer. One thus obtains the condition (2.16).

Another way to get (2.16) is to require that the monodromy for the analytic continuation of the expression on the right-hand side of (2.14) with respect to the common variable u_1 (cf. Fig. 3) be trivial, so that the contour of integration over u_1 around z closes. Otherwise the integrated screenings would not actually commute with $T(z)$ and with other operators of the chiral algebra. It can be checked that this trivial monodromy requirement leads equivalently to the condition (2.16) [6].

This condition, with $\vec{\beta}'$ defined by (2.15) and $\Delta_{\vec{\beta}}, \Delta_{\vec{\beta}'}$ given by (2.5), fixes special values of $\vec{\beta}$ for which the module of $V_{\vec{\beta}}$ contains singular states, i.e. is degenerate.

Note that we have only applied the "+" screenings in Eqs. (2.14)–(2.15). The reason is that once the solution for $\vec{\beta}$ is found, it will automatically satisfy the corresponding condition with negative screenings. For the analysis of singular states the "+" and "−" screenings are therefore complimentary. A similar remark holds true for a mixed mapping, i.e. one involving both "+" and "−" screenings: The corresponding conditions do not lead to new solutions. These statements should become more clear from the analysis that will follow.

We remark that this way of defining the degenerate representations, using the mapping by screenings between vertex operators and vertex operator modules, has probably been used first in Ref. [7] to reproduce the Kac formula for conformal dimensions for the degenerate representations of the Virasoro algebra. This method has also been used in [8, 9] to define the degenerate representations of W-algebra CFTs based on classical Lie algebras.

Until now we have assumed a set of screenings, as depicted in Figs. 1–2, with no restrictions yet, except for those on the lengths of screenings, cf. Eqs. (2.7)–(2.9). As we are looking for Coulomb gas theories which should eventually be endowed with extended chiral algebras, we shall next identify the vertex operators $\{V_{\vec{\beta}}(z,\bar{z})\}$ which induce the degenerate representations of that chiral algebra.

Our approach can be outlined as follows: Without knowing yet the chiral algebra, for a given set of screening operators we shall define, by using the mappings described above, the "would be degenerate representations". The screening operators are assumed to commute with the (presently unknown) chiral algebra. We shall define, for given screenings and background charge, the discrete set of values of $\vec{\beta}$ for which the operators $V_{\vec{\beta}}(z,\bar{z})$ would induce degenerate representations, if the chiral algebra really existed. In this way we shall arrive at a necessary condition for the existence of degenerate representations. The condition will also become sufficient when the corresponding extended chiral algebra is found.

The advantage of this approach will be that, having defined such special values of $\vec{\beta}$ corresponding to given screenings, it will allow us to derive next a set of constraints "acting backwards", i.e. that limits the allowed configurations of screenings and the background charge.

From the vast literature on W-algebras it is well-known that the proper way to look for singular states in the module of $V_{\vec{\beta}}(z,\bar{z})$ is to apply mappings of the type (2.14) by shifting $\vec{\beta}'$ away from $\vec{\beta}$ only along the "principal" directions defined by the individual screenings, and not as has been sketched schematically in Eqs. (2.14)–(2.15), where several different screenings are involved simultaneously.

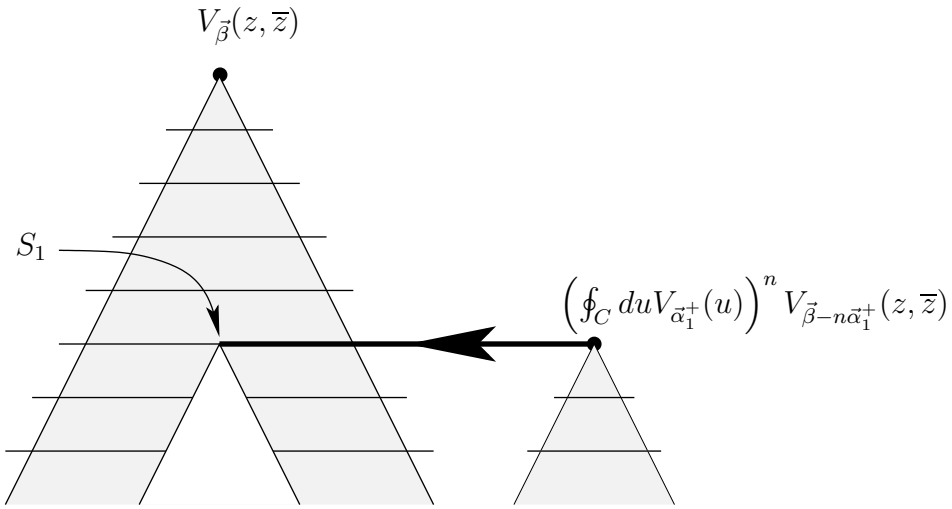

Figure 4: Mapping of the module of $V_{\tilde{\vec{\beta}}} \equiv V_{\vec{\beta}-n\vec{\alpha}_1^+}$ into the module of $V_{\vec{\beta}}$, in the direction of the first screening $\vec{\alpha}_1^+$.

For the direction associated with the first screening, the mapping takes the form

$$\begin{aligned} S_1(V_{\vec{\beta}}(z,\bar{z})) &= \left(\oint_C du V_{\vec{\alpha}_1^+}(u)\right)^n V_{\vec{\beta}-n\vec{\alpha}_1^+}(z,\bar{z}) \\ &= \prod_{i=1}^{n} \oint du_i V_{\vec{\alpha}_1^+}(u_i) V_{\vec{\beta}-n\vec{\alpha}_1^+}(z,\bar{z}), \end{aligned} \tag{2.17}$$

cf. Figs. 3–4. The condition (2.16) becomes

$$\Delta_{\vec{\beta}-n\vec{\alpha}_1^+} - \Delta_{\vec{\beta}} = N, \tag{2.18}$$

where N should be a positive integer. Substituting the formula (2.5) for $\Delta_{\vec{\beta}-n\vec{\alpha}_1^+}$ and $\Delta_{\vec{\beta}}$, one finds

$$(\vec{\beta}-n\vec{\alpha}_1^+)^2 - 2\vec{\alpha}_0(\vec{\beta}-n\vec{\alpha}_1^+) - \vec{\beta}^2 + 2\vec{\alpha}_0\vec{\beta} = N, \tag{2.19}$$

which gives

$$-2n(\vec{\beta}-\vec{\alpha}_0)\vec{\alpha}_1^+ + n^2(\alpha_1^+)^2 = N. \tag{2.20}$$

The general solution of this equation for the allowed values of $\vec{\beta}$ takes the form

$$2(\vec{\beta}-\vec{\alpha}_0) = (n\alpha_1^+ + n'\alpha_1^-)\vec{\omega}_1, \tag{2.21}$$

where the vector $\vec{\omega}_1$ should verify the equation $\vec{\omega}_1\vec{e}_1 = 1$ with $\vec{e}_1 = \vec{\alpha}_1^+/\alpha_1^+$, and n' in (2.21) is an another positive integer. By substituting (2.21) into (2.20) one finds $N = nn'$ so that, for $\vec{\beta}$ of the form (2.21), the constraint (2.18) is indeed verified.

