On the relation between the anyon and Calogero models

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In order to achieve a dimensional reduction from dimension two to one not only in phase space but also in configuration space, the lowest Landau level (LLL) projection is not sufficient. One has also, in the LLL, to take the vanishing magnetic field limit, a procedure which can be given a non ambiguous meaning by means of a long distance regulator. As an illustration, the equivalence of the LLL anyon model in the vanishing magnetic field limit to the Calogero model is established. A thermodynamical argument is proposed which supports this claim. Some general considerations in favor of an intimate connexion between anyon and Haldane statistics are also given.

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A few years ago there has been some discussions aiming at relating the two dimensional anyon model [1] to the one dimensional Calogero model [2]. These efforts [3] were partly motivated by the fact that both models describe identical particles with statistics continuously interpolating between Bose-Einstein and Fermi-Dirac statistics. In the 2d anyon case one speaks of braiding or anyon statistics, in the 1d Calogero case one speaks of Haldane or exclusion statistics [4]. The relation between these statistics is itself an open question since they appear on a quite different footing. Anyon statistics is microscopically defined in terms of a quantum Hamiltonian whose spectrum, eventhough not explicitely known, interpolate continuously between the Bose and Fermi spectra. The interchange properties of the N-body eigenstates generalize the Bose and Fermi $\pm$ signature. On the other hand, Haldane statistics is defined through a Hilbert space counting argument which generalizes the Bose and Fermi counting for N identical particles in G quantum states of a given energy: the number of quantum states available for an additional particle decreases linearly with the number of particles already present.

It happens that the thermodynamics [5] of the 1d Calogero model yields a mean occupation number which coincides with the one obtained [6] from Haldane counting for particles with a free 1d density of states. It is however not clear why particles on a line interacting via $1/x^2$ interactions should have an exclusion like statistics.

On the other hand, if the spectrum of the N-anyon model is unknown, a simplification [3,7,8] arises when projecting the anyon model onto the lowest Landau level of an external magnetic field. A complete eigenstate basis, interpolating between the LLL bosonic and the LLL fermionic basis, can be found in the screening regime [7] where the flux ϕ carried by the anyons is antiparallel to the external magnetic field, i.e. when the statistics parameter $\alpha = \phi/\phi_o$, which varies from $\alpha = 0$ (Bose) to $\alpha = \pm 1$ (Fermi), is such that $\alpha \in [-1, 0]$ if $eB > 0$, or equivalently $\alpha \in [0, 1]$ if $eB < 0$. In this situation, the LLL anyon thermodynamics [7] turns out to be similar to those of the Calogero model: the mean occupation number at energy $\omega_c = |eB/2m|$, i.e. the filling factor in the LLL, coincides with the one obtained from Haldane counting for particles with a infinitely degenerate LLL spectrum. Note however that, contrary to the Calogero case, the relation to Haldane statistics seems natural here, since adding an extra anyon implies an additional screening of the external magnetic field, and thus a Landau degeneracy for the total field (external plus anyon mean field) decreasing linearly with the number of particles.

In this Letter, and to come back to the original question of the relation of the anyon and Calogero models, one would like to argue that the statement generally advanced in the literature [3], namely the equivalence of the Calogero model and the LLL anyon model, cannot be satisfactory. The LLL anyon model is clearly two dimensional: in the thermodynamic limit, its thermodynamical potential diverges as the surface of the plane [7], with a 1-body infinitely degenerate spectrum at energy ω_c. It cannot by any means be identical to the 1d Calogero model with a continuous 1-body spectrum. Also, the LLL anyon model is defined in a screening regime which emphasizes the importance of the sign of the statistics parameter α with respect to the orientation of the magnetic field, whereas no track of this feature can be found in the Calogero model.

More generally, it is commonly understood that projecting a 2d system in the LLL makes it essentially 1d, due to the dimensional reduction of the 1-body phase space from four to two dimensions. Numerous applications have used this line of reasonning, usually refered to as the Peierles substitution [9]. One here argue that in order to achieve an actual dimensional reduction, i.e. not only in phase space but also in configuration space, the LLL projection is not sufficient per se. One has also, once the system has been projected onto the LLL, to take the vanishing magnetic field limit.

