Discrete Thermodynamic Bethe Ansatz

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Abstract

We propose discrete TBA equations for models with discrete spectrum. We illustrate our construction on the Calogero-Moser model and determine the discrete 2-body TBA function which yields the exact N-body Calogero-Moser thermodynamics. We apply this algorithm to the Lieb-Liniger model in a harmonic well, a model which is relevant for the microscopic description of harmonically trapped Bose-Einstein condensates in one dimension. We find that the discrete TBA reproduces correctly the N-body groundstate energy of the Lieb-Liniger model in a harmonic well at first order in perturbation theory, but corrections do appear at second order.

I. INTRODUCTION

It has been known for a long time that the spectrum of the Calogero model [1], defined here as particles on an infinite line interacting via $\alpha(\alpha-1)/(x_i-x_j)^2$ 2-body interactions, can be found [2] by the Bethe Ansatz (BA) which assumes periodic boundary conditions, and that its thermodynamics can be obtained, in the thermodynamic limit, by the Thermodynamic Bethe Ansatz (TBA) [3]. It is however not necessary to rely on the BA (or the TBA) to get the Calogero spectrum (or its thermodynamics). Indeed, the Calogero model is exactly solvable

- either by confining the particles in a harmonic well of frequency ω: the Calogero-Moser model [4] with discretized harmonic well quantum numbers and energies, and Hamiltonian

$$H_N = -\frac{1}{2}\sum_{i=1}^{N}\frac{d^2}{dx_i^2} + \alpha(\alpha-1)\sum_{i<j}\frac{1}{(x_i-x_j)^2} + \frac{1}{2}\omega^2 x_i^2 \tag{1}$$

- or by confining the particles in a periodic box of length L: the Calogero-Sutherland model [5] with discretized momenta and energies, and Hamiltonian

$$H_N = -\frac{1}{2}\sum_{i=1}^{N}\frac{d^2}{dx_i^2} + \alpha(\alpha-1)(\frac{\pi}{L})^2\sum_{i<j}\frac{1}{\sin^2[\frac{\pi}{L}(x_i-x_j)]} \tag{2}$$

(the $1/\sin^2[\pi(x_i-x_j)/L]$ interactions are nothing but the periodic version of the infinite line interactions). It is not a surprise that Bethe ansatz equations yield the Calogero-Sutherland spectrum, since they also assume, as stressed above, periodic boundary conditions.

Both parameters ω and L can be considered as long distance regulators, the thermodynamic limit, i.e. the infinite line limit, being obtained either by $\omega \to 0$ or $L \to \infty$, resulting in continuous momenta and energies.

The Calogero model describes particules with intermediate statistics, which is natural due to the topological (statistical) nature of the $1/(x_i-x_j)^2$ interaction in 1d. In the thermodynamic limit indeed [6], the Calogero thermodynamics realizes microscopically Haldane (Hilbert space counting) statistics [7]. Moreover, the Calogero model has been shown to be obtained as the vanishing magnetic field limit [8] of the lowest Landau level anyon model [9]

(LLL-anyon model) in the regime where the flux tubes carried by the anyons screen the flux of the external magnetic field (screening regime). Not surprisingly, the LLL-anyon model also realizes microscopically Haldane statistics, the Hilbert space counting argument being manifest here via a mean field argument (adding anyons screen the external magnetic field, and thus diminish the Landau degeneracy of the total -mean+external- magnetic field): thus a clear relation between Haldane [10] and anyon statistics [11].

Starting from the BA spectrum, and following Yang and Yang footsteps [3], one can compute à la TBA the thermodynamics of the Calogero model in the thermodynamic limit $L \to \infty$. The thermodynamical potential $\ln Z$ -where $Z = \sum_{N=0}^{\infty} z^N Z_N$ is the grand partition function- ends up to be those of a Fermi gas[1]

$$\log Z = \frac{L}{2\pi}\int_{-\infty}^{\infty} dk \log[1 + ze^{-\beta\epsilon(k)}], \tag{3}$$

but with a 1-body energy $\epsilon(k)$ defined in terms of the free continuous 1-body quadratic spectrum $\epsilon_o(k) = k^2/2$ as

$$\beta\epsilon(k_1) = \beta\epsilon_o(k_1) - \frac{L}{2\pi}\int_{-\infty}^{\infty} dk_2 \Phi(k_1 - k_2) \log[1 + ze^{-\beta\epsilon(k_2)}] \tag{4}$$

In the Calogero case [12],

$$\Phi(k_1 - k_2) = \frac{2\pi}{L}(1-\alpha)\delta(k_1 - k_2) \tag{5}$$

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[1]There is an equivalent formulation in terms of a free Bose gas, namely

$$\log Z = \frac{L}{2\pi}\int_{-\infty}^{\infty} dk \log \frac{1}{1 - ze^{-\beta\tilde{\epsilon}(k)}},$$

but with a 1-body energy $\tilde{\epsilon}(k)$ defined as

$$\beta\tilde{\epsilon}(k_1) = \beta\epsilon_o(k_1) - \frac{L}{2\pi}\int_{-\infty}^{\infty} dk_2 \tilde{\Phi}(k_1 - k_2) \log \frac{1}{1 - ze^{-\beta\tilde{\epsilon}(k_2)}}$$

One has obviously

$$\tilde{\Phi}(k_1 - k_2) - \Phi(k_1 - k_2) = -\frac{2\pi}{L}\delta(k_1 - k_2)$$

is intimately related to the 2-body scattering angle, and encodes, if one thinks in terms of statistics, the statistical exclusion between two quantum states, here with the same momentum.