By similarly requiring that singular states be produced for each of the D screening directions, defined by the vectors $\{\vec{\alpha}_a^+\}$, one finds that $\vec{\beta}$ has to satisfy

$$2(\vec{\beta} - \vec{\alpha}_0) = \sum_{a=1}^{D} (n_a \alpha_a^+ + n'_a \alpha_a^-) \vec{\omega}_a. \tag{2.22}$$

Here the D vectors $\{\vec{\omega}_a\}$ are defined by

$$\vec{\omega}_a \vec{e}_b = \delta_{a,b}, \tag{2.23}$$

and $\{\vec{e}_a\}$ are the unit vectors which define the directions of the screenings:

$$\vec{\alpha}_a^{\pm} = \alpha_a^{\pm} \vec{e}_a \text{ for } a = 1, 2, ..., D. \tag{2.24}$$

Eq. (2.22) generalises (2.21). Alternatively it can be presented as

$$\vec{\beta} = \vec{\beta}_{(n'_1,n_1)(n'_2,n_2)\cdots(n'_D,n_D)} = \sum_{a=1}^{D} \left(\frac{n_a}{2} \alpha_a^+ + \frac{n'_a}{2} \alpha_a^- \right) \vec{\omega}_a + \vec{\alpha}_0. \tag{2.25}$$

This expression can be simplified by developing $\vec{\alpha}_0$ in terms of the vectors $\{\vec{\omega}_a/2\}$,

$$\vec{\alpha}_0 = \sum_a X_a \frac{\vec{\omega}_a}{2}. \tag{2.26}$$

The expansion coefficients $\{X_a\}$ are then found by using the orthogonality property (2.23):

$$X_b = 2\vec{\alpha}_0 \vec{e}_b = 2\alpha_0 \cos\Theta_b = \alpha_b^+ + \alpha_b^-, \tag{2.27}$$

where in the last step we have used the relations (2.10) between α_a^+ and α_a^-. The angles $\{\Theta_a\}$ have been defined in Figs. 1–2. Substituting (2.27) into (2.26), we arrive at

$$\vec{\alpha}_0 = \sum_a \frac{\alpha_a^+ + \alpha_a^-}{2} \vec{\omega}_a, \tag{2.28}$$

which can finally be inserted in Eq. (2.25) to yield the following set of values of $\vec{\beta}$:

$$\vec{\beta}_{(n'_1,n_1)(n'_2,n_2)\cdots(n'_D,n_D)} = \sum_{a=1}^{D} \left(\frac{1+n_a}{2} \alpha_a^+ + \frac{1+n'_a}{2} \alpha_a^- \right) \vec{\omega}_a. \tag{2.29}$$

For the "conjugate" values of $\vec{\beta}$, viz. $\tilde{\vec{\beta}} = 2\vec{\alpha}_0 - \vec{\beta}$, one finds:

$$\tilde{\vec{\beta}}_{(n'_1,n_1)(n'_2,n_2)\cdots(n'_D,n_D)} = \sum_{a=1}^{D} \left(\frac{1-n_a}{2} \alpha_a^+ + \frac{1-n'_a}{2} \alpha_a^- \right) \vec{\omega}_a. \tag{2.30}$$

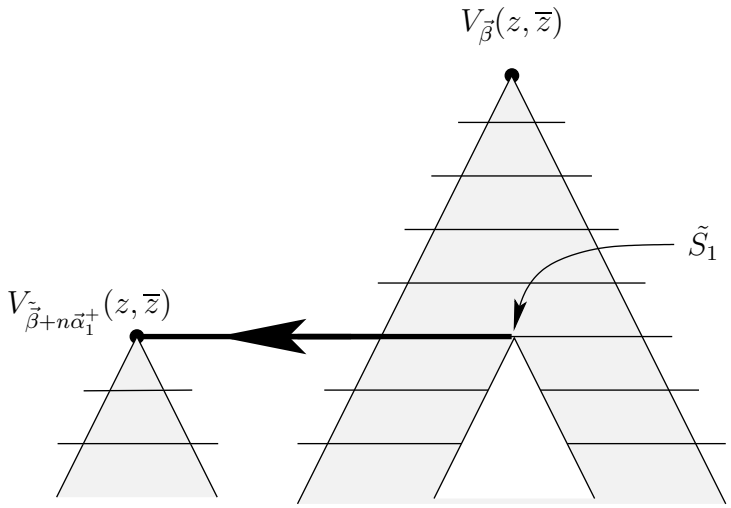

Figure 5: The mapping is of the form $\left(\oint_C \mathrm{d}u\, V_{\vec{\alpha}_1^+}(u)\right)^n \tilde{S}_1 = V_{\tilde{\vec{\beta}}+n\vec{\alpha}_1^+}$.

The manifestation of singular states in the module of the vertex operator $V_{\tilde{\vec{\beta}}}(z,\bar{z})$ is so to say dual to that of $V_{\vec{\beta}}(z,\bar{z})$. If the module of $V_{\tilde{\vec{\beta}}}$ is degenerate, it contains particular primary states which produce, as images under appropriate mappings, the vertex operators $V_{\vec{\beta}'}$ several screenings away, as indicated in Fig. 5. More details on these properties can be found in the paper [6]. Physically, in theories with a chiral algebra, the set of dual operators $\{V_{\vec{\beta}}, V_{\tilde{\vec{\beta}}}\}$ can either represent the same (singlet) operator (such as the energy operator) or a doublet of conjugate operators (an example being the $\{\sigma, \sigma^\dagger\}$ spin operators in the three-state Potts model).

For the conformal dimensions of the vertex operators $V_{\vec{\beta}}(z,\bar{z})$, with the above values for $\vec{\beta}$, one obtains

$$\begin{aligned}\Delta_{\vec{\beta}_{(\cdots)}} &= \Delta_{2\vec{\alpha}_0-\vec{\beta}_{(\cdots)}} \equiv \Delta_{(n_1',n_1)(n_2',n_2)\cdots(n_D',n_D)} \qquad (2.31)\\ &= \sum_{a=1}^{D}\left(u^{(a)}_{n_a',n_a}\right)^2(\vec{\omega}_a)^2 + 2\sum_{a<b}^{D} u^{(a)}_{n_a',n_a}u^{(b)}_{n_b',n_b}\,\vec{\omega}_a\vec{\omega}_b - \sum_{a=1}^{D}\sum_{b=1}^{D}(\alpha_a^+ + \alpha_a^-)u^{(a)}_{n_a',n_a}\,\vec{\omega}_a\vec{\omega}_b,\end{aligned}$$

where we have defined

$$u^{(a)}_{n_a',n_a} = \frac{1-n_a}{2}\,\alpha_a^+ + \frac{1-n_a'}{2}\,\alpha_a^-. \qquad (2.32)$$

The procedure that we followed above to define $\beta_{(n_1'n_1)(n_2'n_2)\cdots(n_D'n_D)}$ in Eqs.(2.29)–(2.30), is perfectly standard. Initially it has been used in Refs. [7, 8, 9].

Note that if the chiral algebra consisting of D chiral operators commuting with the screenings is found, Eq. (2.32) together with Eqs.(2.29)–(2.30) would become the Kac formula for the degenerate representations of the Coulomb gas theory at hand. But in our approach it should rather be interpreted as a *necessary condition* for the existence of degenerate representations, for any given configuration of the screenings. The existence of the extended chiral algebra is not yet guaranteed.

It should be noticed that the degeneracy of the module of $V_{\vec{\beta}}$ in all D screening directions, with $\vec{\beta}$ taking its values in the discrete set (2.29), must be required in order for the operator algebra of the operators $\{V_{\vec{\beta}}(z,\bar{z})\}$ to be well defined. We shall consider the properties of this vertex-operator algebra shortly.

Another remark is on a detail which was actually implicit in the above arguments: For a theory with an extended chiral algebra to be acceptable we want it to contain one free parameter. Put differently, we want the theory to exist for general values of α_0, or, alternatively, of the central charge c. In the class of CFTs in which we are interested, c must remain a free parameter until we eventually impose the much stronger constraint of unitarity, in which case it will be forced to take its values in a discrete (but still denumerably infinite) set. In general we expect the structure of a CFT to be very rigid, in the sense that once we impose additional constraints that limit the generality of the theory, we will immediately either generate inconsistencies or, in the best case, fix c to take a finite number of values. We can illustrate this remark through the example of the two-flavoured loop model of Jacobsen and Kondev [2]. In the special cases where the fugacities of either loop flavour take the value 2 (resp. 1), the model is known to be equivalent to the $\mathrm{su}(4)_{k=1}$ Wess-Zumino-Witten model [10] (resp. to the equally-weighted six-vertex model), both of which are bona fide CFTs with fixed $c = 3$ (resp. $c = 1$). But it is not because of these two special cases that we shall accept the general model (i.e. with generic values of the loop fugacities) as a theory with an extended chiral algebra.

Returning to the general study, one could define the central charge from the two-point function of $T(z)$

$$\langle T(z)T(z')\rangle = \frac{c/2}{(z-z')^4}, \tag{2.33}$$

where $T(z)$ is given by (2.2), and the correlation functions of the fields $\{\varphi_a(z,\bar{z})\}$ are

normalised as in Eq. (2.3). From (2.2), (2.3) and (2.33) one then finds

$$c = D - 24(\alpha_0)^2. \tag{2.34}$$

The Coulomb gas theory itself is defined for general values of α_0. Naturally, then, we are looking for conditions on the theory under which it would contain an extended chiral algebra for general values of α_0, or, equivalently, of the central charge.