It might be objected that taking this limit in the LLL is counter intuitive: the LLL projection is physically justified when the temperature is sufficiently small compared to the cyclotron gap so that the excited states above the

LLL can be ignored. Thus a strong B field limit is naturally associated to the LLL projection, and clearly such an interpretation becomes meaningless when the magnetic field vanishes.

Leaving asides its physical interpretation, the vanishing B field limit might not always be formally defined per se, as we will see below. Still, the procedure proposed here can be given a non ambiguous meaning if some precautions are taken: one has to regularize the system at long distance, for instance by means of a harmonic well of frequency ω [10], and, only after i) projecting in the LLL ii) taking the limit $B \to 0$, can one take the thermodynamic limit $\omega \to 0$. Under these conditions, a dimensional reduction of the configuration space from dimension two to one is properly achieved.

To illustrate this line of reasonning -but it should be operative for other systems as well-, I will indeed show that, in the vanishing magnetic field limit, the LLL anyon model is equivalent to the Calogero model. I will conclude by advocating in favor of an intimate connexion between anyon and Haldane statistics.

Let me first start by a short reminder on the anyon model which can be defined in the singular gauge by a free N-body Pauli Hamiltonian ($\hbar = m = 1$) $H^u_{\rm free} = -2\sum_{i=1}^N \partial_i\bar\partial_i \quad , \quad H^d_{\rm free} = -2\sum_{i=1}^N \bar\partial_i\partial_i$, where the index u, d refers here to the spin degree of freedom. The coupling to an external magnetic field amounts, in the symmetric gauge, to $\partial \to \partial - eB\bar z/4$ and $\bar\partial \to \bar\partial + eBz/4$. The N-body eigenstates $\psi_{\rm free}$ of $H_{\rm free}$ have a non trivial monodromy encoded in the multivalued phase $\exp(-i\alpha\sum_{k<l}\theta_{kl})$ where $\sum_{k<l}\theta_{kl}$ is the sum of the relative angles between pairs of particles in the plane. Looking at the monodromy as a (singular) gauge transformation, one obtains, in the regular gauge, a N-anyon Aharonov-Bohm Hamiltonian acting on monovalued wavefunctions (bosonic by convention) with statistics parameter $\alpha = 0$ for Bose statistics, and $\alpha = \pm 1$ for Fermi statistics, with additional $\mp\pi\alpha\sum_{i<j}\delta^2(z_i - z_j)$ interactions and $\mp\sum_i eB/2$ shifts induced by the spin up or spin down coupling to the local magnetic field of the vortices and to the homogeneous magnetic field. The short range (contact) interactions $\mp\pi\alpha\sum_{i<j}\delta^2(z_i - z_j)$ should implement the exclusion of the diagonal of the configuration space, and thus have to be repulsive. So, depending of the sign of α, the spin up Hamiltonian ($\alpha \in [-1, 0]$) or spin down Hamiltonian ($\alpha \in [0, 1]$), is used.

Let us concentrate without loss of generality on $\alpha \in [-1, 0]$, i.e. on $H^u_{\rm free}$, and, in order to compute its thermodynamical properties, let us add a harmonic well as a long distance regulator. Thus from now on one considers

$$H_{\rm free} = -2\sum_{i=1}^N(\partial_i - \frac{eB}{4}\bar z_i)(\bar\partial_i + \frac{eB}{4}z_i) + \sum_{i=1}^N \frac{\omega^2}{2}\bar z_i z_i \quad (1)$$

which describes two different Bose-Fermi interpolations, depending on the orientation of the magnetic field.

To materialize in the eigenstates the short range repulsion and the long distance Landau and harmonic exponential damping one sets $\psi_{\rm free} = \prod_{k<l}(z_k - z_l)^{-\alpha}\exp(-\frac{1}{2}\omega_t\sum_{i=1}^N z_i\bar z_i)\psi$ to obtain the Hamiltonian acting on ψ

$$H = -2\sum_{i=1}^N\left[\partial_i\bar\partial_i - \frac{\omega_t \pm \omega_c}{2}\bar z_i\bar\partial_i - \frac{\omega_t \mp \omega_c}{2}z_i\partial_i\right] + 2\alpha\sum_{i<j}\left[\frac{1}{z_i - z_j}(\bar\partial_i - \bar\partial_j) - \frac{\omega_t \mp \omega_c}{2}\right] + \sum_{i=1}^N(\omega_t \mp \omega_c) \quad (2)$$

where the $\pm$ refers to the orientation of the magnetic field (if $eB > 0$, $eB/2 = \omega_c$, but if $eB < 0$, $eB/2 = -\omega_c$) and, in the presence of the harmonic well, $\omega_{\rm t} = \sqrt{\omega_{\rm c}^2 + \omega^2}$.

