

# Singular statistics

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## Abstract

We consider the statistical distribution of zeros of random meromorphic functions whose poles are independent random variables. It is demonstrated that correlation functions of these zeros can be computed analytically and explicit calculations are performed for the 2-point correlation function. This problem naturally appears in e.g. rank-one perturbation of an integrable Hamiltonian and, in particular, when a  $\delta$ -function potential is added to an integrable billiard.

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## I. INTRODUCTION

The investigation of statistical properties of quantum energy levels of a given system is a long-standing problem (see e.g. [1]- [3]). According to the accepted conjectures energy levels of integrable systems behave as independent random variables (i.e. they obey the Poisson statistics) [4] and those of generic chaotic systems follow the random matrix predictions [5]. The proof of these conjectures in the full generality is without doubt quite difficult and is still lacking though partial results (concerning mostly integrable models) are available (see e.g. [6] and references therein).

But there are systems which are neither integrable nor completely chaotic for which quantum energy levels are defined by an equation

$$f(E) = 0 \tag{1}$$

with a well defined (and simple) function  $f(E)$ .

In [7] the case of polynomial equation

$$f(E) = \sum_{n=0}^N a_n E^n \tag{2}$$

has been considered and statistical properties of solution of  $f(E) = 0$  have been calculated provided  $a_n$  be independent random variables.

The purpose of this paper is to consider the case of random meromorphic functions of the form

$$f(E) = P(E) + \sum_{j=1}^N \frac{r_j}{E - e_j}, \tag{3}$$

where  $P(E)$  is a polynomial and  $e_j, r_j$  are, correspondingly, poles and residues of  $f(E)$ .

The natural example leading to the quantization condition in this form is the perturbation of a Hamiltonian by rank-one perturbation. If  $H_{\mu\nu}^{(0)}$  is an unperturbed Hamiltonian then the Hamiltonian after perturbation is

$$H_{\mu\nu} = H_{\mu\nu}^{(0)} + v_\mu v_\nu, \tag{4}$$

where  $v_\mu$  is a perturbation vector.

Solutions of the ‘Schrödinger’ equation

$$H_{\mu\nu} \psi_\nu = E \psi_\mu \tag{5}$$

can be expressed through solutions of unperturbed equation

$$H_{\mu\nu}^{(0)} \psi_\nu^{(0)}(n) = e_n \psi_\mu^{(0)}(n) \tag{6}$$

as follows

$$\psi_\mu = \sum_n c_n \psi_\mu^{(0)}(n), \tag{7}$$

where (up to a factor)

$$c_n = \frac{\langle v|\psi^{(0)}(n)\rangle}{E - e_n} \quad (8)$$

provided new eigenvalues,  $E$ , obey the following quantization condition

$$\sum_n \frac{|\langle v|\psi^{(0)}(n)\rangle|^2}{E - e_n} = 1. \quad (9)$$

Here  $\langle v|\psi^{(0)}(n)\rangle = \sum_\mu v_\mu \psi_\mu^{(0)}(n)$ .

This equation has the form of Eq. (3) with  $P(E)=\text{const}$  while unperturbed energy levels play the role of poles, and the residues are projections of unperturbed wave functions in the directions of perturbation vector

$$r_n = |\langle v|\psi^{(0)}(n)\rangle|^2. \quad (10)$$

The addition of a  $\delta$ -function potential

$$V(x) = \lambda\delta(\vec{x} - \vec{x}_0) \quad (11)$$

corresponds exactly to a rank-one perturbation. In this case (see e.g. [8], [9]) Eq. (9) takes the form

$$\lambda \sum_n \frac{|\psi_n^{(0)}(\vec{x}_0)|^2}{E - e_n} = 1, \quad (12)$$

where  $\psi_n^{(0)}(\vec{x})$  and  $e_n$  are eigenfunctions and eigenvalues of the problem without the  $\delta$ -function potential.

Another model which leads to similar equations is the Bohr-Mottelson model [10] which describes the interaction of one level (denoted below by index 0) with all other levels. The model is defined by the Hamiltonian

$$H = H_0 + V, \quad (13)$$

where the interaction potential has non-zero matrix elements only between the chosen level and all other levels

$$V_{0i} = V_{i0}, \quad V_{00} = V_{ij} = 0. \quad (14)$$

The energy levels of the Hamiltonian (13) obey the equation [10]

$$\sum_j \frac{|V_{0j}|^2}{E - e_j} - (E - e_0) = 0, \quad (15)$$

which is again of the form of Eq. (3) with linear polynomial part.

A quite natural question appears: What is the statistical distribution of new eigenvalues (i.e. solutions of Eq. (3)) provided that statistical distributions of poles and residues are

known? In [11] it was proved that, if unperturbed system is described by random matrix theory, the distribution of new eigenvalues will also be of random matrix type.

The main purpose of this paper is to compute analytically the statistical distribution of solutions of Eq. (3) when the poles,  $e_j$ , are independent random variables (i.e. they obey the Poisson statistics). We shall show that in this case the resulting statistics exhibits the level repulsion and differs from known distributions.

The plan of the paper is the following. In Section II the general formalism is described. In Section III the calculation of the mean density is presented. In Section IV the 2-point correlation function is computed when all residues,  $r_j$ , in Eq. (3) are the same. Generalization to different residues is discussed in Section VII. As the exact expression of the 2-point correlation function is cumbersome, in Section V the series expansion of the results is given. In Section VI the limiting behavior of the 2-point correlation function for small and large energy difference is obtained without the knowledge of the exact solution. The details of the calculation of a certain important integral are presented in Appendix.

## II. GENERAL FORMALISM

We consider the most interesting case of Eq. (3) when the mean separation of the poles is much smaller than a characteristic scale of polynomial  $P(E)$ . Under such condition this polynomial can be considered as a constant and after dividing by it Eq. (3) takes the form

$$\sum_{j=1}^N \frac{r_j}{E - e_j} = 1. \quad (16)$$

Our goal is to find the statistical distribution of solutions,  $E$ , of this equation provided  $r_j$  are constants and  $N$  numbers  $e_j$  are independent random variables with a common distribution  $d\mu(e)$  which for simplicity we choose as follows

$$d\mu(e) = \begin{cases} \frac{1}{2W} de & \text{if } -W \leq e_j \leq W, \\ 0 & \text{otherwise} \end{cases}. \quad (17)$$

As the density of these poles is a constant they can be considered as eigenvalues of a 2-dimensional integrable billiard and we shall often call them energy levels (or unperturbed energy levels). All our calculations remain also valid in more general case when the mean density of poles is not a constant but is not changed noticeable in the scale of the mean pole separation (e.g for 3-dimensional integrable models). The only difference is that  $N/2W$  below should be substituted by the local mean density of poles,  $\bar{\rho}$  (see the end of Section III).