2.3 Closure of the operator algebra of the operators $\{V_{\vec{\beta}_{(\dots)}}\}$

Having defined the "would be degenerate" operators $\{V_{\vec{\beta}_{(\dots)}}\} \equiv \{V_{(n'_1,n_1)(n'_2,n_2)\cdots(n'_D,n_D)}\}$, with $\{\vec{\beta}_{(\dots)}\}$ defined by Eqs. (2.29)–(2.30), we shall impose next the condition that their operator algebra closes. In a Coulomb gas theory with background charge and screenings the OPE for a product of two vertex operators $V_{\vec{\beta}_1}(z,\bar{z})$ and $V_{\vec{\beta}_2}(z',\bar{z}')$ produces by conservation of electric charge the vertex operator $V_{\vec{\beta}_1+\vec{\beta}_2}(z',\bar{z}')$, but also the vertex operators where the charge $\vec{\beta}_1+\vec{\beta}_2$ has been shifted by any number of screening vectors. Thus, schematically,

$$\begin{aligned} V_{\vec{\beta}_1}(z,\bar{z})V_{\vec{\beta}_2}(z',\bar{z}') \;&\sim\; V_{\vec{\beta}_1+\vec{\beta}_2}(z',\bar{z}') + \text{descendents} \\ &+\; V_{\vec{\beta}_1+\vec{\beta}_2+\vec{\alpha}_1^+}(z',\bar{z}') + \text{descendents} + \cdots . \end{aligned} \tag{2.35}$$

In general the OPE of $V_{\vec{\beta}_1}V_{\vec{\beta}_2}$ will produce the entire set of operators

$$\{V_{\vec{\beta}_1+\vec{\beta}_2+\sum_a (k_a\vec{\alpha}_a^+ + l_a\vec{\alpha}_a^-)}\} \tag{2.36}$$

as well as their descendents. If $\vec{\beta}_1$ and $\vec{\beta}_2$ belong to the Kac table, i.e. to the sets $\{\vec{\beta}_{(n'_1,n_1)(n'_2,n_2)\cdots(n'_D,n_D)}\}$, $\{\tilde{\vec{\beta}}_{(n'_1,n_1)(n'_2,n_2)\dots(n'_D,n_D)}\}$ given by Eqs. (2.29)–(2.30), then in order that the operators (2.36) also belong to the Kac table, the screening vectors $\{\vec{\alpha}_a^+\}$ must decompose as a integer linear combination of the basis vectors spanning (2.29)–(2.30), i.e. the set $\{\alpha_a^+\frac{\vec{\omega}_a}{2}\}$. It is sufficient to ensure this decomposition for the "+" screenings, since the corresponding decomposition of the "−" screenings over the basis $\{\alpha_a^-\frac{\vec{\omega}_a}{2}\}$ will then follow automatically. Writing

$$\vec{\alpha}_a^+ = \sum_b A_{ab}\alpha_b^+\frac{\vec{\omega}_b}{2}, \tag{2.37}$$

the integer coefficients A_{ab} can be computed by projecting onto the normalised screening vectors, cf. Eq. (2.24):

$$\vec{\alpha}_a^+ \vec{e}_c = \sum_b A_{ab} \alpha_b^+ \frac{1}{2} \vec{\omega}_b \vec{e}_c = \sum_b A_{ab} \alpha_b^+ \frac{1}{2} \delta_{bc} = \frac{1}{2} A_{ac} \alpha_c^+ \tag{2.38}$$

$$A_{ac} = \frac{2(\vec{\alpha}_a^+, \vec{e}_c)}{\alpha_c^+} = \frac{2\vec{\alpha}_a^+ \vec{\alpha}_c^+}{(\vec{\alpha}_c^+)^2}. \tag{2.39}$$

We observe that when expressing the matrix of coefficients A_{ab} in terms of screening vectors, one recovers the same form as the Cartan matrix in the theory of classical Lie algebras, expressed in terms of the simple root vectors. The classification condition is the same: Its components have to be integers.

It is well-known that this condition implies that the screening vectors $\{\vec{\alpha}_a^+\}$ must belong to the root lattice of one of the classical Lie algebras. This criterion appears here as a *necessary condition* for the closure of the operator algebra of the vertex operators $V_{(n_1',n_1)(n_2',n_2)\ldots(n_D',n_D)}(z,\bar{z})$. Clearly, since we are interested in the most general Coulomb gas theory, and not just a closed sub-theory, we can further require the $\{\vec{\alpha}_a^+\}$ to be a set of *basis vectors* of the root lattice. However, there is for the moment no need that we should constrain further to the *simple roots* of a classical Lie algebra, in which case we would immediately limit ourselves to the standard classification of W-type extended CFTs.

Several remarks are in order.

In the Coulomb gas theory there are additional constraints on the lengths of the vectors $\{\vec{\alpha}_a^+\}$, in order to ensure the marginality of the screening operators. These constraints are expressed by Eq. (2.9).

For the case of simply laced algebras, all the lengths of $\{\vec{\alpha}_a^+\}$ have to be equal. In this case, to maintain the marginality constraint (2.9) upon varying α_0, all the angles $\{\Theta_a\}$ have to be equal.

For non-simply laced algebras, e.g. B_n, the ratio of the lengths of long and short roots is $\sqrt{2}$. This is incompatible with Eq. (2.9), at least for small values of α_0. We conclude that for Coulomb gas models defined in terms of bosonic fields only, the screening vectors $\{\vec{\alpha}_a^+\}$ have to correspond to the basic vectors of the root lattice of simply laced algebras only: A_n, D_n, E_6, E_7 or E_8. It is in fact well-known that the free-field representation for

W-theories based on non-simply laced algebras uses fermionic fields, in addition to the bosonic ones [9].

We emphasise once again that we are analysing the generic case of a Coulomb gas model of D bosonic fields $\{\varphi_a(z,\bar{z})\}$, with a non-degenerate set of D screening vectors $\{\vec{\alpha}_a^+\}$ which couple to *all* the fields $\{\varphi_a(z,\bar{z})\}$. On the contrary, if the set of the screening operators is reduced and some of the fields $\{\varphi_a(z,\bar{z})\}$ are decoupled from screenings, our arguments do not apply. For instance one could imagine having an extra bosonic field which bosonises a couple of fermionic fields. Evidently, there is a wide range of possibilities of constructing non-generic Coulomb gas models.

As has been concluded above, in our Coulomb gas models all $\{\Theta_a\}$ have to be equal, i.e. the vector $\vec{\alpha}_0$ has to be "equidistant" from all the screening vectors, cf. Fig. 1. All the lengths of the screening vectors $\{\vec{\alpha}_a^+\}$ have to be identical, as have those of $\{\vec{\alpha}_a^-\}$. We shall henceforth denote them as α_+ and α_-. Eqs. (2.9)–(2.11) now take the simpler form

$$\alpha_\pm = \alpha_0\cos\Theta \pm \sqrt{\alpha_0^2\cos^2\Theta + 1}, \tag{2.40}$$

$$\alpha_+ + \alpha_- = 2\alpha_0\cos\Theta, \tag{2.41}$$

$$\alpha_+\alpha_- = -1. \tag{2.42}$$

Similarly, Eq. (2.28) for the background charge vector becomes

$$2\vec{\alpha}_0 = \sum_a(\alpha_a^+ + \alpha_a^-)\vec{\omega}_a = (\alpha_+ + \alpha_-)\sum_a \vec{\omega}_a, \tag{2.43}$$

and Eq. (2.44) now reads

$$\vec{\alpha}_a^\pm = \sum_b A_{ab}\alpha_b^\pm\frac{\vec{\omega}_b}{2} = \frac{\alpha_\pm}{2}\sum_b A_{ab}\vec{\omega}_b. \tag{2.44}$$

From this latter equation one finds

$$\vec{\omega}_a = \sum_b A^{-1}_{ab}\frac{2\vec{\alpha}_b^\pm}{\alpha_\pm} = \sum_b 2A^{-1}_{ab}\vec{e}_b, \tag{2.45}$$

where A^{-1}_{ab} is a matrix inverse of A_{ab}. Substituting (2.45) into (2.43) one obtains

$$2\vec{\alpha}_0 = (\alpha_+ + \alpha_-)\sum_a\sum_b 2A^{-1}_{ab}\vec{e}_b \equiv (\alpha_+ + \alpha_-)\sum_b m_b\vec{e}_b, \tag{2.46}$$

with the notation $m_b = \sum_a 2A^{-1}_{ab}$. We finally arrive at the following relation between the background charge and the screenings:

$$2\vec{\alpha}_0 = \sum_a m_a(\vec{\alpha}^+_a + \vec{\alpha}^-_a). \tag{2.47}$$

Multiplying by $\vec{\alpha}_0$ we also deduce that

$$\frac{1}{\cos^2\theta} = \sum_a m_a \ . \tag{2.48}$$

Another remark can be made concerning the degeneracy assumption of the module of $V_{\vec{\beta}}(z,\bar{z})$, which has been used throughout the analysis. Its justification is that an extended chiral algebra must necessarily possess degenerate representations realised by the vertex operators, which are primaries as has been argued in the beginning of this Section.