Note that if one denotes $y(k) = 1 + ze^{-\beta\epsilon(k)}$, which can be regarded, in view of (3), as the grand partition function at momentum k, then (4) can be rewritten as

$$y(k) - ze^{-\beta\epsilon_o(k)}y(k)^{1-\alpha} = 1 \tag{6}$$

a particular case of Ramanujan equations [13].

As already said, equations of the type (3,6) were first obtained directly by i) considering the exact N-body Calogero spectrum in a harmonic well [6], or by considering the exact N-body LLL-anyon spectrum in a harmonic well [9], ii) and then taking the thermodynamic limit.

Now we might ask the following question: are the TBA equations (3,4) specific to the thermodynamic limit with continuous momenta and continuous dressed energies, or can they also describe the thermodynamic of the Calogero model in a harmonic well or in a periodic box with discretized energies? In other words, can we find a discretized version of the function $\Phi(k_1 - k_2)$ in (5) such that the harmonic well or periodic box Calogero thermodynamics narrow down to a set of defining equations analogous to (3,4)?

We will show that the thermodynamic of the Calogero model in a harmonic well can indeed be rewritten "à la TBA" in terms of a discretized function Φ which will encode the statistical Calogero exclusion between different discrete harmonic energy levels and which will, as it should, reproduce, in the thermodynamic limit $\omega \to 0$, $\Phi(k_1 - k_2)$ in (5). Not surprisingly, the same conclusion will be reached for the LLL-anyon model in an harmonic well, whose thermodynamics will obey the same TBA equations as the Calogero-Moser thermodynamics.

We will also argue that the same logic applies in the Calogero-Sutherland case, provided that a global shift of the bare quantum numbers is made in order to maintain a symmetric repartition of the dressed quantum numbers around zero.

Finally, we will look at possible applications of discrete TBA thermodynamics beyond the Calogero-Moser and harmonic LLL-anyon cases, by considering the Lieb-Liniger model in a harmonic well [14], [15]. This model is interesting because of its relevance to the description of one dimensional trapped Bose condensates [16]. We will show that the N-body groundstate energy is correctly reproduced at first order in perturbation theory by the discrete TBA equations, but corrections do appear at second order.

II. THE CALOGERO CASE

For a system with a discrete 1-body harmonic spectrum, the TBA equations (3,4) should rewrite quite generally as

$$\log Z = \sum_{n=0}^{\infty} \log[1 + z e^{-\beta\epsilon(n)}], \tag{7}$$

where the 1-body dressed energy $\epsilon(n)$ should now be defined in terms of the 1-body 1d harmonic spectrum (bare spectrum) $\epsilon_o(n) = (n + 1/2)\omega, \quad 0 \le n$ as

$$\beta\epsilon(n_1) = \beta\epsilon_o(n_1) - \sum_{n_2=0}^{\infty} \Phi_{n_1,n_2} \log[1 + z e^{-\beta\epsilon(n_2)}] \tag{8}$$

(7,8) are just the discretized versions of (3,4). In (8), Φ_{n_1,n_2} has to be understood as acting on the free harmonic spectrum, i.e as acting on the power series in z obtained from (8) by expanding $y(n) = 1 + z e^{-\beta\epsilon(n)}$ as

$$y(n) = \sum_{N=0}^{\infty} y_N(n) z^N \tag{9}$$

The lowest order terms of (9) are

$$\begin{aligned}
y_0(n_1) &= 1, \\
y_1(n_1) &= e^{-\beta\epsilon_o(n_1)}, \\
y_2(n_1) &= \sum_{n_2=0}^{\infty} e^{-\beta\epsilon_o(n_1)} \Phi_{n_1,n_2} e^{-\beta\epsilon_o(n_2)}, \\
y_3(n_1) &= \sum_{n_2,n_3=0}^{\infty} e^{-\beta\epsilon_o(n_1)} \left(\Phi_{n_1,n_2} e^{-\beta\epsilon_o(n_2)} \Phi_{n_2,n_3} + \frac{1}{2} \Phi_{n_1,n_2} e^{-\beta\epsilon_o(n_2)} \Phi_{n_1,n_3} \right) e^{-\beta\epsilon_o(n_3)} \\
&\quad - \frac{1}{2} \sum_{n_2=0}^{\infty} e^{-\beta\epsilon_o(n_1)} \Phi_{n_1,n_2} e^{-2\beta\epsilon_o(n_2)},
\end{aligned} \tag{10}$$

where $0 \leq n_1, n_2, \cdots$ and the summation should be taken for all possible independant integers $n_2, n_3, \cdots$.