In general, if one is interested in solutions of equation

$$f(x_n) = 0, \quad (18)$$

it is often convenient to express the exact density of such solutions

$$\rho(x) = \sum_n \delta(x - x_n), \quad (19)$$

in the following manner

$$\rho(x) = \delta(f(x)) \left| \frac{df(x)}{dx} \right|. \quad (20)$$

The main advantage of such representation is the possibility of calculating the statistical distribution of roots,  $x_n$ , directly from statistical distribution of coefficients of  $f(x)$ . E.g. this method has been used for deriving the distribution of roots of random polynomials [7].

In our case

$$\rho(E) = \delta\left(\sum_{j=1}^N \frac{r_j}{E - e_j} - 1\right) \sum_{k=1}^N \frac{r_k}{(E - e_k)^2}. \quad (21)$$

Representing the  $\delta$ -function as the Fourier integral (i.e. considering the characteristic function of the roots) one gets

$$\rho(E) = \int_{-\infty}^{\infty} \frac{d\alpha}{2\pi} \exp(i\alpha(\sum_{j=1}^N \frac{r_j}{E - e_j} - 1)) \sum_{k=1}^N \frac{r_k}{(E - e_k)^2}. \quad (22)$$

It is this representation of the exact density that we shall use throughout the paper.

As all  $e_j$  are considered as independent random variables this expression can be rewritten in the form

$$\rho(E) = \int \frac{d\alpha}{2\pi} e^{-i\alpha} \prod_{j=1}^N \exp(i \frac{\alpha r_j}{E - e_j}) \sum_{k=1}^N \frac{r_k}{(E - e_k)^2}, \quad (23)$$

where all factors are also independent random variables which clearly permits to find all mean values by straightforward integration.

### III. MEAN DENSITY

Let us start with the calculation of the mean density

$$\langle \rho(E) \rangle = \int \frac{d\alpha}{2\pi} e^{-i\alpha} \prod_{j=1}^N \int d\mu(e_j) \exp(i \frac{\alpha r_j}{E - e_j}) \sum_{k=1}^N \frac{r_k}{(E - e_k)^2}. \quad (24)$$

The integrals can be transformed in the following way

$$\langle \rho(E) \rangle = \int \frac{d\alpha}{2\pi} e^{-i\alpha} \left( \sum_{k=1}^N r_k g(r_k \alpha) \right) \prod_{j \neq k} f(r_j \alpha), \quad (25)$$

where

$$f(\alpha) = \int d\mu(e) \exp(i \frac{\alpha}{E - e}), \quad (26)$$

and

$$g(\alpha) = \int d\mu(e) \frac{1}{(E-e)^2} \exp(i\frac{\alpha}{E-e}) = -\frac{\partial^2}{\partial\alpha^2} f(\alpha). \quad (27)$$

Let us rewrite the expression for  $f(\alpha)$  in the form

$$f(\alpha) = 1 - \frac{1}{2W} I(\alpha), \quad (28)$$

where

$$I(\alpha) = \int_{-W}^W de (1 - \exp(i\frac{\alpha}{E-e})), \quad (29)$$

As

$$g(\alpha) = \frac{1}{2W} \frac{\partial^2}{\partial\alpha^2} I(\alpha), \quad (30)$$

it is necessary to compute only  $I(\alpha)$ .

Though the above steps are exact for finite  $N$ , the most interesting case is the case  $N \rightarrow \infty$ . In this limit only small values of  $\alpha$  are important ( $\alpha \approx 1/N$ ) and it is necessary to take into account in  $I(\alpha)$  only terms linear in  $\alpha$ .

Due to the singular character of the integral  $I(\alpha)$  (29) one cannot just expand the integrand in power of  $\alpha$ . If  $E$  belongs to the support of the measure,  $-W < E < W$ , the change of variable

$$t = \frac{1}{E-e}, \quad (31)$$

reduces the integral for  $I(\alpha)$  (29) to a sum of two integrals

$$I(\alpha) = \left( \int_{(E+W)^{-1}}^{\infty} + \int_{-\infty}^{(E-W)^{-1}} \right) (1 - e^{i\alpha t}) \frac{dt}{t^2}, \quad (32)$$

which can be transform as follows

$$I(\alpha) = \left( \int_{-\infty}^{\infty} - \int_{(E-W)^{-1}}^{(E+W)^{-1}} \right) (1 - e^{i\alpha t}) \frac{dt}{t^2}. \quad (33)$$

The first integral equals  $\pi|\alpha|$  and in the second integral one can safely use perturbation theory in  $\alpha$ . The final result is

$$I(\alpha) = \pi|\alpha| + i\alpha \ln \frac{W-E}{W+E} + \alpha^2 \frac{W}{E^2 - W^2} + O(\alpha^3), \quad (34)$$

and

$$g(\alpha) = \frac{\pi}{W} \delta(\alpha) + \frac{1}{E^2 - W^2}. \quad (35)$$

For small values of  $\alpha$

$$e^{-i\alpha} \prod_{j=1}^N f(r_j \alpha) = \exp\left(-\frac{N}{2W} v (\pi|\alpha| + i\frac{\alpha}{v'})\right), \quad (36)$$

where  $v$  plays the role of a ‘bare’ coupling constant,

$$v = \frac{1}{N} \sum_{j=1}^N r_j, \quad (37)$$

and  $v'$  is a ‘renormalized’ coupling constant

$$\frac{1}{v'} = \frac{2W}{Nv} + \ln \frac{W-E}{W+E}. \quad (38)$$

The necessity of renormalization for such type of equations is well known when a  $\delta$ -function potential is added to a  $d$ -dimensional system with  $d \geq 2$  (see e.g. [8] and Eqs. (98 and (131)) where it is connected with one-parameter self-adjoint extension of a singular Hamiltonian. Physically the renormalization means that the limit of infinite small size impurity is not uniquely defined and depends on internal details of the scatterer. All physically measurable quantities (like the cross-section) depend only on renormalized coupling constant,  $v'$ . The bare coupling constant,  $v$ , is not observable and can be arranged to produce any  $v'$ . When a specific model of small-size scatterer is considered (e.g. a hard disk with a small radius) one gets a concrete form of the bare (and renormalized) coupling constant. Below we consider the most interesting case when renormalised coupling constant is assumed to be independent of  $N$  (or energy). All other limits can be derived from this one. Note that in our calculations the appearance of such renormalization (i.e. the fact that the bare coupling constant,  $v$ , and the renormalization factor  $\log(W-E)/(W+E)$  appear only in the combination (38)) is automatic.

Finally, when  $-W < E < W$  the density of state is the sum of two terms

$$\rho_{in}(E) = \frac{N}{2W} - \frac{2W}{(W^2 - E^2)(\pi^2 + 1/v'^2)}. \quad (39)$$

As  $N$  is assumed to be large, the first term dominates and the mean density of levels is

$$\bar{\rho} = \frac{N}{2W}, \quad (40)$$

as it should be.