That the operator algebra of $\{V_{\vec{\beta}}\}$, corresponding to such degenerate representations, should close can be shown by using, for instance, the differential equations for correlation functions. These equations could be derived if the singular states were realised explicitly by the chiral algebra operators. In the analysis we have just used these assumptions.

One particular feature in the construction of "would be singular states" by mappings produced by the integrated screenings is that we have demanded the degeneracy in all D directions, corresponding to the D vectors $\{\vec{\alpha}^+_a\}$. This condition is necessary to ensure that the OPE of two operators $V_{\vec{\beta}_1}(z,\bar{z})$, $V_{\vec{\beta}_2}(z',\bar{z}')$ produces only a finite number of primary operators out of an, in principle, infinite set of operators:

$$\{V_{\vec{\beta}_1+\vec{\beta}_2+\sum_a(k_a\vec{\alpha}^+_a+l_a\vec{\alpha}^-_a)}\}, \tag{2.49}$$

cf. the arguments given above in connection with Eqs. (2.35)–(2.36).

The support for this way of implementing the degeneracy assumption can be found, on one hand, in the fact that a finite OPE (in a sense of a finite number of primaries appearing in the OPE) is also a consequence of the above-mentioned differential equations. On the other hand, a subset of the primaries produced by the OPE (2.35) by adding an appropriate amount of screenings will consist of vertex operators $V_{\vec{\beta}'}(z,\bar{z})$ which are preimages of singular states, in the sense of the mapping (2.14). This subset, which constitutes the unphysical part of the Kac table, have to decouple from the rest

in correlation functions as well as in the operator algebra of physical operators. In this way the infinite set of operators appearing in (2.49) will be restricted to a finite number of physical operators. The fact that we demand degeneracy in all D directions is thus tantamount to bordering the physical domain of the Kac table in all these directions.

It is interesting to apply the above result to the two-flavoured loop model introduced by Jacobsen and Kondev [2]. Its configurations are those of two colours of closed loops ($N_{\rm b}$ black loops and $N_{\rm g}$ grey ones) placed on the edges of a square lattice, in such a way that every vertex of the lattice touches exactly one loop of either colour. Introducing loop fugacities $n_{\rm b}$ and $n_{\rm g}$ for the loop flavours, the partition function of the discrete model reads

$$Z = \sum_{\mathcal{G}} n_{\rm b}^{N_{\rm b}} n_{\rm g}^{N_{\rm g}}, \tag{2.50}$$

where $\mathcal{G}$ are the fully-packed loop configurations just defined. Interestingly, this model is critical for any $0 \leq |n_{\rm b}|, |n_{\rm g}| \leq 2$.

In the Coulomb gases obtained from loop models [2] the conditions defined above are not satisfied in general. In particular, the components of the matrix A_{ab}, cf. Eq. (2.39), are not integer-valued for generic values of $n_{\rm b}$ and $n_{\rm g}$ belonging to the critical manifold.[2] As a result, defining the "would be degenerate representations" and the Kac table would not make sense: the operator algebra of vertex operators $\{V_{\vec{\beta}_{(\ldots)}}\}$ would not close.

In these models one defines physical operators differently, by using physical arguments and constructions that are directly linked to quantities in the discrete model of oriented loops. The operators coupling to the electric part[3] of the Coulomb gas again take the form of vertex operators $\{V_{\vec{\beta}}(z,\bar{z})\}$, with $\vec{\beta}$ belonging to a particular lattice, which e.g. in

[2]It should be remarked that in the Coulomb gas models derived from the models of loops [1, 2], the normalisation of free fields, of their action, and of the two-point function, is usually different from the one that we are using, defined by the two-point function (2.3). This implies a different definition for scalar products of vectors, like the screening vectors $\{\vec{\alpha}_a^{\pm}\}$, background charge $2\vec{\alpha}_0$, and the electric charges $\{\vec{\beta}\}$ of the physical operators $V_{\vec{\beta}}(z,\bar{z})$. This also results in a different formula for the conformal dimensions of the vertex operators. To compare the formulas in a general setting, independent of the details of a particular model, the matrix of elastic constants in the free field action of loop models has to be first diagonalised and then renormalised appropriately, so that the two-point functions of free fields take the form of Eq. (2.3), or, in any case, take a form which is symmetric with respect to the components of the free fields $\{\varphi_a(z,\bar{z})\}$.

[3]Operators with magnetic Coulomb charge are also physically relevant. They correspond to topo-

case of the two-flavoured loop model will be a three-dimensional body-centered cubic lattice. The operator algebra of the operators $\{V_{\vec{\beta}}(z,\bar{z})\}$ is closed, by construction. But one will, in general, have to admit an infinite OPE for a couple of operators $V_{\vec{\beta}_1}V_{\vec{\beta}_2}$. As it stands, a model with such properties will not have an extended chiral algebra, will not have extended symmetries. It is just a Coulomb gas endowed with conformal invariance, nothing more.

On the other hand, one can check that in the special case $n_{\rm b} = n_{\rm g}$ of the above-mentioned model the matrix A_{ab} appearing in Eq. (2.39) actually *does* have integer valued components:

$$A_{ab} = \frac{2(\vec{\alpha}_a, \vec{\alpha}_b)}{(\vec{\alpha}_b)^2} = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix}. \tag{2.51}$$

The relation (2.47) between $2\vec{\alpha}_0$ and the screenings is also nicely satisfied, with $m_a = (1,0,1)$. The physical operators of this model (the electric ones) coincide with $\{V_{\vec{\beta}_{(\dots)}}\}$, and they satisfy the necessary condition for the degeneracy of the modules in the way it has been described above. (Incidentally, in the loop model language, for $n_{\rm b} = n_{\rm g}$ the general model can be rewritten in a way so that all three elasticity constants are equal [3].)

So far, so good. The worrying point is that the matrix A_{ab} in Eq. (2.51) does not correspond to a Cartan matrix of any classical Lie algebra. The closest one would be the algebra A_3, with the Cartan matrix of the form

$$A_{ab} = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}. \tag{2.52}$$

The fact that the off-diagonal elements of A_{ab} are now negative is a general property of the Cartan matrix associated with a classical Lie algebra, it being the matrix of scalar products of simple root vectors. On the other hand, the condition for the closure of the algebra of operators $\{V_{\vec{\beta}_{(\dots)}}\}$ requires, in its general form presented above, the integer-valuedness only.

logical defects, vortices, in the interfacial representation, and they are reminiscent of disorder operators attached to a defect line.

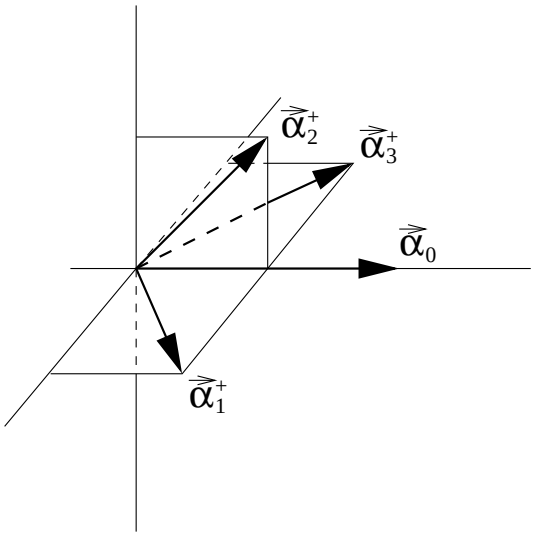

Figure 6: Screening geometry in the two-flavoured loop model.