If the TBA cluster coefficients obtained from expanding $\log Z = \sum b_n z^n$

$$b_1 = \sum_{n=0}^{\infty} y_1(n),$$
$$b_2 = \sum_{n=0}^{\infty} \left[y_2(n) - \frac{1}{2} y_1^2(n) \right],$$
$$b_3 = \sum_{n=0}^{\infty} \left[y_3(n) - y_1(n) y_2(n) + \frac{1}{3} y_1^3(n) \right], \tag{11}$$

have to match against the Calogero-Moser cluster coefficients $b_1 = \frac{e^{-\frac{\beta\omega}{2}}}{1-e^{-\beta\omega}}$, $b_2 = e^{-\beta\omega}\left[\frac{e^{-\beta\omega\alpha}-e^{-\beta\omega}}{(1-e^{-2\beta\omega})(1-e^{-\beta\omega})} - \frac{1}{2}\frac{1}{1-e^{-2\beta\omega}}\right]$, $\ldots$, Φ_{n_1,n_2} should be defined as

$$\Phi_{n_1,n_2} = P_{n_1,n_2}(\alpha) - P_{n_1,n_2}(\alpha = 1) \tag{12}$$

where $P_{n_1,n_2}(\alpha)$ projects the two independant quantum numbers $0 \leq n_1, n_2$ on dressed quantum numbers which, not surprisingly, obey exclusion statistics. More precisely, evaluating in (7,8,10) expressions of the type

$$\sum_{n_2=0}^{\infty} P_{n_1,n_2}(\alpha) e^{-\beta\epsilon(n_2)}, \tag{13}$$

$P_{n_1,n_2}(\alpha)$ amounts, n_1 being given, to the shift

$$n_2 \to n_1 + \tilde{n}_2 + \alpha, \quad \tilde{n}_2 \geq 0 \tag{14}$$

and the summation over n_2 is replaced by the summation over $\tilde{n}_2 \geq 0$. In other words, in terms of the independant quantum numbers $0 \leq n_1, n_2$, denoting $\mathbf{n_1} = n_1, \mathbf{n_2} = n_1 + \tilde{n}_2$, $P_{n_1,n_2}(\alpha)$ means $n_1 \to \mathbf{n}_1$, and $n_2 \to \mathbf{n}_2 + \alpha$, where $0 \leq \mathbf{n}_1 \leq \mathbf{n}_2$ are now bosonic quantum numbers. Therefore $P_{n_1,n_2}(0)$ projects $0 \leq n_1, n_2$ onto bosonic quantum numbers $0 \leq \mathbf{n}_1 \leq \mathbf{n}_2$, whereas $P_{n_1,n_2}(1)$ projects $0 \leq n_1, n_2$ onto fermionic quantum numbers. Note that in (12) substracting $P_{n_1,n_2}(\alpha = 1)$ is simply a matter of convention, i.e. as stressed above, a fermionic thermodynamical potential (3) with a spectrum which has to coincide with the bare spectrum when $\alpha = 1$ -the Bose convention would yield $\tilde{\Phi}_{n_1,n_2} = P_{n_1,n_2}(\alpha) - P_{n_1,n_2}(\alpha = 0)$.

More generally notice that (12) allows to rewrite (8) as

$$y(n_1) - ze^{-\beta\epsilon_o(n_1)}\frac{\prod_{\tilde{n}_2=0}^{\infty} y(n_1+\tilde{n}_2+\alpha)}{\prod_{\tilde{n}_2=0}^{\infty} y(n_1+\tilde{n}_2+1)} = 1 \tag{15}$$

which can be viewed as the discretized version of (6).

Going one step further one gets

$$\prod_{\tilde{n}_2=0}^{\infty} y(n_1+\tilde{n}_2) = \prod_{\tilde{n}_2=0}^{\infty} y(n_1+\tilde{n}_2+1) + ze^{-\beta\epsilon_o(n_1)}\prod_{\tilde{n}_2=0}^{\infty} y(n_1+\tilde{n}_2+\alpha) \tag{16}$$

which in turn, taken at $n_1 = 0$, rewrites as

$$Z = ze^{-\beta\epsilon_o(0)}\prod_{\tilde{n}_2=0}^{\infty} y(\tilde{n}_2+\alpha) + \prod_{\tilde{n}_2=0}^{\infty} y(\tilde{n}_2+1) \tag{17}$$