When  $E$  is beyond the interval  $[-W, W]$  the calculation is simpler as in this case there is no singularity on the contour of integration and one can simply expand the integrand of  $I(\alpha)$  on series of  $\alpha$

$$I(\alpha) = i\alpha \ln \frac{E-W}{E+W} + \alpha^2 \frac{W}{E^2 - W^2} + O(\alpha^3). \quad (41)$$

Therefore

$$\rho_{out}(E) = \frac{|\phi(E)'|}{\sqrt{2\pi\sigma}} \exp\left(-\frac{\phi^2(E)}{2\sigma^2}\right), \quad (42)$$

where

$$\phi(E) = \ln \frac{E+W}{E-W} - \frac{1}{v'}, \quad \sigma^2 = \frac{4W^2}{(E^2 - W^2)N}. \quad (43)$$

When  $N \rightarrow \infty$ ,  $\sigma \rightarrow 0$  and

$$\rho(E) \rightarrow \delta(E - E_c), \quad (44)$$

where  $E_c$  is a root of equation  $\phi(E_c) = 0$

$$E_c = W \coth \frac{1}{2v'}. \quad (45)$$

These results correspond exactly to what one attends from the simple geometrical picture of roots of Eq. (16). The poles,  $e_j$ , divide the real axis into  $N + 1$  intervals. Due to the pole behavior each interval contains one of solutions,  $E$ . There is only one eigenvalue outside of the support of the initial measure and all other  $(N - 1)$  eigenvalues are distributed practically uniformly inside the initial interval  $[-W, W]$ . The second term in Eq. (39) is a smooth bump which is necessary to insure that

$$\int_{-W}^W \rho_{in}(E) dE = N - 1,$$

which can easily be checked by noting that  $2W/(E^2 - W^2) = \partial(1/v')/\partial E$ .

In Eq. (17) we have assumed the particular form of the distribution of  $d\mu(e)$  but the results will be valid for any form of this measure (provided that it is not changed noticeably in the scale of the mean distance between levels) with the substitution  $N/2W \rightarrow \bar{\rho}$ ,  $E+W \rightarrow E - E_{min}$  and  $W - E \rightarrow E_{max} - E$  where  $\bar{\rho}$  is the local mean density of unpertubated levels,  $E_{min}$  and  $E_{max}$  are minimal and maximal values of levels included in the sum (16).

#### IV. 2-POINT CORRELATION FUNCTION

Using the previously discussed method one can compute higher correlation functions as well. We consider here the calculation of the 2-point correlation function,  $R_2(E_1, E_2)$ , defined in the standard way

$$R_2(E_1, E_2) = \langle \rho(E_1)\rho(E_2) \rangle, \quad (46)$$

where  $\langle \dots \rangle$  denotes the mean value over all random variables.

For clarity we consider first the case where all residues are equal,  $r_j = v$ . This case appears e.g. when a  $\delta$ -function potential is added to a rectangular billiard with periodic boundary conditions (see Eq. (98)). More general case with different  $r_j$  will be considered shortly in Section VII.

When all residues are the same our defining equation takes the form

$$\sum_{j=1}^N \frac{1}{E - e_j} = \frac{1}{v}, \quad (47)$$



and the 2-point correlation function can be expressed as follows

$$R_2(E_1, E_2) = \langle \int \frac{d\alpha_1 d\alpha_2}{4\pi^2} \exp(i \sum_{j=1}^N (\frac{\alpha_1}{E_1 - e_j} + \frac{\alpha_2}{E_2 - e_j})) \times \sum_{k_1, k_2=1}^N \frac{1}{(E_1 - e_{k_1})^2 (E_2 - e_{k_2})^2} e^{-\frac{i}{v}(\alpha_1 + \alpha_2)} \rangle. \quad (48)$$

After a simple algebra this expression can be transformed to

$$R_2(E_1, E_2) = \int \frac{d\alpha_1 d\alpha_2}{4\pi^2} [N(f(\alpha_1, \alpha_2))^{N-1} g(\alpha_1, \alpha_2) + N(N-1)(f(\alpha_1, \alpha_2))^{N-2} \psi_1(\alpha_1, \alpha_2) \psi_2(\alpha_1, \alpha_2)] \exp(-\frac{i}{v}(\alpha_1 + \alpha_2)), \quad (49)$$

where

$$\begin{aligned} f(\alpha_1, \alpha_2) &= \int d\mu(e) \exp(i \frac{\alpha_1}{E_1 - e} + i \frac{\alpha_2}{E_2 - e}), \\ g(\alpha_1, \alpha_2) &= \int d\mu(e) \exp(i \frac{\alpha_1}{E_1 - e} + i \frac{\alpha_2}{E_2 - e}) \frac{1}{(E_1 - e)^2 (E_2 - e)^2}, \\ \psi_1(\alpha_1, \alpha_2) &= \int d\mu(e) \exp(i \frac{\alpha_1}{E_1 - e} + i \frac{\alpha_2}{E_2 - e}) \frac{1}{(E_1 - e)^2}, \\ \psi_2(\alpha_1, \alpha_2) &= \int d\mu(e) \exp(i \frac{\alpha_1}{E_1 - e} + i \frac{\alpha_2}{E_2 - e}) \frac{1}{(E_2 - e)^2}. \end{aligned} \quad (50)$$

We shall be interested in the distribution of eigenvalues inside the interval  $[-W, W]$  and therefore shall assume that both arguments  $E_1$  and  $E_2$  belong to this interval.

Let us as denote

$$f(\alpha_1, \alpha_2) = 1 - \frac{1}{2W} I(\alpha_1, \alpha_2), \quad (51)$$

where

$$I(\alpha_1, \alpha_2) = \int_{-W}^W (1 - \exp(i \frac{\alpha_1}{E_1 - e} + i \frac{\alpha_2}{E_2 - e})) de. \quad (52)$$

Other functions are expressed through  $I(\alpha_1, \alpha_2)$  as follows

$$\begin{aligned} g(\alpha_1, \alpha_2) &= -\frac{1}{2W} \frac{\partial^4}{\partial \alpha_1^2 \partial \alpha_2^2} I(\alpha_1, \alpha_2), \\ \psi_1(\alpha_1, \alpha_2) &= \frac{1}{2W} \frac{\partial^2}{\partial \alpha_1^2} I(\alpha_1, \alpha_2), \\ \psi_2(\alpha_1, \alpha_2) &= \frac{1}{2W} \frac{\partial^2}{\partial \alpha_2^2} I(\alpha_1, \alpha_2). \end{aligned} \quad (53)$$

The integral (52) which defines  $I(\alpha_1, \alpha_2)$  can be split into three terms

$$I(\alpha_1, \alpha_2) = \left( \int_{-\infty}^{\infty} - \int_{-\infty}^{-W} - \int_{\infty}^W \right) (1 - \exp(i \frac{\alpha_1}{E_1 - e} + i \frac{\alpha_2}{E_2 - e})) de. \quad (54)$$

In the first integral (which we denoted by  $J(\alpha_1, \alpha_2)$ ) singular points  $E_1$  and  $E_2$  are on the contour of integration. In the second and the third ones there are no singularities and they can be computed in perturbation theory on  $\alpha_1$  and  $\alpha_2$ . In the later integrals we will see that one needs only linear in  $\alpha$  terms and

$$I(\alpha_1, \alpha_2) = J(\alpha_1, \alpha_2) + i(\alpha_1 \ln \frac{W - E_1}{W + E_1} + \alpha_2 \ln \frac{W - E_2}{W + E_2}). \quad (55)$$