To make more clear the difference between the two sets of vectors $\{\vec{\alpha}_a^+\}$, corresponding to the matrices (2.51) and (2.52), we present them in Figs. 6 and 7. Evidently, the vectors in Fig. 6, cf. Eq.(2.51), belong to the same lattice as the vectors in Fig. 7, but they are not *simple* roots of A_3. The vector $\vec{\alpha}_2^+$ has been switched from one side to another, leading to a sign change in the off-diagonal elements of the matrix A_{ab}.

It might be possible that a more refined analysis could be given to the selection of screenings in the Coulomb gas models, taking into account the properties of representations: Detailed properties of the operator algebra and of the correlation functions of physical operators. The purpose of the analysis would be to decide on the acceptance of the theories with the non-classical matrices A_{ab}, as in Eq. (2.51), which appear in the loop models.

In the absence of such a more detailed analysis, but also to test one more detail of our approach, we have directly calculated the chiral algebra operators, the W's, for the theories with the matrices A_{ab} in (2.51) and in (2.52).

This last part of our approach is in some sense akin to the inverse scattering problem: with the data of scattering one has to reconstruct the Hamiltonian of the problem. Here we have constructed the Coulomb gases starting from the geometry of the screening operators. But finally, having fixed them, it is then perfectly possible to calculate the extended chiral algebra operators. In particular, in this way it is possible to give a

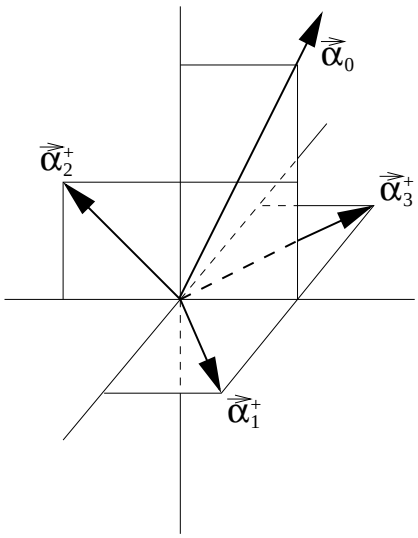

Figure 7: Screening geometry in the WA_3 model.

definitive answer whether an extended chiral algebra exists in the loop model with the matrix of screenings given by Eq. (2.51).

This last part of the analysis is described and applied to a variety of particular Coulomb gas problems in the Section which follows. For the loop model with screening matrix (2.51) the answer will be negative. The model does not have an extended chiral algebra, and its symmetry is only a conformal one.

In the last Section, which will be devoted to discussion and conclusions, we shall discuss the possibilities to define, in the loop models setting, models which might lead to the Coulomb gases related to the classical Lie algebras, with screenings realised by simple roots. This appears to be the only possibility to provide them with extended symmetries.

3 Applications

We have seen in the preceding Section that the general consistency requirements means that the screening vectors must span the root lattice of a classical Lie algebra. In particular, we would like to consider situations where this is the case, without actually taking the screening vectors to be the *simple* roots of the concerned algebra. As we have seen

above, one such situation is of interest to the two-flavoured loop model of Jacobsen and Kondev.

We shall consider various such examples with $D = 2$ and $D = 3$. It should be noticed that the fully-packed loop model on the honeycomb lattice (which has $D = 2$) possesses a unique screening vector, which is proportional to the background charge [2]. We have argued that such a situation is non-generic, and leads to the decoupling of one of the scalar fields from the other *and* from the background charge. Thus, from the point of view of CFT this theory is trivial. Notwithstanding this "triviality", some of the critical exponents related to geometrical properties of the loops do take non-trivial values [2], different from those of the standard $D = 1$ loop model describing the dense phase of the O(n) model [1]. This is possible because these exponents do not belong to the physical part of the Kac table for minimal models. We also point out that despite of the triviality of its CFT, the model actually has an interesting integrable structure, due to its underlying $sl_q(2)$ quantum group symmetry [11].

Discarding such trivial possibilities, for both $D = 2$ and $D = 3$ there are two different choices of D screening vectors of equal length, spanning the root lattice of A_D. After defining the method of computations we turn to the detailed application to those two cases. One of the cases, of course, is the standard WA_D geometry, and it serves as a check of our computations that in this case we reproduce various known results.

3.1 Definitions

As already explained, we consider the case of a D dimensional Coulomb gas with a fixed background charge $\vec{\alpha}_0$ and D screening charges. In addition to T, we must construct $D - 1$ extra chiral operators with integer dimensions. These operators are made as linear combinations of products of derivatives of free fields. We shall choose here the simplest construction, namely to search for chiral operators $W_3, \ldots, W_{D+1}$, where the subscript in W_i indicates its dimension. It is of course possible that the fields W_i should be found on higher levels, but for reason of simplicity we find this unlikely. After building the most general operators on the respective levels, we will impose on them that they commute with the integrated screening operators, as explained in Section 2.2, and that

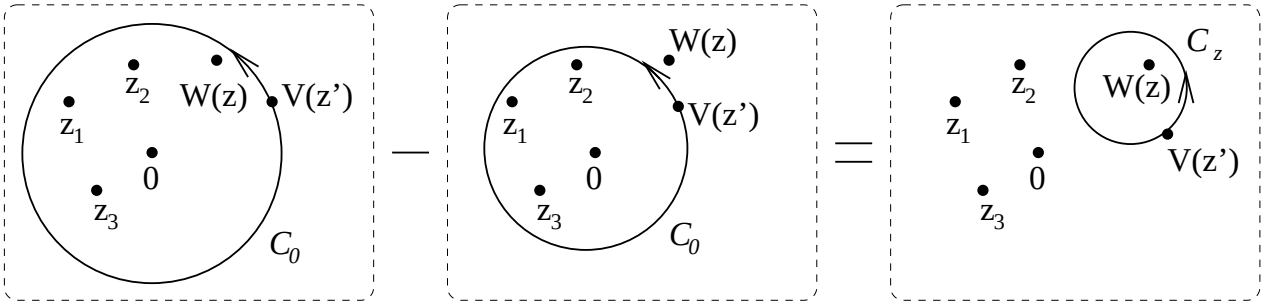

Figure 8: Deformation of the integration contour in (3.1).

they be primary operators. The primarity condition is requested in order to ensure that $\{T(z), W_3(z), \cdots, W_{D+1}(z)\}$ forms a closed algebra.

The requirement that a field W commutes with the currents of Eq. (2.6) can be written

$$[W(z), Q] = [W(z), \oint_{C_0} \mathrm{d}z'\, V(z')] = \oint_{C_z} \mathrm{d}z'\, V(z')W(z) = 0, \tag{3.1}$$

where we have deformed the integration contour as indicated in Fig. 8. This implies that the residue must be zero:

$$\mathrm{Res}(V(z')W(z); z) = 0. \tag{3.2}$$

Second, W must be a primary operator of dimension Δ_W:

$$T(z)W(z') = \frac{\Delta_W}{(z-z')^2}W(z') + \frac{1}{(z-z')}\partial W(z') + \cdots. \tag{3.3}$$

We shall see below that for a specified choice of the screening charges, these two constraints, when properly expressed, suffice to uniquely determine the W operator, up to a global normalisation. To fix the latter, we impose a third constraint:

$$W(z)W(z') = \frac{c/\Delta_w}{(z-z')^{2\Delta_W}} + \cdots. \tag{3.4}$$

Keeping only the holomorphic part in (2.2), (2.3) and (2.4), we have

$$T(z) = -\frac{1}{4}\sum_{i=1}^{D}(\partial\varphi_i)^2 + i\vec{\alpha_0}\cdot\partial^2\vec{\varphi} \tag{3.5}$$

$$\langle\varphi_i(z)\varphi_j(z')\rangle = -2\delta_{ij}\log(z-z') \tag{3.6}$$

with the corresponding central charge:

$$c = D - 24\sum_{i=1}^{D}(\alpha_0^i)^2 \tag{3.7}$$

and the screening operators:

$$V_k(z) = \exp\left(i\sum_i \alpha_k^i \varphi_i(z)\right) = \exp\left(\frac{1}{2}\sum_i g_k^i \varphi_i(z)\right) \tag{3.8}$$

with $g_k^i = 2i\alpha_k^i$. With this last definition, the Wick contraction with a derivative of a field reads simply