Equation (17) can be interpreted by saying that either the first particle is in the groundstate at energy $\frac{1}{2}\omega$ and then the next particle is in the energy level at least higher than $(\frac{1}{2}+\alpha)\omega$, or the groundstate is vacant and the first particle is in the energy level at least higher than $(\frac{1}{2}+1)\omega$. One verifies recursively that Z in (17) is identical to the grand partition of the Calogero-Moser model with N-body spectrum

$$E_N = \omega\sum_{i=1}^{N}\left([\mathbf{n}_i+\alpha(i-1)]+\frac{1}{2}\right) \tag{18}$$

where $0 \leq \mathbf{n}_1 \leq \mathbf{n}_2 \leq \ldots$. In terms of the bare independent quantum numbers $0 \leq n_1, n_2, \ldots$, one has $n_i \to \mathbf{n}_i + \alpha(i-1) = n_{i-1} + \tilde{n}_i + \alpha(i-1)$ with $\tilde{n}_i \geq 0$. This is indeed a BA "like" spectrum, i.e. in terms of the dressed quantum numbers $\mathbf{n}'_i = \mathbf{n}_i + \alpha(i-1)$, one has $\mathbf{n}'_i = \mathbf{n_i} + \alpha\sum_{j\neq i}\theta(\mathbf{n}'_i - \mathbf{n}'_j)$. In the 2-body case, it indeed amounts to $n_1 \to n_1, n_2 \to n_1 + \tilde{n}_2 + \alpha$, i.e. to the the action of the projector $P_{n_1,n_2}(\alpha)$ on the independent quantum numbers $0 \leq n_1, n_2$.

Note that (12) implies that the N-body partition function Z_N obtained from (7) as

$$Z_N = \prod_{n=0}^{\infty} y_{N_n}(n), \qquad 0 \leq N_n, \qquad \sum_n N_n = N \tag{19}$$

has, using (10) to all orders, the simple factorized form

$$Z_N = \sum_{n_1,\ldots,n_N=0}^{\infty} P_{n_1,n_2}(\alpha)P_{n_2,n_3}(\alpha)\ldots P_{n_{N-1},n_N}(\alpha)e^{-\beta(\epsilon_o(n_1)+\epsilon_o(n_2)+\ldots+\epsilon_o(n_N))} \tag{20}$$

In particular in the 2-body case

$$Z_2 - Z_2|_{Fermi} = \sum_{n_1,n_2=0}^{\infty} \left(P_{n_1,n_2}(\alpha) - P_{n_1,n_2}(\alpha=1)\right) e^{-\beta(\epsilon_o(n_1)+\epsilon_o(n_2))} \tag{21}$$

and thus in the thermodynamic limit[2] $\omega \to 0$

$$Z_2 - Z_2|_{Fermi} = (\frac{L}{2\pi})^2 \int_{-\infty}^{\infty} dk_1 dk_2 \Phi(k_1-k_2) e^{-\beta(\frac{k_1^2}{2}+\frac{k_2^2}{2})} \tag{26}$$

where Φ is given in (5).

As far as the LLL-anyon model in a harmonic well [9] is concerned, one finds that (7,8,12) are unchanged, to the exception of the 1-body energy which now reads $\epsilon_o(n) = (\omega_t - \omega_c)n + \omega_c$, where $\omega_c = eB/2$, $\omega_t = \sqrt{\omega_c^2+\omega^2}$, and the statistical anyonic parameter has to be understood as being $-\alpha$, i.e. the screening regime where the flux $\phi = -\alpha\phi_o$ (ϕ_o is the quantum of flux) carried by each anyon is antiparallel to the external magnetic field.

One can easily convince oneself that in the thermodynamic limit $\omega \to 0$, both the Calogero and LLL-anyon TBA thermodynamics narrow down to

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[2]By factorizing the center of mass, (21) rewrites as

$$Z_2 - Z_2|_{Fermi} = \frac{1}{2\sinh\frac{\beta\omega}{2}} \sum_{\tilde{n}_2=2l\geq 0} \left(e^{-\beta\omega(\tilde{n}_2+\frac{1}{2}+\alpha)} - e^{-\beta\omega(\tilde{n}_2+\frac{1}{2}+1)}\right) \tag{22}$$

and (26) as

$$Z_2 - Z_2|_{Fermi} = (\frac{L}{2\pi})^2 \int_{-\infty}^{\infty} dK e^{-\beta\frac{K^2}{4}} \int_{-\infty}^{\infty} dk \Phi(2k) e^{-\beta k^2} \tag{23}$$

Since, in the thermodynamic limit [17] for the N-th cluster coefficient, $\frac{1}{\beta\omega} \to \frac{L}{\sqrt{2\pi\beta}}\sqrt{N}$, one infers that in the 2-body case

$$\frac{1}{2\sinh\frac{\beta\omega}{2}} \to \frac{L}{2\pi}\int_{-\infty}^{\infty} dK e^{-\beta\frac{K^2}{4}} \tag{24}$$

and therefore one should have

$$\sum_{\tilde{n}_2=2l\geq 0} \left(e^{-\beta\omega(\tilde{n}_2+\frac{1}{2}+\alpha)} - e^{-\beta\omega(\tilde{n}_2+\frac{1}{2}+1)}\right) \to \frac{L}{2\pi}\int_{-\infty}^{\infty} dk \Phi(2k) e^{-\beta k^2} \tag{25}$$

a result that can be trivially checked by direct computation.