It is the calculation of the first term which is difficult. The details of this calculation are given in Appendix. The final result for  $J(\alpha_1, \alpha_2)$  is the following

$$\begin{aligned} J(\alpha_1, \alpha_2) = & \pi(\alpha_1 + \alpha_2) \operatorname{sgn}(\alpha_2) - \pi [i(\alpha_1 + \alpha_2) G(-\frac{\alpha_2}{\omega}, \frac{\alpha_1}{\omega}) \\ & + (\alpha_2 J_0(\xi) + i\sqrt{-\alpha_1 \alpha_2} J_1(\xi)) \exp(i\frac{\alpha_1 - \alpha_2}{\omega})] (\operatorname{sgn}(\alpha_1) - \operatorname{sgn}(\alpha_2)), \end{aligned} \quad (56)$$

where  $\omega = E_1 - E_2$ ,  $\xi = \frac{2}{\omega} \sqrt{-\alpha_1 \alpha_2}$  and

$$G(x, y) = e^{iy} \int_x^\infty J_0(2\sqrt{yt}) e^{it} dt. \quad (57)$$

The symmetry relations

$$\begin{aligned} J(\alpha_2, \alpha_1) &= J^*(\alpha_1, \alpha_2), \\ J(-\alpha_1, -\alpha_2) &= J^*(\alpha_1, \alpha_2), \\ J(-\alpha_2, -\alpha_1) &= J(\alpha_1, \alpha_2), \end{aligned} \quad (58)$$

are also useful. We are interested in the situation when the difference of energies,  $\omega = E_1 - E_2$ , is of the order of the mean distance between the levels

$$\omega = \Omega \frac{2W}{N}, \quad (59)$$

and dimensionless frequency  $\Omega$  is a constant. In this case one can check that the important values of  $\alpha$  will also be of the order of  $1/N$  which explains why we have restricted the expansion only up to linear terms. Other simplification comes from the fact that in perturbation theory terms (55) one can put  $E_1 = E_2$  after which they only depend on the sum  $\alpha_1 + \alpha_2$ .

In the limit of large  $N$  one obtains (see Appendix)

$$f^N(\alpha_1, \alpha_2) = \exp(-\frac{N}{2W} \tilde{I}(\alpha_1, \alpha_2)), \quad (60)$$

$$g(\alpha_1, \alpha_2) = \frac{1}{2W} \frac{1}{\omega^2} (\frac{\partial}{\partial \alpha_1} - \frac{\partial}{\partial \alpha_2}) [\exp(i\frac{\alpha_1 - \alpha_2}{\omega}) \Phi(\alpha_1, \alpha_2)], \quad (61)$$

$$\psi_1(\alpha_1, \alpha_2) = \frac{1}{2W} \exp(i\frac{\alpha_1 - \alpha_2}{\omega}) \frac{\partial}{\partial \alpha_1} \Phi(\alpha_1, \alpha_2), \quad (62)$$

$$\psi_2(\alpha_1, \alpha_2) = -\frac{1}{2W} \exp(i\frac{\alpha_1 - \alpha_2}{\omega}) \frac{\partial}{\partial \alpha_2} \Phi(\alpha_1, \alpha_2). \quad (63)$$

Here we introduce

$$\tilde{I}(\alpha_1, \alpha_2) = J(\alpha_1, \alpha_2) + (\alpha_1 + \alpha_2) \frac{i}{v'}, \quad (64)$$

where  $v'$  is the renormalized coupling constant as in (38) and

$$\Phi(\alpha_1, \alpha_2) = 2\pi J_0\left(\left(\frac{2}{\omega} \sqrt{-\alpha_1 \alpha_2}\right)\right) \Theta(-\alpha_1 \alpha_2) \text{sgn}(\alpha_1). \quad (65)$$

Therefore

$$\begin{aligned} R_2(\omega) = & \int \frac{d\alpha_1 d\alpha_2}{(4\pi W)^2} \left\{ N f^{N-1} \frac{2W}{\omega^2} \left( \frac{\partial}{\partial \alpha_1} - \frac{\partial}{\partial \alpha_2} \right) [e^{i\frac{\alpha_1 - \alpha_2}{\omega}} \Phi] \right. \\ & \left. - N(N-1) f^{N-2} e^{2i\frac{\alpha_1 - \alpha_2}{\omega}} \left[ \frac{\partial}{\partial \alpha_1} \Phi \right] \left[ \frac{\partial}{\partial \alpha_2} \Phi \right] \right\}. \end{aligned} \quad (66)$$

It is convenient to integrate the first term by parts

$$\begin{aligned} & \int f^{N-1} \left( \frac{\partial}{\partial \alpha_1} - \frac{\partial}{\partial \alpha_2} \right) [e^{i\frac{\alpha_1 - \alpha_2}{\omega}} \Phi] = \\ & \frac{N-1}{2W} \int f^{N-2} e^{i\frac{\alpha_1 - \alpha_2}{\omega}} \Phi \left( \frac{\partial}{\partial \alpha_1} - \frac{\partial}{\partial \alpha_2} \right) J = \\ & \frac{N-1}{2W} \int f^{N-2} e^{2i\frac{\alpha_1 - \alpha_2}{\omega}} \Phi^2. \end{aligned} \quad (67)$$

Substituting this expression into the previous equation one obtains

$$R_2(\omega) = \frac{N(N-1)}{(4\pi W)^2} \int d\alpha_1 d\alpha_2 \left\{ \frac{\Phi^2}{\omega^2} - \left[ \frac{\partial}{\partial \alpha_1} \Phi \right] \left[ \frac{\partial}{\partial \alpha_2} \Phi \right] \right\} f^{N-2} e^{2i\frac{\alpha_1 - \alpha_2}{\omega}}. \quad (68)$$

The second useful form can be derived by the following transformation of the second term

$$e^\Psi \left[ \frac{\partial}{\partial \alpha_2} \Phi \right] \left[ \frac{\partial}{\partial \alpha_1} \Phi \right] = \left[ \frac{\partial^2}{2\partial \alpha_1 \partial \alpha_2} + \frac{i}{\omega} \left( \frac{\partial}{\partial \alpha_1} - \frac{\partial}{\partial \alpha_2} \right) + \frac{1}{\omega^2} \right] \Phi^2 e^\Psi, \quad (69)$$

where

$$\Psi = 2i \frac{\alpha_1 - \alpha_2}{\omega}. \quad (70)$$

Combining these two expressions one gets

$$R_2(\omega) = -\frac{N(N-1)}{(4\pi W)^2} \int d\alpha_1 d\alpha_2 e^{-\frac{N}{2W} \tilde{I}} \left[ \frac{\partial^2}{2\partial \alpha_1 \partial \alpha_2} + \frac{i}{\omega} \left( \frac{\partial}{\partial \alpha_1} - \frac{\partial}{\partial \alpha_2} \right) \right] \Phi^2 e^\Psi. \quad (71)$$