If one puts bluntly $\omega = 0$ in (2), one easily realizes that, if $eB > 0$, i.e. the screening regime, the Hamiltonian (2) acts trivially on N-body eigenstates ψ made of products of the 1-body LLL holomorphic eigenstates

$$(\frac{\omega_c^{\ell_i+1}}{\pi\ell_i!})^{\frac{1}{2}}z_i^{\ell_i}, \quad \ell_i \geq 0 \quad (3)$$

of zero energy (remember the LLL spectrum has been shifted downward by the spin induced $-\omega_c$, thus the LLL has zero energy). This is the LLL anyon model with an infinitely degenerate N-body spectrum $E_N = 0$. Note on the other hand that if $eB < 0$, (2) would not have a simple form when acting on products of 1-body LLL anti-holomorphic eigenstates.

To proceed further in the case of interest, i.e. the screening regime, one has to recognize [7] that the virtue of the harmonic confinement $\omega \neq 0$ is precisely to lift the degeneracy with respect to the ℓ_i's and dress the N-body spectrum with an explicit α dependence. In a harmonic well, the 1-body LLL eigenstates (3) become the 1-body harmonic LLL eigenstates

$$(\frac{\omega_t^{\ell_i+1}}{\pi\ell_i!})^{\frac{1}{2}}z_i^{\ell_i}, \quad \ell_i \geq 0 \quad (4)$$

with now a non degenerate spectrum

$$(\omega_t - \omega_c)(\ell_i + 1), \quad \ell_i \geq 0 \quad (5)$$

Up to a ω_t dependant normalization, the LLL anyonic eigenstates in a harmonic well are symmetrized products of the 1-body harmonic LLL eigenstates (4) ($0 \leq \ell_1 \leq \ldots \leq \ell_N$)

$$\psi_{\rm free} = \prod_{i<j}(z_i - z_j)^{-\alpha}\prod_{i=1}^N z_i^{\ell_i}\exp(-\frac{1}{2}\omega_t\sum_{i=1}^N z_i\bar z_i) \quad (6)$$

Acting on this basis, the Hamiltonian (2) rewrites

$$H_{LLL} = (\omega_t - \omega_c)\left[\sum_{i=1}^{N} z_i\partial_i - \alpha N(N-1)/2 + N\right] \quad (7)$$

with a harmonic LLL N-body spectrum

$$E_N = (\omega_t - \omega_c)\left[\sum_{i=1}^{N} \ell_i - \frac{1}{2}N(N-1)\alpha + N\right] \quad (8)$$

which is the sum of the 1-body harmonic LLL spectra shifted by the 2-body statistics term $-\frac{1}{2}N(N-1)\alpha(\omega_\mathrm{t} - \omega_\mathrm{c})$. The spectrum and the eigenstates (6,8) interpolate from the harmonic LLL bosonic to the harmonic LLL fermionic basis when $\alpha : 0 \to -1$ and lead, in the thermodynamic limit $\omega \to 0$, to Haldane exclusion statistics with parameter $g = -\alpha$ for a degenerate 1-body LLL spectrum [7].

At this point, being in the LLL and a harmonic well, let us take, as advocated above, the $B \to 0$ limit, i.e. one considers (2,4-8) with $B = 0$ and $\omega_t = \omega$. In this limit, (2) becomes of course the N-anyon Hamiltonian in a harmonic well, here for $\alpha \in [-1, 0]$, bearing in mind that in the absence of an external magnetic field the Bose-Fermi interpolations $\alpha : 0 \to 1$ and $\alpha : 0 \to -1$ are equivalent. The harmonic LLL basis (4) become

$$(\frac{\omega^{\ell_i+1}}{\pi\ell_i!})^{\frac{1}{2}} z_i^{\ell_i}, \quad \ell_i \geq 0 \quad (9)$$