$$\log Z = \int_0^\infty d\epsilon_o \rho_o(\epsilon_o) \ln(1 + ze^{-\beta\epsilon(\epsilon_o)}) \tag{27}$$

where the dressed energy $\epsilon(\epsilon_o)$ is implicitely defined à la TBA in terms of the bare energy ϵ_o as

$$\beta\epsilon = \beta\epsilon_o - \int_0^\infty d\epsilon' \Phi(\epsilon, \epsilon') \ln(1 + ze^{-\beta\epsilon'}) \tag{28}$$

and

$$\Phi(\epsilon, \epsilon') = (1 - \alpha)\delta(\epsilon - \epsilon') \tag{29}$$

Here, $\rho_o(\epsilon_o)$ is the 1-body density of states of the bare spectrum of the system considered, i.e. in the Calogero case the 1d free density of states $\rho_o(\epsilon_o) = L/(\pi\sqrt{2\epsilon_o})$, and, in the LLL-anyon case, the 2d LLL density of states, $\rho_o(\epsilon_o) = (BV/\phi_o)\delta(\epsilon_o - \omega_c)$, where V is now the infinite surface of the 2d plane.

In the case of the Calogero model in a periodic box -the Calogero-Sutherland model-, one can still propose (7) and (8), but now the 1-body dressed energy $\epsilon(n)$ should be defined in terms of the 1-body spectrum in a 1d periodic box of length L, $\epsilon_o(n) = k^2/2$, with discretized momentum $k = 2\pi n/L$ and n positive, null or negative integer.

However, and contrary to the harmonic case, the very fact that the bare quantum numbers in a periodic box can be of both signs lead to some adjustments. If one looks at the N-body Calogero-Sutherland spectrum[3]

$$E_N = \frac{1}{2}(\frac{2\pi}{L})^2 \sum_{i=1}^{N} \left([\mathbf{n}_i + \alpha(i-1)] - \alpha\frac{(N-1)}{2} \right)^2 \tag{30}$$

where $\mathbf{n}_1 \leq \mathbf{n}_2 \leq \ldots$, one finds that in terms of the bare quantum numbers $n_i \to \mathbf{n}_i + \alpha(i-1) - \alpha(N-1)/2 = n_{i-1} + \tilde{n}_i + \alpha(i-1) - \alpha(N-1)/2$ with $\tilde{n}_i \geq 0$. This is quite similar to the Calogero-moser spectrum, up to a global shift, $n_i \to n_i - \alpha(N-1)/2$, a N-dependant periodic boundary condition adjustment insuring that the dressed spectrum

[3]with a BA spectrum $\mathbf{n}'_i = \mathbf{n_i} + \frac{\alpha}{2}\sum_{j\neq i} sign(\mathbf{n'}_i - \mathbf{n'}_j)$

remain symmetric around 0 in order to minimize the N-body energy. In the 2-body case $n_1 \to n_1 - \alpha/2, n_2 \to n_1 + \tilde{n}_2 + \alpha/2$, it amounts to the the action of the Calogero-Moser projector $P_{n_1,n_2}(\alpha)$ as given in (12) on the a priori two independent quantum numbers n_1, n_2, again up to the 2-body shift $n_{1,2} \to n_{1,2} - \alpha/2$. This being considered, altogether with the fact that the Calogero-Moser and Calogero-Sutherland models originate from the same model, up to a long distance regularisation, it is natural to take for both models the same TBA function (12) to obtain, in view of (20), the correct Calogero-Sutherland N-body partition function, but in addition the shift $n_i \to n_i - \alpha(N-1)/2$ has to be made a posteriori.

At this point, one can remark that in all cases studied so far, the Calogero-Moser model, as well as the Calogero -Sutherland model up to periodic boundary conditions adjustments, and their thermodynamic limit, the Calogero model, the TBA functions Φ_{n_1,n_2} and $\Phi(k_1 - k_2)$ are intimately related to the relative 2-boson density of states for the problem at hand. Indeed, in a harmonic well, the spectrum for a relative particle with bosonic statistics and interacting with a Calogero potential at the origin is

$$\epsilon = \omega(n + \frac{1}{2} + \alpha) \quad 0 \leq n \tag{31}$$

with n even, i.e. with symmetric eigenstates under $x \to -x$.

Let us first consider, in the thermodynamic limit, the Calogero model: when $\omega \to 0$, the relative 2-boson density of states reads

$$\rho_\alpha(\epsilon) - \rho_{\alpha=1}(\epsilon) = \frac{1-\alpha}{2}\delta(\epsilon) \tag{32}$$

It rewrites in terms of the relative momentum k such that $\epsilon = k^2$

$$\rho_\alpha(k) - \rho_{\alpha=1}(k) = \frac{1-\alpha}{2}\delta(k) \tag{33}$$

Now one has to map the relative 2-body momentum k on the "momentum" $k_2 - k_1$ the function $\Phi(k_2 - k_1)$ is concerned with. Since $k_2 - k_1 = 2k$, one gets for the density of states in terms of $k_2 - k_1$

$$\rho_\alpha(\frac{k_2 - k_1}{2}) - \rho_{\alpha=1}(\frac{k_2 - k_1}{2}) = (1-\alpha)\delta(k_2 - k_1) \tag{34}$$

i.e. precisely (5) up to a factor $2\pi/L$.