It is easy to check that under the scale transformation (assuming  $\lambda > 0$ )

$$\omega \rightarrow \lambda \omega, \quad \alpha_i \rightarrow \lambda \alpha_i, \quad (72)$$

the pre-factor does not change and  $\tilde{I} \rightarrow \lambda \tilde{I}$ . Therefore after the transformations

$$\Omega = \frac{N}{2W} \omega, \quad (73)$$

and

$$R_2(\omega) = \frac{N(N-1)}{4W^2} r_2(\Omega), \quad (74)$$

plus the corresponding change of  $\alpha$  the dependence of  $N$  will disappear and after the substitution

$$\alpha_i = \Omega \alpha_i \quad (75)$$

the resulting expression for the 2-point correlation function takes the form

$$r_2(\Omega) = - \int \frac{d\alpha_1 d\alpha_2}{4\pi^2} e^{-2\pi\Omega\bar{J}} \left[ \frac{\partial^2}{2\partial\alpha_1\partial\alpha_2} + i \left( \frac{\partial}{\partial\alpha_1} - \frac{\partial}{\partial\alpha_2} \right) \right] \Phi^2 e^{2i(\alpha_1 - \alpha_2)}. \quad (76)$$

where

$$\tilde{J}(\alpha_1, \alpha_2) = \bar{J}(\alpha_1, \alpha_2) + i(\alpha_1 + \alpha_2) \frac{1}{2\pi v'}, \quad (77)$$

and from Eq. (191)

$$\begin{aligned} \bar{J}(\alpha_1, \alpha_2) &= -\frac{1}{2}(\alpha_1 + \alpha_2) \text{sgn}(\alpha_2) \\ &+ \frac{i}{2} [(\alpha_1 + \alpha_2)G(\alpha_1, -\alpha_2) - (i\alpha_1 J_0(2\sqrt{-\alpha_1\alpha_2}) \\ &+ \sqrt{-\alpha_1\alpha_2} J_1(2\sqrt{-\alpha_1\alpha_2})) e^{i(\alpha_1 - \alpha_2)}] (\text{sgn}(\alpha_1) - \text{sgn}(\alpha_2)). \end{aligned} \quad (78)$$

When  $\Omega \rightarrow 0$  it is convenient to perform the integration by part

$$r_2(\Omega) = - \int d\alpha_1 d\alpha_2 e^{-2\pi\Omega\tilde{J}} \left[ \frac{\Omega^2}{2} \frac{\partial\tilde{J}}{\partial\alpha_1} \frac{\partial\tilde{J}}{\partial\alpha_1} + \frac{3\Omega i}{8\pi^2} \Phi e^{i(\alpha_1 - \alpha_2)} \right] \Phi^2 e^{2i(\alpha_1 - \alpha_2)}, \quad (79)$$

and take into account only the linear in  $\Omega$  term

$$r_2(\Omega) \rightarrow \Omega A, \quad (80)$$

where

$$A = -\frac{3i}{8\pi^2} \int d\alpha_1 d\alpha_2 \Phi^3 e^{3i(\alpha_1 - \alpha_2)}, \quad (81)$$

As in the region  $\alpha_1\alpha_2 < 0$   $\Phi = 2\pi J_0(2\sqrt{-\alpha_1\alpha_2})$ , after the change of variables

$$\xi = 2\sqrt{-\alpha_1\alpha_2}, \quad \eta = -\frac{\alpha_1}{\alpha_2}, \quad (82)$$

one gets

$$A = -\frac{3\pi i}{2} \int_0^\infty \xi J_0^3(\xi) d\xi \int_0^\infty \frac{d\eta}{\eta} e^{\frac{3\xi i}{2}(\eta + \eta^{-1})} + c.c.. \quad (83)$$

The integral over  $\eta$  equals  $i\pi H_0^{(1)}(3\xi)$  (see (169)) and the final expression for  $A$  is

$$A = 3\pi^2 \lim_{\epsilon \rightarrow 0} \int_0^\infty \xi J_0((3 + \epsilon)\xi) J_0^3(\xi) d\xi. \quad (84)$$

We write here  $(3 + \epsilon)$  (where  $\epsilon$  is proportional to  $\Omega$ ) as this integral is a discontinuous integral and its value when  $\epsilon = 0$  is a half of the value for  $\epsilon \rightarrow 0$ . The last value can be computed using the following integral ([15], p. 414)

$$\int_0^\infty \prod_{n=1}^4 J_0(a_n t) t dt = \frac{1}{\pi^2 \sqrt{a_1 a_2 a_3 a_4}} \begin{cases} K(x), & \text{if } x < 1 \\ \frac{1}{x} K(x), & \text{if } x > 1 \end{cases}, \quad (85)$$

where  $K(x)$  is the full elliptic integral of the second kind

$$x = \frac{\Delta}{\sqrt{a_1 a_2 a_3 a_4}} \quad (86)$$

and

$$16\Delta^2 = \prod_{n=1}^4 (a_1 + a_2 + a_3 + a_4 - 2a_n). \quad (87)$$

If the left hand side is negative the above integral equals zero.

In our case  $\Delta \rightarrow 0$  and  $K(0) = \pi/2$ , therefore

$$\lim_{\epsilon \rightarrow 0} \int_0^\infty \xi J_0((3 + \epsilon)\xi) J_0^3(\xi) d\xi = \frac{1}{2\pi\sqrt{3}}. \quad (88)$$

Hence

$$A = \frac{\pi\sqrt{3}}{2} \approx 2.72 \dots \quad (89)$$

Note that the slope at the origin is independent on the coupling constant and differs from the prediction of the Gaussian Orthogonal Ensembles of random matrices ( $r_2(\Omega) \rightarrow (\pi^2/6)\Omega$  [2]).

To find the asymptotics of the 2-point correlation function when  $\Omega \rightarrow \infty$  it is convenient to use Eq. (68). After rescaling of this expression one obtains (the constant term comes from the  $\delta$ -function contribution of derivatives)

$$r_2(\Omega) = 1 + \left\{ \int_0^\infty d\alpha_1 \int_{-\infty}^0 d\alpha_2 [J_0^2(2\sqrt{-\alpha_1\alpha_2}) + J_1^2(2\sqrt{-\alpha_1\alpha_2})] \right. \\ \left. \times \exp(-2\pi\Omega\tilde{J} + 2i(\alpha_1 - \alpha_2)) + c.c. \right\}. \quad (90)$$

When  $\Omega \rightarrow \infty$  the dominant contribution comes from region of small  $\alpha$ . Taking into account that when  $\alpha \rightarrow 0$

$$\tilde{J} \rightarrow \frac{1}{2}(\alpha_1 - \alpha_2) + i\frac{\alpha_1 + \alpha_2}{2\pi v'}, \quad (91)$$

one concludes that the corresponding asymptotics of the 2-point correlation function is

$$r_2(\Omega) \rightarrow 1 + \frac{2}{\Omega^2(\pi^2 + 1/v'^2)}. \quad (92)$$

Note the absence of oscillation on large  $\Omega$  typical for standard random matrix ensembles.