$$V_k(z')\partial^i\varphi_j(z) = \frac{a_i g_k^j}{(z'-z)^i}V(z') \tag{3.9}$$

with $a_1 = a_2 = 1, a_3 = 2$ and $a_4 = 6$. Forming suitable linear combinations, this formula will allow us to compute (3.2) for any operator $W(z)$ built out of the fields $\varphi_i(z)$ and their derivatives. However, we still need to develop the screening operator around z:

$$\begin{aligned} V_k(z') &= \exp\left(\sum_i \frac{g_k^i}{2}\varphi_i(z')\right) \\ &= \exp\left(\sum_i \frac{g_k^i}{2}(\varphi_i(z) + (z'-z)\varphi_i'(z) + \frac{(z'-z)^2}{2}\varphi_i''(z) + \frac{(z'-z)^3}{6}\varphi_i'''(z) + \cdots)\right) \\ &= V_k(z)\left(1 + (z'-z)\sum_i \frac{g_k^i}{2}\varphi_i'(z) + (z'-z)^2\left[\sum_i \frac{g_k^i}{4}\varphi_i''(z) + \sum_{i,j}\frac{g_k^i g_k^j}{8}\varphi_i'(z)\varphi_j'(z)\right]\right. \\ &\quad \left. +(z'-z)^3\left[\sum_i \frac{g_k^i}{12}\varphi_i'''(z) + \sum_{i,j}\frac{g_k^i g_k^j}{8}\varphi_i'(z)\varphi_j''(z) + \sum_{i,j,k}\frac{g_k^i g_k^j g_k^l}{48}\varphi_i'(z)\varphi_j'(z)\varphi_l'(z)\right] + \cdots\right), \end{aligned} \tag{3.10}$$

where we have used the notation $\varphi_i'(z') = \partial\varphi_i(z')$, $\varphi_i''(z') = \partial^2\varphi_i(z')$ and $\varphi_i'''(z') = \partial^3\varphi_i(z')$.

We consider an operator W_N of dimension N, built as a linear combination of the derivatives of the D scalar fields. Let us define by $\mathcal{N}_D(N)$ the number of terms in the linear combination which forms the most general operator of dimension N. We then impose the constraint that this operator commutes with the D screening charges. As shown above, this corresponds to canceling the residue in z of the product $V(z')W_N(z)$, which for a general operator W_N amounts to canceling an operator of dimension $N-1$. We then expect $D \times \mathcal{N}_D(N-1)$ constraints. Next we have to impose the primarity. In the product $T(z)W_N(z')$, we have to cancel all the powers in $(z-z')^{-i}$ for $i = N+2, N+$

$1, \ldots, 3$. Since the power $(z - z')^{-i}$ comes with an operator of dimension $N + 2 - i$, we will get an additional number of constraints equal to $\mathcal{N}_D(0) + \mathcal{N}_D(1) + \cdots + \mathcal{N}_D(N-1)$. Thus for a W_n operator with $\mathcal{N}_D(N)$ parameters, we have $(D+1)\mathcal{N}_D(N-1) + \mathcal{N}_D(N-2) + \cdots + \mathcal{N}_D(0)$ constraints. Of course, this is just the maximal number of constraints that we can expect, since in general not all of the corresponding linear equations will be independent.

3.2 Two scalar fields

The most general operator of dimension three built from two scalar fields is:

$$W_3 = \sum_i a_i \partial^3 \varphi_i + \sum_i b_i \partial\varphi_i \partial^2\varphi_i + \sum_i c_i (\partial\varphi_i)^3 + \sum_{i\neq j} d_{ij} (\partial^2\varphi_i)\partial\varphi_j + \sum_{i\neq j} e_{ij} (\partial\varphi_i)^2 \partial\varphi_j. \tag{3.11}$$

We shall choose the coordinate system so that the background electric charge is directed along the 1-direction. The stress-energy tensor then reads

$$T(z) = -\frac{1}{4}(\partial\varphi_1)^2 - \frac{1}{4}(\partial\varphi_2)^2 + i\alpha_0 \partial^2\varphi_1. \tag{3.12}$$

For this case, we have $\mathcal{N}_2(0) = 1$, $\mathcal{N}_2(1) = 2$, and $\mathcal{N}_2(2) = 5$. Thus we will have 8 constraints from the primarity of W_3 and 10 constraints from the commutation of the W_3 operator with the two screening operators.

By applying first the constraints from the primarity, we obtain seven equations on the 10 constants entering in (3.11) :

$$\begin{aligned} &a_1 = -4\alpha_0^2 c_1 \qquad b_1 = -6i\alpha_0 c_1 \qquad b_2 = 6i\alpha_0(1 - 8\alpha_0^2)c_1 \\ &e_{21} = -3(1 - 8\alpha_0^2)c_1 \quad e_{12} = \tfrac{i}{4\alpha_0}(d_{12} + d_{21}) \quad a_2 = -\tfrac{2}{3} i\alpha_0 d_{12} \\ &12 i\alpha_0 c_2 = (1 - 24\alpha_0^2)d_{12} + (1 + 8\alpha_0^2)d_{21}. \end{aligned} \tag{3.13}$$

3.2.1 Classical WA_2 geometry

Turning now to the commutation of W_3 with the screening operators, we begin by considering the geometry of the classical WA_2 theory:

$$\vec{\alpha}_1^+ = \alpha_+ \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right) \qquad \vec{\alpha}_2^+ = \alpha_+ \left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right), \tag{3.14}$$

where α^+ is related to α_0 by the relation (2.7), i.e. $\alpha_0 = \frac{1}{2}(\alpha^+ - \frac{1}{\alpha^+})$.

The two screenings are related by a reflectional symmetry, and it is thus no surprise that considering just $\vec{\alpha}_1^+$ leads to a single-parameter solution

$$\begin{array}{lllll} a_1 = 0 & a_2 = 24\alpha_0^2\tilde{c} & b_1 = 0 & b_2 = 0 & c_1 = 0 \\ c_2 = 4\tilde{c} & qd_{12} = 12i\alpha_0\tilde{c} & d_{21} = 36i\alpha_0\tilde{c} & e_{12} = -12\tilde{c} & e_{21} = 0\,. \end{array} \tag{3.15}$$

If we also impose the normalisation condition (3.4), then we end up with the following expression of the parameters in terms of the central charge ($c = 2 - 24\alpha_0^2$):

$$a_2 = \frac{i(2-c)}{12\sqrt{5c+22}} \;\;;\;\; c_2 = \frac{i}{3\sqrt{5c+22}} \;\;;\;\; d_{12} = \pm\sqrt{\frac{(2-c)}{24(5c+22)}} \tag{3.16}$$

$$d_{21} = 3d_{12} = \pm\sqrt{\frac{3(2-c)}{8(5c+22)}} \;\;;\;\; e_{12} = -\frac{i}{\sqrt{5c+22}}\,. \tag{3.17}$$

Thus, our final result reads

$$T = -\frac{1}{4}(\partial\varphi_1)^2 - \frac{1}{4}(\partial\varphi_2)^2 + i\sqrt{\frac{2-c}{24}}\partial^2\varphi_1 \tag{3.18}$$

$$W_3 = \frac{1}{\sqrt{5c+22}}\left(\frac{i(2-c)}{12}\partial^3\varphi_2 + \frac{i}{3}(\partial\varphi_2)^3 \pm \sqrt{\frac{2-c}{24}}(\partial^2\varphi_1)\partial\varphi_2 \right.$$
$$\left. \pm\sqrt{\frac{3(2-c)}{8}}(\partial^2\varphi_2)\partial\varphi_1 - i(\partial\varphi_1)^2\partial\varphi_2\right). \tag{3.19}$$

Up to the normalization constant, this coincides with the solution of Fateev and Zamolodchikov [8].

3.2.2 Alternative geometry

Consider next the alternative geometry of the screening charges

$$\vec{\alpha}_1^+ = \alpha_+\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right) \qquad \vec{\alpha}_2^+ = \alpha_+\left(\frac{\sqrt{3}}{2}, -\frac{1}{2}\right). \tag{3.20}$$

These vectors still span a triangular lattice, i.e. the root lattice of the Lie algebra A_2. They also correspond to a "Cartan matrix" where the signs of the off-diagonal elements have been changed:

$$\frac{2(\vec{\alpha}_i, \vec{\alpha}_j)}{(\vec{\alpha}_j)^2} \equiv A_{ij} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}. \tag{3.21}$$

In this case we find only the trivial (null) solution for W.