When ω is kept finite, the same logic applies: the relative spectrum (31) yields

$$n \to n + \alpha \tag{35}$$

One has yet to map the relative 2-body bosonic even quantum number n on the "quantum number" $n_2 - n_1$ that the function Φ_{n_1,n_2} is concerned with. For a given 2-body energy, i.e. for $\mathbf{n}_1 + \mathbf{n}_2$ given, one has $\mathbf{n}_2 - \mathbf{n}_1 = n$ -then the center of mass quantum number is $2\mathbf{n}_1$, or $\mathbf{n}_2 - \mathbf{n}_1 = n + 1$ -then the center of mass quantum number is $2\mathbf{n}_1 + 1$, depending if $\mathbf{n}_2 - \mathbf{n}_1$ is even or odd. One finds that $n \to n + \alpha$ rewrites, in terms of the independent n_1, n_2 as $n_1 \to n_1$, $n_2 \to n_1 + \tilde{n}_2 + \alpha$, where now $\tilde{n}_2 = n$ or $\tilde{n}_2 = n + 1$, i.e. any positive integer. Then (35) is indeed identical to the action of Φ_{n_1,n_2} in (12).

This is not a surprise, scattering 2-body phase shifts are known to be linked to the 2-body density of states via S-matrix arguments [18].

III. THE LIEB-LINIGER CASE

In the Lieb-Liniger model in the thermodynamic limit, the same conclusion happens to be true. The model, defined as

$$H_N = -\frac{1}{2}\sum_{i=1}^{N}\frac{d^2}{dx_i^2} + c\sum_{i<j}\delta(x_i - x_j) \tag{36}$$

is solvable by Bethe ansatz [14] and has a TBA thermodynamics [3] obtained from

$$\Phi(k_1 - k_2) = \frac{1}{L}\frac{2c}{(k_2 - k_1)^2 + c^2} \tag{37}$$

It interpolates between the Bose ($c = 0$) and Fermi ($c = \infty$) thermodynamics **and describes particles with intermediate statistics [15].** For a relative particle with bosonic statistics interacting with a δ potential at the origin the density of states is

$$\rho_c(\epsilon) - \rho_{c=\infty}(\epsilon) = \frac{1}{4\pi\sqrt{\epsilon}}\frac{c}{\epsilon + \frac{c^2}{4}} \tag{38}$$

which in terms of $\epsilon = k^2$, $k > 0$ rewrites as

$$\rho_c(k) - \rho_{c=\infty}(k) = \frac{1}{2\pi} \frac{c}{k^2 + \frac{c^2}{4}} \tag{39}$$

Now, one has again to map k on the "momentum" $k_2 - k_1$ the function $\Phi(k_1, k_2)$ is concerned with, i.e. $k_2 - k_1 = 2k$, and since $k_2 - k_1$ can be either positive or negative, one gets for the density of states in terms of $k_2 - k_1$

$$\rho_c(\frac{k_2 - k_1}{2}) - \rho_{c=\infty}(\frac{k_2 - k_1}{2}) = \frac{1}{2\pi} \frac{2c}{(k_2 - k_1)^2 + c^2} \tag{40}$$

i.e. nothing but (37), again up to a factor $2\pi/L$.

If one follows the same line of reasoning which was operative in the Calogero-Moser case to obtain the discrete TBA function Φ_{n_1,n_2} (12) from the relative 2-body spectrum (31), one might try, for the Lieb-Liniger model in a harmonic well, discrete TBA thermodynamics (7,8) equations with a TBA function Φ_{n_1,n_2} deduced from the 2-body relative bosonic spectrum in a harmonic well [15]. It rewrites as

$$\epsilon = \omega \left(n + \frac{1}{2} + \frac{2}{\pi} \arctan \left(\frac{c}{2\sqrt{2\omega}} \frac{\Gamma(\frac{\epsilon}{2\omega} + \frac{1}{4})}{\Gamma(\frac{\epsilon}{2\omega} + \frac{3}{4})} \right) \right) \quad 0 \leq n \tag{41}$$

with n even, i.e. as $\epsilon = \omega(n + \frac{1}{2} + f_c(n))$ with[4]

$$0 \leq f_c(n) = \frac{2}{\pi} \arctan \left(\frac{c}{2\sqrt{2\omega}} \frac{\Gamma(\frac{n+1}{2} + \frac{f_c(n)}{2})}{\Gamma(\frac{n+2}{2} + \frac{f_c(n)}{2})} \right) \leq 1 \tag{43}$$

and interpolates between the relative 2-body bosonic ($c = 0$, $f_0(n) = 0$, $g_0(n) = 1$) and fermionic ($c = \infty$, $f_\infty(n) = 1$, $g_\infty(n) = 0$) spectra.