with 1-body spectrum

$$\omega(\ell_i + 1), \quad \ell_i \geq 0 \quad (10)$$

that is one picks up on each 2d harmonic energy level $(j+1)\omega$, $j \geq 0$, with degeneracy $j+1$, the state of maximal angular momentum $\ell = j$, and consequently zero radial quantum number, yielding (10) which happens to coincide with a 1d harmonic spectrum. The N-body eigenstates become, up to a ω-dependant normalization

$$\psi_\mathrm{free} = \prod_{i<j}(z_i - z_j)^{-\alpha} \prod_{i=1}^{N} z_i^{\ell_i} \exp(-\frac{1}{2}\omega\sum_{i=1}^{N} z_i\bar{z}_i) \quad (11)$$

with the projected Hamiltonian

$$H_\omega = \omega\left[\sum_{i=1}^{N} z_i\partial_i - \alpha N(N-1)/2 + N\right] \quad (12)$$

and N-anyon spectrum

$$E_N = \omega\left[(\sum_{i=1}^{N} \ell_i - \frac{1}{2}N(N-1)\alpha + N\right] \quad (13)$$

Now, looking at (13), one recognizes (up to a global one-body shift $\omega/2$) the N-body 1d Calogero spectrum in a harmonic well. Moreover, one can show that the Hamiltonian (12) is equivalent to the 1d harmonic Calogero Hamiltonian by formally following the steps of [3]: the algebra of annihilation-creation operators of the Calogero model in a harmonic well with coupling constant-exclusion parameter $g = -\alpha \in [0, 1]$ can be realized in a 2d holomorphic representation which precisely yields the Hamiltonian (12).

What has been obtained here by taking the $B \to 0$ limit in the LLL is a projection from a 2d model to a 1d model, contrary to what a LLL projection alone can achieve. Note that, looking at the eigenstates (9-11) and forgetting that they were obtained from the $B \to 0$ limit in the LLL, one has here a "harmonic" projection which maps the anyon model on the Calogero model, without any reference to a magnetic field.

The equivalence obtained at the Hamiltonian and spectrum levels between the $B \to 0$ LLL anyon model and the Calogero model can be also seen in a thermodynamical approach. Since, in the thermodynamic limit $\omega \to 0$, the 2d harmonic well regulator 1-body partition function should become the 2d free partition function $Z_o^{d=2} = V/(2\pi\beta)$, one infers that [10]

$$Z = \frac{e^{-\beta\omega}}{(2\sinh\frac{\beta\omega}{2})^2} \underset{\omega\to 0}{\simeq} \frac{1}{(\beta\omega)^2} \to \frac{V}{2\pi\beta} \quad (14)$$

where V stands for the infinite area of the plane. It follows that the harmonic LLL 1-body partition function corresponding to (5) becomes in the thermodynamic limit $\omega \to 0$ (ignoring the global shift)

$$\frac{e^{-\beta\omega_t}}{1 - e^{-\beta(\omega_t - \omega_c)}} \underset{\omega\to 0}{\simeq} \frac{e^{-\beta\omega_c}}{\beta(\omega_t - \omega_c)} \to e^{-\beta\omega_c}\left|\frac{eB}{2\pi}\right| V \quad (15)$$

i.e. of course the 2d LLL partition function $Z_{LLL} = e^{-\beta\omega_c}\,|eB/(2\pi)|\,V$ with infinite Landau degeneracy BV/ϕ_o. On the contrary, the "projected" 1-body partition function obtained by restricting the 2d harmonic spectrum to (10)

$$\frac{e^{-\frac{\beta\omega}{2}}}{2\sinh\frac{\beta\omega}{2}} \underset{\omega\to 0}{\simeq} \frac{1}{\beta\omega} \to \sqrt{\frac{V}{2\pi\beta}} \quad (16)$$

behaves as the free 1d partition function $Z_o^{d=1} = L/\sqrt{2\pi\beta}$ provided that $\sqrt{V}$ is interpreted as the infinite length L of the resulting 1d system.