To check the above results we compute the statistical distribution of energy levels of a rectangular billiard with a  $\delta$ -function potential inside (sometimes called the Seba billiard [9]).

For a rectangle of sides  $a$  and  $b$  solutions of the Schrödinger equation

$$(e_{\vec{n}} - \Delta)\psi_{\vec{n}}(\vec{x}) = 0 \quad (93)$$

in 2 dimensions with periodic boundary conditions have the form

$$\psi_{\vec{n}}(\vec{x}) = \frac{1}{\sqrt{ab}} \exp(i\frac{2\pi}{a}nx + i\frac{2\pi}{b}my), \quad (94)$$

and

$$e_{\vec{n}} = (\frac{2\pi}{a}n)^2 + (\frac{2\pi}{b}m)^2 \quad (95)$$

for all (positive and negative) integers  $n$  and  $m$ .

As  $|\psi_{\vec{n}}(\vec{x})|^2 = 1/ab$  for all levels Eq. (12) which determines energy levels after the introduction of a  $\delta$ -function potential (11) takes the form

$$v \sum_{\vec{n}} \frac{1}{E - e_{\vec{n}}} = 1 \quad (96)$$

with  $v = \lambda/ab$ .

Unperturbed eigenvalues have multiplicity 4 (for non-zero  $m, n$ ) due to the existence of positive and negative values of  $m, n$ . To remove this degeneracy we consider in the above sum only positive integers and to have the same mean density ( $\bar{\rho} = ab/4\pi$ ) we divide all eigenvalues by 4 after which eigenvalues included in the sum are

$$e_{\vec{n}} = (\frac{\pi}{a}n)^2 + (\frac{\pi}{b}m)^2 \quad (97)$$

and  $m, n > 0$ .

The sum (96) formally diverges and for computation we consider the following renormalization

$$\frac{v'}{\bar{\rho}} \left( \sum_{\vec{n}} \frac{1}{E - e_{\vec{n}}} - \bar{\rho} \int_{E_{min}}^{E_{max}} de \frac{1}{E - e} \right) = 1, \quad (98)$$

where  $E_{min}$  and  $E_{max}$  are minimal and maximal values of energy included in the sum. The subtracted integral (considered as principal value) equals  $\log(E_{max} - E)/(E - E_{min})$  and one obtains the same relation between bare and renormalized coupling constants as before (cf. (38))

$$\frac{1}{v'} = \frac{1}{\bar{\rho}v} + \log \frac{E_{max} - E}{E - E_{min}}. \quad (99)$$

We take  $v' = 1$  and compute 100000 energy levels for such model. In Fig. 1 the cumulative nearest-neighbor distribution of these levels,  $N(s)$ , is presented. This quantity equals the integral over the nearest-neighbor distribution

$$N(s) = \int_0^\infty p(s') ds' \quad (100)$$

and it is better defined numerically than the usual nearest-neighbor distribution. On the same figure two other curves are presented. The dashed line corresponds to the Wigner surmise for the cumulative nearest-neighbor distribution in the Gaussian Orthogonal Ensemble (GOE) of random matrices [2]

$$N_{GOE}(s) = 1 - e^{-\pi s^2/4}. \quad (101)$$

The thin solid line represents the cumulative nearest-neighbor distribution for the so-called semi-Poisson model [12], [13] which serves as a reference point in models with intermediate statistics

$$N_{s,P}(s) = 1 - (2s + 1)e^{-2s}. \quad (102)$$

It is clearly seen that the cumulative nearest-neighbor distribution for the Seba billiard is quite far from GOE result and it is in between the semi-Poisson curve and the GOE one.

The numerically computed 2-point correlation function for this model is plotted in Fig. 2. The two curves in this figure correspond to theoretical predictions for small and large values of the argument given by Eqs. (80) and (92) respectively.

## V. SERIES EXPANSIONS

The above expressions are quite cumbersome. Therefore it is of interest to represent them as power expansions. We start with function  $G(x, y)$  defined in (193). It is convenient to define

$$G(x, y) = ig(t, s), \quad (103)$$

where

$$t = ix, s = iy. \quad (104)$$

Using the standard formula for the Bessel function

$$J_n(x) = \sum_{m=0}^{\infty} \left(\frac{x}{2}\right)^{2m+n} \frac{1}{m!(m+n)!}, \quad (105)$$

one gets

$$\begin{aligned} g(t, s) &= \sum_{m=0}^{\infty} \frac{(-t)^m}{m!} \sum_{n=m}^{\infty} \frac{(-s)^n}{n!} e^{t+s} \\ &= 1 - \sum_{m=0}^{\infty} \frac{(-t)^m}{m!} \sum_{n=0}^{m-1} \frac{(-s)^n}{n!} e^{t+s}. \end{aligned} \quad (106)$$

Expanding the exponent leads

$$g(t, s) = 1 - \sum_{m,n=0}^{\infty} \frac{(-t)^m (-s)^n}{m!n!} R(m, n), \quad (107)$$

$$R(m, n) = \sum_{l=0}^n C_n^l (-1)^l \sum_{k=0}^{m-n+l-1} C_m^k (-1)^k, \quad (108)$$

and  $C_m^n$  are the binomial coefficients. Only terms for which the upper limits in these sums are non-negative are included in the summation. But

$$\sum_{k=0}^L C_m^k (-1)^k = (-1)^L C_{m-1}^L, \quad (109)$$

therefore

$$R(m, n) = (-1)^{m+n-1} \sum_{l=0}^n C_n^l C_{m-1}^{m-n+l-1} = (-1)^{m+n-1} C_{n+m-1}^n. \quad (110)$$

Finally we get

$$g(t, s) = 1 + \sum_{m,n=0}^{\infty} \frac{t^m s^n}{m!n!} C_{n+m-1}^n. \quad (111)$$

Using Eq.(194) one can show that

$$-\frac{1}{\pi i \omega} J(\alpha_1, \alpha_2) = s + t + 2 \sum_{n,m \geq 1}^{\infty} \frac{t^m s^n}{m!n!} C_{n+m-2}^{m-1}, \quad (112)$$

where, as before,  $t = -i\alpha_2/\omega$  and  $s = i\alpha_1/\omega$ .