Therefore let us try for the Lieb and Liniger in a harmonic well the discrete TBA function

$$\Phi_{n_1,n_2} = P_{n_1,n_2}(c) - P_{n_1,n_2}(c = \infty) \tag{44}$$

should be defined in terms of $P_{n_1,n_2}(c)$ such that, n_1 being left unchanged,

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[4]Equivalently, starting from the Fermi spectrum by rewriting $f_c(n) = 1 - g_c(n)$

$$0 \leq g_c(n) = \frac{2}{\pi} \arctan \left(\frac{2\sqrt{2\omega}}{c} \frac{\Gamma(\frac{n+3}{2} - \frac{g_c(n)}{2})}{\Gamma(\frac{n+2}{2} - \frac{g_c(n)}{2})} \right) \leq 1 \tag{42}$$

$$n_2 \to n_1 + \tilde{n}_2 + f_c(\tilde{n}_2) \tag{45}$$

if $\tilde{n}_2 \geq 0$ is even, and

$$n_2 \to n_1 + \tilde{n}_2 + f_c(\tilde{n}_2 - 1) \tag{46}$$

if $\tilde{n}_2$ is odd. Note again that substracting $P_{n_1,n_2}(c = \infty)$ in (44) originates, as in the Calogero case, from the fermionic convention (obviously $P_{n_1,n_2}(c = \infty) = P_{n_1,n_2}(\alpha = 1)$.)

It is easy to check that the 2-body partition function is reproduced by the discrete TBA equations

$$Z_2 - Z_2|_{Fermi} = \sum_{n_1=n_2=0}^{\infty} (P_{n_1,n_2}(c) - P_{n_1,n_2}(c = \infty))e^{-\beta(\epsilon_o(n_1)+\epsilon_o(n_2))} \tag{47}$$

and thus, in the thermodynamic limit $\omega \to 0$,

$$\sum_{\tilde{n}_2=2l\geq 0} \left(e^{-\beta(\tilde{n}_2+\frac{1}{2}+f_c(\tilde{n}_2))} - (c = \infty)\right) \to \frac{L}{2\pi} \int_{-\infty}^{\infty} dk\Phi(2k)e^{-\beta k^2} \tag{48}$$

where the function Φ is given in (37), a result that can be checked by direct computation, order by order in $1/c$. There are two independant dimensionless parameters, $\beta\omega$ (thermodynamic limit) and $\sqrt{\beta}c$ ("coupling constant"). Clearly, for a given coupling constant $\sqrt{\beta}c$, looking at $f_c(n) = 1 - g_c(n)$ in (43,42), one has to consider, in the thermodynamic limit $\beta\omega \to 0$, the spectrum close to the Fermi point ($c = \infty$),

$$g_c(2l) = 4\sqrt{2}\frac{\sqrt{\beta\omega}}{\pi\sqrt{\beta}c}\frac{\Gamma(l+\frac{3}{2})}{l!} + \dots \tag{49}$$

from which (48) can be recovered, here at first order $1/(\sqrt{\beta}c)$.

Note also that in the thermodynamic limit for the relative spectrum, with $(n+1/2)\omega \to k^2$, i.e. $n\omega$ fixed, (43) becomes

$$f_c(k) = \frac{2}{\pi} \arctan \frac{c}{2k} \tag{50}$$

which is indeed reminiscent of the Lieb and Liniger BA spectrum [14].

What about the N-body problem? A possible way to check the discrete TBA is to see if the perturbative TBA thermodynamics coincide with the exact (standard) Hamiltonian perturbative thermodynamics [19], which can be computed with the Lieb and Liniger Hamiltonian from the Bose point $c = 0$ (from the Fermi point $c = \infty$ standard perturbation theory is meaningless). Perturbation theory yields

$$\log Z = \log Z|_{Bose} + \sqrt{\beta}c \sum_{s,t=1}^{\infty} \frac{z^{s+t}}{4\sqrt{\pi}} \frac{\sqrt{\beta\omega}}{\sqrt{\sinh\frac{s\beta\omega}{2}\sinh\frac{t\beta\omega}{2}\sinh\frac{(s+t)\beta\omega}{2}}} + \dots \tag{51}$$