Therefore, in the thermodynamic limit $\omega \to 0$, the harmonic LLL partition function obtained with the spectrum (5) leads, when $B \neq 0$, to Z_{LLL}, and when $B = 0$, to $Z_o^{d=1}$ meaning that the following identity for the partition functions holds

$$Z_{LLL} = e^{-\beta\omega_c}\left|\frac{eB}{2\pi}\right| V \underset{B\to 0}{\to} Z_o^{d=1} = \sqrt{\frac{V}{2\pi\beta}} \quad (17)$$

and accordingly for the density of states

$$\rho_{LLL} = \left|\frac{eB}{2\pi}\right| V\delta(E - \omega_c) \underset{B\to 0}{\to} \rho_o^{d=1} = \frac{\sqrt{V}}{\pi\sqrt{2E}} \quad (18)$$

where ρ_{LLL} and $\rho_o^{d=1}$ stand respectively for the LLL and the free 1d density of states. Clearly, setting bluntly $B = 0$ in Z_{LLL} has no meaning whatsoever. Still, (17,18) have been given a non ambiguous meaning through the long distance harmonic regularization. Accordingly, when $B = 0$, the LLL eigenstates basis (3) is mapped on the free 1d plane wave basis.

The same reasonning equally applies to the thermodynamics of the LLL anyon model, with in dimension two a thermodynamic limit prescription [7,10] at order n in the cluster expansion $1/(\beta\omega)^2 \to nV/(2\pi\beta)$ that generalizes (14). The cluster coefficients of the harmonic LLL anyon model, corresponding to the N-body spectrum (8), rewrites in the small ω limit [7]

$$b_n^{LLL} = \frac{1}{\beta(\omega_t - \omega_c)} \frac{e^{-n\beta(\omega_t-\omega_c)}}{n^2} \prod_{k=1}^{n-1} \frac{k+n\alpha}{k} \tag{19}$$

Again, one can take the thermodynamic limit, either with $B \neq 0$, or $B = 0$, to arrive respectively at the thermodynamical potential of the 2d LLL anyon model and of the 1d Calogero model

$$\ln Z = \int dE \rho(E) \ln \Xi(E) \tag{20}$$

where ρ stand respectively for the 2d LLL density of states ρ_{LLL} and the 1d free density of states $\rho_o^{d=1}$, the latter being as advocated above, the vanishing magnetic field limit of the former. Ξ, which may be interpreted as the grand partition function at energy E, satisfies the transcendental equation [5–7]

$$\Xi - x\Xi^{1+\alpha} = 1 \tag{21}$$

x being the Gibbs factor at energy E. Equation (21), considered as the cornerstone of Haldane-exclusion statistics, has just been shown to come directly from the thermodynamics of the 2d LLL anyon model projected either on the harmonic LLL basis, or on the "harmonic" basis, the latter projection being also understood as the vanishing magnetic field limit of the former.

To conclude, let us mention that, in the paradygmatic LLL anyon case, the basic structure of the cluster coefficient (19) is, in the thermodynamic limit,

$$b_n = G\frac{e^{-n\beta E_o}}{n} \prod_{k=1}^{n-1} \frac{k-ng}{k} \tag{22}$$

where the infinite Landau degeneracy has been denoted ed by G, the number of quantum states at a given energy E_o, and the anyonic parameter $-\alpha$ has been replaced by the more familiar Haldane exclusion parameter g. (22) as well of the resulting N-body partition function (for a now finite G)

$$Z_N = Ge^{-N\beta E_o} \prod_{k=2}^{N} \frac{k+G-1-Ng}{k} \tag{23}$$

are a direct consequence of the transcendental equation (21), expanded in powers of x, with the grand-partition function given from (20) by $Z = \Xi^G$. Note that

i) the counting deduced from (23) differs [11] from the standard Haldane counting

ii) if one insists on interpreting Ξ as the grand partition for a system of exclusion particles in a single state of energy E_o, i.e. $G = 1$, which is in principle not allowed since from thermodynamics on average $< N > /G < 1/g$, then one has to face in (23) the so-called problem of negative probabilities [12,11], which would be avoided if one sticks, as it should, to $G > Ng$.

So one has on the one hand the 2d anyon model microscopically defined from first principles, on the other hand Haldane statistics coming from counting considerations in the Hilbert space. The thermodynamics of Haldane statistics leads to the basic equations (21,22,23), which, on the other hand, follow directly from the LLL-anyon model, a particular limit of which yields, as already said, yet another microscopic example of Haldane statistics, the Calogero model. It is thus tempting to propose quite generally that Haldane statistics might in fact be a particular limit of anyon statistics, each time it can be realized in terms of a microscopic Hamiltonian.

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