The expansion of the pre-exponent factor in (68) can be simplified by the following identity ([15], p.32)

$$J_n^2(z) = \sum_{m=0}^{\infty} \frac{(-1)^m (2m+2n)!}{m!(m+2n)![(m+n)!]^2} \left(\frac{z}{2}\right)^{2m+2n}. \quad (113)$$

One gets

$$J_0^2(\xi) + J_1^2(\xi) = 1 + \sum_{m=1}^{\infty} \frac{(2m)!}{[m!]^3 (m+1)!} (\alpha_1 \alpha_2)^m. \quad (114)$$

Changing  $\alpha_2 \rightarrow -\alpha_2$  we can rewrite Eq.(90) in the form

$$r_2(\Omega) = 1 + \int_0^{\infty} d\alpha_1 d\alpha_2 P(\alpha_1, \alpha_2) \exp(-\Omega(\pi + \frac{i}{v'})\alpha_1 - \Omega(\pi - \frac{i}{v'})\alpha_2) \exp(2i(\alpha_1 + \alpha_2) + 2\pi i \Omega Q(\alpha_1, \alpha_2)) + c.c., \quad (115)$$

where

$$P(\alpha_1, \alpha_2) = 1 + \sum_{m=1}^{\infty} \frac{(2m)!}{[m!]^3 (m+1)!} (-\alpha_1 \alpha_2)^m, \quad (116)$$

and

$$Q(\alpha_1, \alpha_2) = \sum_{n,m \geq 1}^{\infty} i^{m+n} \frac{\alpha_2^m \alpha_1^n}{m!n!} C_{n+m-2}^{m-1}. \quad (117)$$



## VI. LIMITING BEHAVIOR

The above formulas give the exact expressions for the 2-point correlation function for the problem considered but they are quite cumbersome and suitable mostly for numerical calculations. The most interesting information which one can extract from them is the behavior of the 2-point correlation function at small and large  $\Omega$ . The purpose of this section is to discuss methods which permit to find these asymptotics without knowledge of the exact solution.

It is clear that in order to find the behavior of the 2-point correlation function in the limit  $\omega \rightarrow 0$  it is necessary to consider only the case when three initial levels (which we shall denote  $e_1, e_2, e_3$ ) are close each to others and all other levels are far from this triple. In other words only three terms in Eq. (47) are big. In such a case Eq. (47) which should determine the positions of two nearest levels can be approximated as follows

$$\frac{1}{E - e_1} + \frac{1}{E - e_2} + \frac{1}{E - e_3} = 0. \quad (118)$$

(Note the absence of the coupling constant.) The solution of this equation is

$$E_{1,2} = \frac{e_1 + e_2 + e_3}{3} \pm \frac{1}{3} \sqrt{e_1^2 + e_2^2 + e_3^2 - e_1 e_2 - e_1 e_3 - e_2 e_3}. \quad (119)$$

This expression is translationally invariant, therefore one can assume that  $e_1 + e_2 + e_3 = 0$  and the difference between two adjacent levels is

$$\Delta E = \frac{2}{\sqrt{3}} \sqrt{e_1^2 + e_2^2 - e_1 e_2}. \quad (120)$$

After the corresponding rescaling the 2-point correlation function at the limit  $\Omega \rightarrow 0$  takes the form

$$r_2(\Omega) = \frac{1}{2} \int \delta(\Omega - \frac{2}{\sqrt{3}} \sqrt{e_1^2 + e_2^2 - e_1 e_2}) de_1 de_2. \quad (121)$$

The factor 1/2 comes from the restriction  $e_1 < e_2$ . Changing variables  $e_1 = r \cos \theta$ ,  $e_2 = r \sin \theta$  and performing the integral over  $r$  one gets

$$r_2(\Omega) = \frac{3}{8} \Omega \int \frac{d\theta}{1 - \sin \theta \cos \theta}. \quad (122)$$

The last integral equals  $4\pi/\sqrt{3}$  and finally we obtain that in the limit of small  $\Omega$

$$r_2(\Omega) = \frac{\pi\sqrt{3}}{2} \Omega, \quad (123)$$

which coincides with the result (89) obtained above by different method.

To compute the behavior of the 2-point correlation function at large  $\Omega$  it is convenient to use a method based on the usual trace formula

Let us define

$$G(z) = \sum_{i=1}^N \frac{1}{z - e_i}, \quad (124)$$

where all  $e_i$  are independent random variables as before.

We need to calculate the density of levels  $E_j$  defined by the equation

$$vG(E_j) = 1. \quad (125)$$

Formally this density can be expressed in the following way

$$\rho(E) = -\frac{1}{\pi} \text{Im}[G(E) + \frac{\partial}{\partial E} \log(1 - vG(E))], \quad (126)$$

where the symbol  $\text{Im}[F(E)]$  means the following limit

$$\text{Im}[F(E)] = \lim_{\epsilon \rightarrow 0} \frac{1}{2i} (F(E + i\epsilon) - F(E - i\epsilon)), \quad (127)$$

taken over positive  $\epsilon$ .

The derivation of Eq.(126) is simple. The function  $1 - \lambda G(E)$  has zeros at  $E_j$  and poles at  $e_k$ , therefore

$$\frac{\partial}{\partial E} \ln(1 - vG(E)) = \sum_j \frac{1}{E - E_j} - \sum_k \frac{1}{E - e_k}. \quad (128)$$

The first term in (126) cancels the poles from unperturbed levels and the imaginary part produces  $\delta$ -function singularity at the required positions.

Let us denote

$$G_{\pm}(E) = G(E \pm i\epsilon), \quad (129)$$

where  $\epsilon$  is positive and  $\epsilon \rightarrow 0$ .

In calculating the mean values it is useful to take explicitly into account the mean values of  $G_{\pm}(E)$ . Using the relation

$$\frac{1}{x \pm i\epsilon} = P \frac{1}{x} \mp i\pi\delta(x), \quad (130)$$

one finds

$$\langle G_{\pm}(E) \rangle = \mp \pi i \bar{\rho} + \bar{\rho} \log\left(\frac{W + E}{W - E}\right), \quad (131)$$

where  $\bar{\rho}$  is the mean level density of non-perturbed states. Introducing

$$g_{\pm}(E) = G_{\pm}(E) - \langle G_{\pm}(E) \rangle, \quad (132)$$

one can write

$$1 - vG_{\pm}(E) = (1 - v \langle G_{\pm}(E) \rangle)(1 - \lambda g_{\pm}(E)), \quad (133)$$

where

$$\lambda = \frac{v'}{\bar{\rho}(1 \pm i\pi v')}, \quad (134)$$

and  $v'$  is the renormalized coupling constant the same as in Eq.(38)

$$\frac{1}{v'} = \frac{1}{\bar{\rho}v} + \ln \frac{W - E}{W + E}. \quad (135)$$

The density of states (ignoring small correction to the mean density of state as in Eq. (39) now will take the form

$$\rho(E) = \bar{\rho} - \frac{1}{2\pi i}(g_+(E) - g_-(E)) - \frac{1}{2\pi i} \frac{\partial}{\partial E} (\log(1 - \lambda g_+(E)) - \log(1 - \lambda^* g_-(E))). \quad (136)$$

The 2-point correlation function is the mean value of the product of two such expressions at different energies. The computation of the mean value can be done in perturbation theory by expanding this expression into powers of  $g_{\pm}(E)$  and using a formula

$$\begin{aligned} \langle g_+^n(E_1)g_-^m(E_2) \rangle &\approx \bar{\rho} \int \frac{de}{(E_1 - e + i\epsilon)^n (E_2 - e - i\epsilon)^m} \\ &= 2\pi i (-1)^{n-1} \bar{\rho} C_{m+n-2}^{m-1} \frac{1}{(\omega)^{m+n-1}} (1 + O(\omega)), \end{aligned} \quad (137)$$

where  $\omega = E_1 - E_2$ . Therefore one can organize the perturbation series in a series of inverse power of  $\omega$ .