Let us now consider the $\sqrt{\beta}c$ expansion from the TBA point of view, again from the Bose point. One has to consider (43) at first order in $\sqrt{\beta}c$

$$f_c(2l) = \frac{1}{\pi\sqrt{2}} \frac{\sqrt{\beta}c}{\sqrt{\beta\omega}} \frac{\Gamma(l+\frac{1}{2})}{l!} + \dots \tag{52}$$

and compute from the discrete TBA equations

$$\log Z = \log Z|_{Bose} + \sum_{n_1=0}^{\infty} \frac{ze^{-\beta\omega(n_1+\frac{1}{2})}}{1-ze^{-\beta\omega(n_1+\frac{1}{2})}} \sum_{n_2=0}^{\infty} \Phi_{n_1,n_2} \ln\frac{1}{1-ze^{-\beta\omega(n_2+\frac{1}{2})}}|_1 + \dots \tag{53}$$

where $\Phi_{n_1,n_2} \ln\frac{1}{1-ze^{-\beta\omega(n_2+\frac{1}{2})}}|_1$ means evaluating this expression at first order in $\sqrt{\beta}c$ using (52). One finds

$$\log Z = \log Z|_{Bose} + \sqrt{\beta}c \sum_{s,t=1}^{\infty} \frac{z^{s+t}}{4\sqrt{2\pi}} \frac{\sqrt{\beta\omega}}{\sinh\frac{(s+t)\beta\omega}{2}} \left(\sqrt{\coth\frac{s\beta\omega}{2}} + \sqrt{\coth\frac{t\beta\omega}{2}}\right) + \dots \tag{54}$$

which coincides with (51) only in the limit $\beta\omega \to \infty$, i.e. for a given ω, in the zero temperature limit[5], i.e the groundstate.

In fact, discrete TBA gives in the vanishing temperature limit direct information on the N-body groundstate energy : in the Calogero-Moser case, it is obtained, in the bosonic based formulation, by restricting the discrete TBA equations

$$\log Z = \sum_{n_1=0}^{\infty} \log\frac{1}{1-ze^{-\beta\bar{\epsilon}(n_1)}}, \tag{55}$$

[5]The limit $\omega \to \infty$ for a given temperature is not considered here. In this limit all particles are confined at $x_i = 0$. But δ interactions actually forbid this unless the effective coupling constant vanishes, which is precisely happening in the 2-body case (43). In other words the $\omega \to \infty$ limit is the trivial bosonic limit.

$$\beta\tilde{\epsilon}(n_1) = \beta\epsilon_o(n_1) - \sum_{n_2=0}^{\infty} \tilde{\Phi}_{n_1,n_2} \log \frac{1}{1 - ze^{-\beta\tilde{\epsilon}(n_2)}} \tag{56}$$

with $\tilde{\Phi}_{n_1,n_2} = P_{n_1,n_2}(\alpha) - P_{n_1,n_2}(\alpha = 0)$ to the groundstate quantum numbers $n_1 = 0$ and $\tilde{n}_2 = 0$. One obtains

$$E_N^{G.S.} = \omega \left(\frac{N}{2} + \frac{N(N-1)}{2}\alpha \right) \tag{57}$$

In the Lieb and Liniger case a similar approach gives

$$E_N^{G.S.} = \omega \left(\frac{N}{2} + \frac{N(N-1)}{2} f_c(0) \right) \tag{58}$$

a result which is consistent with the groundstate energy at first order in c

$$E_N^{G.S.} = \omega \left(\frac{N}{2} + \frac{N(N-1)}{2} f_c^{(1)}(0)c + \dots \right) \tag{59}$$

where $f_c^{(1)}(0) = 1/\sqrt{2\pi\omega}$ stands for the first order term in the expansion of $f_c(0)$ in power of c. However, second order standard perturbation theory gives corrections to the Calogero-Moser like energy $\omega N(N-1) f_c(0)/2$. For example in the $N = 3$ case

$$E_3^{G.S.} = \omega \left(\frac{3}{2} + 3 f_c^{(1)}(0)c + 3 \left(f_c^{(2)}(0) - \frac{1}{\pi\omega} \log \frac{4}{2+\sqrt{3}} \right) c^2 + \dots \right) \tag{60}$$

where $f_c^{(2)}(0) = -\frac{\log 2}{2\pi\omega}$ stands for the second order term in the expansion of $f_c(0)$.

IV. CONCLUSION

We have shown how the Caloger-Moser thermodynamics can be rewritten in terms of discrete TBA equations. In the Calogero-Sutherland model, the same TBA equations were shown to be operative, up to a global shift of the bosonic quantum numbers. Since the Lieb-Liniger model shares common features with the Calogero model -BA solvability, TBA thermodynamics in the thermodynamic limit, intermediate statistics- it might also have, when considered in a harmonic well, a discrete TBA thermodynamics. We tried to illustrate this point of view by proposing discrete TBA equations for the harmonic Lieb and Liniger model in analogy with the Calogero-Moser TBA thermodynamics. However the groundstate energy shows deviations from this TBA framework at second order in perturbation theory.

We leave to a further study to find analytical or numerical ways to improve and give a stronger basis to the discrete TBA thermodynamics for the Lieb-Liniger model, and in particular extract a useful information on the groundstate for a given density of particles. It would however certainly be interesting to understand more in detail the zero temperature limit of a system which is supposed to describe the physics of 1d Bose Einstein condensates in harmonic traps.

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