3.3 Three scalar fields

In this section we have found it convenient to impose the following parametrisation for the screening operators

$$g_1^i = 2ia(1,0,-1)\ ;\ g_2^i = 2ia(-1,1,0)\ ;\ g_3^i = 2ia(1,0,1) \tag{3.22}$$

and for the stress-energy tensor

$$\alpha_0^1 = X/2\ ;\ \alpha_0^2 = X\ ;\ \alpha_0^3 = 0. \tag{3.23}$$

Here the variable $X = (2a - 1/a)$ is related to the central charge by $c = 3 - 30X^2$.

We now have to consider the construction of two operators of dimensions 3 and 4. The most general such operators built from three scalar fields read

$$\begin{aligned} W_3 \;=\; & \sum_i a_i \partial^3\varphi_i + \sum_i b_i \partial\varphi_i \partial^2\varphi_i + \sum_i c_i (\partial\varphi_i)^3 + \sum_{i\neq j} d_{ij} (\partial^2\varphi_i)\partial\varphi_j \\ & + \sum_{i\neq j} e_{ij} (\partial\varphi_i)^2 \partial\varphi_j + f\,\partial\varphi_1\partial\varphi_2\partial\varphi_3 \end{aligned} \tag{3.24}$$

and

$$\begin{aligned} W_4 \;=\; & \sum_i a_i \partial^4\varphi_i + \sum_i b_i \partial\varphi_i \partial^3\varphi_i + \sum_{i\neq j} c_{ij} \partial^3\varphi_i \partial + \varphi_j \sum_i d_i (\partial^2\varphi_i)^2 \\ + \;& \sum_{i<j} e_{ij} \partial^2\varphi_i \partial^2\varphi_j + \sum_i f_i (\partial^2\varphi_i)(\partial\varphi_i)^2 + \sum_{i\neq j} g_{ij} (\partial^2\varphi_i)(\partial\varphi_i)(\partial\varphi_j) \\ + \;& \sum_{i\neq j} h_{ij} (\partial^2\varphi_i)(\partial\varphi_j)^2 + \sum_{i\neq j,k;j<k} i_{ijk} (\partial^2\varphi_i)(\partial\varphi_j)(\partial\varphi_k) + \sum_i j_i (\partial\varphi_i)^4 \\ + \;& \sum_{i\neq j} k_{ij} (\partial\varphi_i)^3 (\partial\varphi_j) + \sum_{i<j} l_{ij} (\partial\varphi_i)^2 (\partial\varphi_j)^2 + \sum_{i\neq j,k;j<k} m_{ijk} (\partial\varphi_i)^2 (\partial\varphi_j)(\partial\varphi_k). \end{aligned} \tag{3.25}$$

Note that there is some overlap among the symbols used to designate the constants entering in the definitions of W_3 and W_4. These constants are of course independent in the two cases, but in order to avoid complicating the notation we do not distinguish them by an extra index. This convention should lead to no confusion since we shall consider the operators W_3 and W_4 separately in the following.

3.3.1 Classical WA_3 geometry

In this case we have $\mathcal{N}_3(0) = 1$, $\mathcal{N}_3(1) = 3$, $\mathcal{N}_3(2) = 9$, $\mathcal{N}_3(3) = 22$ and $\mathcal{N}_3(4) = 51$. Thus, Eq. (3.24) for W_3 contains 22 free parameters. Imposing the commutation of W_3 with the screening charges produces 3×9 constraints on these parameters.

Solving the corresponding equations gives the following relations, leaving just two free parameters, b_1 and d_{13} :

$$a_1 = -iXb_1 \;\; ; \;\; a_2 \;\; = \;\; -2iXb_1 \;\; ; \;\; a_3 = -id_{13}X \tag{3.26}$$

$$b_3 = b_2 = b_1 \;\; ; \;\; d_{32} \;\; = \;\; d_{31} = d_{13} \;\; ; \;\; f = i\frac{d13}{X}. \tag{3.27}$$

Next we impose the primarity of W_3, yielding $\mathcal{N}_3(0)+\mathcal{N}_3(1)+\mathcal{N}_3(2) = 13$ additional constraints. It turns out that the only additional condition which result from these constraints is $b_1 = 0$. Thus we end with the following result :

$$T(z) \;\; = \;\; -\frac{1}{4}\sum_{i=1}^{3}(\partial\varphi_1)^2 + i\sqrt{\frac{3-c}{120}}(\partial^2\varphi_1 + 2\partial^2\varphi_2) \tag{3.28}$$

$$W_3(z) \;\; \propto \;\; (\frac{3-c}{30}\partial^3\varphi_3 + i\sqrt{\frac{3-c}{30}}\left((\partial^2\varphi_1)\partial\varphi_3 + (\partial^2\varphi_3)\partial\varphi_1 + (\partial^2\varphi_3)\partial\varphi_2\right) - \partial\varphi_1\partial\varphi_2\partial\varphi_3).$$

To normalise this properly, we demand that $W(z)W(z') = \frac{c/3}{(z-z')^6} + \cdots$. It follows that the above solution should be divided by $-\frac{4}{5}(7+c)$. The singularity at $c = -7$ will reappear in W_4 (see below).

The computation for the W_4 operator goes along the same line as for the W_3 case, but with much more parameters and constraints. In this case, we have $\mathcal{N}_3(4) = 51$ parameters in the definition of W_4 (see (3.25)). There are $3\times\mathcal{N}_3(3) = 66$ constraints produced by the commutation of the screening operators with W_4, and $\mathcal{N}_3(0)+\mathcal{N}_3(1)+\mathcal{N}_3(2)+\mathcal{N}_3(3) = 35$ additional constraints coming from the requirement that W_4 be a primary operator. We thus have 101 constraints on the 51 parameters. Solving these constraints is rather straightforward since they are just linear equations. Our result (up to a multiplicative factor) can be expressed as a function of the central charge c (all the terms not present being zero):

$$a_1 = i\frac{\sqrt{3-c}(-26+c+2c^2)}{\sqrt{30}} \qquad ; \qquad a_2 = -i\frac{\sqrt{3-c}(c-2)(c+7)}{\sqrt{30}} \tag{3.29}$$

$$b_1 = b_3 = -2(c-6)(c+2) \quad ; \quad b_2 = 3(c-2)(c+7) \tag{3.30}$$

$$c_{12} = -(c-3)(5c+22) \quad ; \quad d_1 = \frac{1}{5}(-18-63c-2c^2) \tag{3.31}$$

$$d_2 = \frac{1}{10}(c+7)(19c-102) \quad ; \quad d_3 = -\frac{9}{2}(c-2)(c+7) \tag{3.32}$$

$$e_{12} = -\frac{2}{5}(c-3)(9c-2) \quad ; \quad f_1 = -i8\sqrt{\frac{6}{5}}\sqrt{3-c}(c+7) \tag{3.33}$$

$$f_2 = 2f_1 \;\; ; \;\; g_{12} = g_{31} \;\; = \;\; g_{32} = i\sqrt{30}\sqrt{3-c}(5c+22) \tag{3.34}$$

$$h_{12} = i\sqrt{\frac{6}{5}}\sqrt{3-c}(17c+54) \quad ; \quad h_{13} = -i8\sqrt{\frac{6}{5}}\sqrt{3-c}(c+7) = f1 \tag{3.35}$$

$$h_{21} = h_{23} = i\sqrt{\frac{6}{5}}\sqrt{3-c}(9c-2) \quad ; \quad j_1 = j_2 = j_3 = 12(c+7) \tag{3.36}$$

$$l_{12} = l_{13} = l_{23} \;\; = \;\; -3(17c+54) \tag{3.37}$$

with the following solution:

$$\begin{aligned}
W_4 \;\; = \;\; & i\frac{\sqrt{3-c}(-26+c+2c^2)}{\sqrt{30}}\partial^4\varphi_1 + \frac{1}{5}(-18-63c-2c^2)(\partial^2\varphi_1)^2 \\
+ \;\; & (c+7)\{-i\frac{\sqrt{3-c}}{\sqrt{30}}[(c-2)\partial^4\varphi_2 + 48(\partial^2\varphi_1(\partial\varphi_1)^2 + 2\partial^2\varphi_2(\partial\varphi_2)^2 + \partial^2\varphi_1(\partial\varphi_3)^2)] \\
& \quad +3(c-2)[\partial\varphi_2\partial^3\varphi_2 - \frac{3}{2}(\partial^2\varphi_3)^2] + \frac{1}{10}(19c-102)(\partial^2\varphi_2)^2\} \\
& \quad +12((\partial\varphi_1)^4 + (\partial\varphi_2)^4 + (\partial\varphi_3)^4)) \\
+ \;\; & (17c+54)\{i\sqrt{\frac{6}{5}}\sqrt{3-c}\partial^2\varphi_1(\partial\varphi_2)^2 - 3((\partial\varphi_1)^2(\partial\varphi_2)^2 + (\partial\varphi_1)^2(\partial\varphi_3)^2 + (\partial\varphi_2)^2(\partial\varphi_3)^2)\} \\
+ \;\; & (9c-2)i\sqrt{3-c}\{-i\frac{2}{5}\sqrt{3-c}\partial^2\varphi_1\partial^2\varphi_2 + \sqrt{\frac{6}{5}}(\partial^2\varphi_2(\partial\varphi_1)^2 + \partial^2\varphi_2(\partial\varphi_3)^2)\} \\
+ \;\; & i(5c+22)\sqrt{3-c}\{-i\sqrt{3-c}\partial^3\varphi_1\partial\varphi_2 + \sqrt{30}(\partial^2\varphi_1\partial\varphi_1\partial\varphi_2 \\
& \quad +\partial^2\varphi_3\partial\varphi_3\partial\varphi_1 + \partial^2\varphi_3\partial\varphi_3\partial\varphi_2)\} \; .
\end{aligned} \tag{3.38}$$

Finally, we impose the standard normalisation $W(z)W(z') = \frac{c/4}{(z-z')^8} + \cdots$, which means that all the above should be divided by the factor

$$192(2+c)(7+c)(22+5c)(114+7c). \tag{3.39}$$

In particular, the W_4 operator becomes singular at a set of special values of the central charge:

$$c = -7 \qquad c = -2 \qquad c = -\frac{22}{5} \qquad c = -\frac{114}{7}. \tag{3.40}$$

These singularities are exactly those found by Blumenhagen et al. [12]. Kausch and Watts [13] find in addition the singularities $c = 1/2$, $c = -68/27$ and $c = -24$ of which we see no sign.

3.3.2 Alternative geometry

Finally, we consider the geometry of the two-flavoured loop model (cf. Eq. (2.51)), for which we use the following parametrisation for the screening operators

$$g_1^i = 2ia(1, 0, -1) \; ; \; g_2^i = 2ia(1, 1, 0) \; ; \; g_3^i = 2ia(1, 0, 1) \tag{3.41}$$

and for the stress-energy tensor

$$\alpha_0^1 = X/2 \; ; \; \alpha_0^2 = 0 \; ; \; \alpha_0^3 = 0 \; . \tag{3.42}$$

In this case, very tedious computations show that for W_3 as well as for W_4, all the parameters are zero under the application of the constraints.

4 Discussion

In this paper we have discussed the general construction of CFTs based on Coulomb gases with several bosonic fields. The requirement that the CFT be consistent has dictated us two physical guiding principles: The existence of degenerate representations, and the closure condition on the vertex operator algebra. From these principles we have obtained the classification condition (2.39) stating that the screening vectors must belong to the root lattice of a classical Lie algebra. The screenings are however not required to be *simple* roots, and therefore the matrix A_{ac} in Eq. (2.39) is more general than the Cartan matrix in the theory of W-algebras.

One of the main motivations of our work has been the attempt to identify the CFT underlying two different model of fully-packed loops (FPL) [2]: The single-flavoured FPL model on the honeycomb lattice and the two-flavoured FPL model on the square lattice. Both these loop models are known to be critical for any value of the loop fugacities in the interval $[-2, 2]$. In both cases, the underlying Coulomb gas furnishes exact values of the central charge, the thermal scaling dimension, and the scaling dimensions of an

infinite set of topological defects linked to the propagation of a set of strings between two points ("watermelon dimensions"). In the honeycomb case this information is further supported by a Bethe ansatz solution, the integrability of the model being assured by its $sl_q(2)$ quantum group symmetry [11], whereas in the square case no Bethe ansatz solution is known, except at the point $(n_{\rm b}, n_{\rm g}) = (2, 2)$ [14].

However, without access to the associated CFT, our knowledge of these models cannot be considered complete. To illustrate this point it is useful to compare with the finite-temperature O(n) model on the honeycomb lattice [1]. Its low-temperature critical phase is described by the dense phase of a (non-fully packed) loop model, the loops being defined in terms of the diagrammatic expansion of the associated spin model. This loop model is solvable, both as a Coulomb gas and by Bethe ansatz techniques, and it furnishes information on critical indices analogous to the FPL cases mentioned above. On the other hand, when $n = -2\cos(\pi g)$ with $m = g/(1-g)$ a positive integer ($m \geq 3$) the model is known to coincide with the series of minimal models of conventional CFT. For these unitary cases a wealth of further information is available, and many interesting applications become possible. To mention but one important example, the knowledge of exact operator product expansions makes it possible to study perturbatively the coupling of quenched randomness to the local energy density of such models. For the FPL cases one could imagine addressing the problem of a compact polymer in a random environment by similar techniques, but to do so knowing the corresponding CFT becomes indispensable. Another illustration of the complementarity of the two approaches (Coulomb gas versus CFT) is that the watermelon dimensions, which have a straightforward interpretation in terms of the loop model, coincide with CFT operators that are *outside* the physical part of the Kac table [15]. Conversely, local physical operators in the CFT description do not in general seem to have an analogue in the loop approach.

In the present work we have argued that the honeycomb FPL model is in fact trivial from the CFT point of view, since one of the scalar fields decouples from the other, and from the background charge. On the other hand, we have showed that the non-trivial two-flavoured FPL model satisfies the classification condition (2.39). However, we have also found that the extended chiral operators needed to control the $D-1$ remaining degrees of freedom (with $D = 3$) do not exist, at least not on level 3 and 4 of the identity

module. We can of course not rule out the eventuality that such operators exist at higher levels, but we find it rather unlikely.

To acquire a consistent CFT description of fully-packed loops, it thus seems to us that the most natural thing to do would be to somehow modify the definition of the loop models in question, so that the corresponding CFTs become the classical W-theories, WA_2 and WA_3 respectively. We recall that in the Coulomb gas formalism, the operator assigning the proper weights to the loops is a periodic function on the ideal state graph. The lattice to which screenings must belong is therefore fixed by standard Fourier analysis [2]. One generally assumes that all of the Fourier modes come with non-zero amplitudes, and the actual screenings are therefore singled out as being those closest to the background charge vector, since this ensures the lowest conformal dimension. For loop models with further adjustable parameters one may however imagine that the amplitude corresponding to the closest vectors can be made to vanish, in which case the screenings will have to be chosen from the vectors second-closest to the background charge, and so on. This is actually what happens in the O(n) model, where the temperature T acts as an adjustable parameter. By making the obvious ("closest") choice for the screening vector, the one-dimensional Coulomb gas describes the dense (critical) phase, as mentioned above. But by fine-tuning the temperature it is possible to access another dilute (tricritical) phase, corresponding to the choice of the next-closest screening [15, 16]. This situation is rather analogous to standard Ginzburg-Landau theory, where fine-tuning may serve to make the φ^4 term vanish, thus giving access to multi-critical behaviour governed by a more general φ^{2n} term. In a certain sense, successive fine-tuning of more and more parameters is tantamount to augmenting the symmetry of the corresponding critical theory. We would expect that the CFTs of the two FPL models under consideration may be turned into the classical WA_2 and WA_3 theories by fine-tuning suitable extra parameters, thus driving the systems to multi-criticality.

For the moment we do not have any concrete proposal for the construction of such tunable parameters. The introduction of temperature-like vacancies in FPL models is known [17] to introduce a flow towards the dense phase of the standard O(n) model [1], so clearly the temperature is not a suitable parameter for endowing the model with a *higher* symmetry. A more promising possibility would be to progress in analogy with the

ferromagnetic Ising model, which can be driven to tricriticality by introducing a staggered magnetic field and fine-tuning its strength (in addition to the critical temperature). For the fully-packed loop models, it is possible to impose several staggered fields that act so as to distinguish between the various ideal states [2].

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