Taking into account the first terms in the expansion of the logarithm in the above expression one gets

$$\rho(E) = \bar{\rho} - \frac{1}{2\pi i} [(1 - \lambda \frac{\partial}{\partial E})g_+(E) - (1 - \lambda^* \frac{\partial}{\partial E})g_-(E)]. \quad (138)$$

At large  $\omega$

$$\begin{aligned} R_2(\omega) &= \bar{\rho}^2 - \frac{1}{4\pi^2} (\lambda \frac{\partial}{\partial E_1} g_+(E_1)g_-(E_2) + (E_1 \leftrightarrow E_2) + c.c.) \\ &= \bar{\rho}^2 + \frac{2}{\omega^2(\pi^2 + 1/v'^2)}, \end{aligned} \quad (139)$$

which agrees with Eq. (92) derived from the general formula.

We stress that the methods used in this Section are not restricted to particular cases considered. They also can be used in more general situations where exact solution are not available, e.g. for rank-two perturbations (2 short-range impurities) and similar problems.

## VII. GENERAL CASE

In the previous Sections we considered the calculation of the 2-point correlation function under the assumption of equality of all residues. Here the generalization of these calculations to the case of different residues is presented.

When the residues are different one has instead of Eq. (47) the following equation

$$\sum_{j=1}^N \frac{r_j}{E - e_j} = 1, \quad (140)$$

and, consequently, instead of Eq. (49) one gets a more general relation

$$\begin{aligned} R_2(E_1, E_2) &= \int_{-\infty}^{\infty} \frac{d\alpha_1 d\alpha_2}{4\pi^2} e^{-i(\alpha_1 + \alpha_2)} \sum_{k=1}^N r_k^2 g(r_k \alpha_1, r_k \alpha_2) \prod_{j \neq k} f(r_j \alpha_1, r_j \alpha_2) \\ &+ \sum_{k_1 \neq k_2} r_{k_1} r_{k_2} \psi_1(r_{k_1} \alpha_1, r_{k_1} \alpha_2) \psi_2(r_{k_2} \alpha_1, r_{k_2} \alpha_2) \prod_{j \neq k_1, k_2} f(r_j \alpha_1, r_j \alpha_2), \end{aligned} \quad (141)$$

where  $f(\alpha_1, \alpha_2)$ ,  $g(\alpha_1, \alpha_2)$ , and  $\psi_i(\alpha_1, \alpha_2)$  are the same as in Eqs. (50). Repeating the same steps as in Section IV one gets exact expressions for the 2-point correlation function. The analog of Eq. (76) (which is convenient for calculation of the small- $\Omega$  series of the 2-point correlation function) has the following form

$$\begin{aligned} r_2(\Omega) &= - \int \frac{d\alpha_1 d\alpha_2}{4\pi^2} e^{-2\pi\Omega < \bar{J}(r\alpha_1, r\alpha_2) > - i\Omega v(\alpha_1 + \alpha_2)/v'} \\ &\times \left[ \frac{\partial^2}{2\partial\alpha_1 \partial\alpha_2} < \Phi(r\alpha_1, r\alpha_2) e^{ir(\alpha_1 - \alpha_2)} >^2 \right. \\ &+ i \left( \frac{\partial}{\partial\alpha_1} - \frac{\partial}{\partial\alpha_2} \right) < \Phi(r\alpha_1, r\alpha_2) e^{ir(\alpha_1 - \alpha_2)} > \\ &\left. \times < \Phi(r\alpha_1, r\alpha_2) r e^{ir(\alpha_1 - \alpha_2)} > \right]. \end{aligned} \quad (142)$$

Instead of Eq. (90) useful for large- $\Omega$  asymptotics one obtains

$$\begin{aligned} r_2(\Omega) &= 1 + \left\{ \int_0^{\infty} d\alpha_1 \int_{-\infty}^0 d\alpha_2 e^{-2\pi\Omega < \bar{J}(r\alpha_1, r\alpha_2) > - i\Omega v(\alpha_1 + \alpha_2)/v'} \right. \\ &\times [ < r J_0^2(2r\sqrt{-\alpha_1\alpha_2}) e^{ir(\alpha_1 - \alpha_2)} >^2 \\ &+ < r J_1^2(2r\sqrt{-\alpha_1\alpha_2}) e^{ir(\alpha_1 - \alpha_2)} >^2 ] + c.c. \} \end{aligned} \quad (143)$$

Here  $< f(r) >$  denotes the mean value over all residues

$$< f(r) > = \frac{1}{N} \sum_{j=1}^N f(r_j), \quad (144)$$

functions  $\bar{J}(\alpha_1, \alpha_2)$  and  $\Phi(\alpha_1, \alpha_2)$  are defined in Eqs. (78) and (172),  $v$ , and  $v'$  are ‘bare’ and ‘renormalized’ coupling constants (see (37) and (38)).

As in Section IV it is of interest to compute the behavior of the 2-point correlation function at small and large energy difference. When  $\Omega \rightarrow 0$  the integration by parts as in Section IV leads to

$$r_2(\Omega) \rightarrow \Omega A, \quad (145)$$

where

$$\begin{aligned}
A &= \int \frac{d\alpha_1 d\alpha_2}{4\pi^2} \left[ \frac{i}{2} \langle \Phi(r\alpha_1, r\alpha_2) r^2 e^{ir(\alpha_1 - \alpha_2)} \rangle \langle \Phi(r\alpha_1, r\alpha_2) e^{ir(\alpha_1 - \alpha_2)} \rangle^2 \right. \\
&\quad \left. + \langle \Phi(r\alpha_1, r\alpha_2) r e^{ir(\alpha_1 - \alpha_2)} \rangle^2 \langle \Phi(r\alpha_1, r\alpha_2) e^{ir(\alpha_1 - \alpha_2)} \rangle \right].
\end{aligned} \tag{146}$$

Using (172) this triple sum is transformed to the form (cf. with Eq. (83))

$$\begin{aligned}
A &= -i \frac{\pi}{N^3} \sum_{i,j,k=1}^N \left( \frac{1}{2} r_j^2 + r_j r_k \right) \int_0^\infty \xi d\xi J_0(r_i \xi) J_0(r_j \xi) J_0(r_k \xi) \\
&\quad \times \int_0^\infty \frac{d\eta}{\eta} e^{i(r_i + r_j + r_k)(\eta + 1/\eta)\xi/2} + c.c.
\end{aligned} \tag{147}$$

According to (169) the last integral equals  $i\pi H_0^{(1)}((r_i + r_j + r_k)\xi)$  therefore

$$A = \frac{2\pi^2}{N^3} \sum_{i,j,k=1}^N \left( \frac{1}{2} r_j^2 + r_j r_k \right) \int_0^\infty \xi d\xi J_0(r_i \xi) J_0(r_j \xi) J_0(r_k \xi) J_0((r_i + r_j + r_k)\xi). \tag{148}$$

Taking into account Eq. (85) and symmetrising the answer one gets

$$A = \frac{\pi}{6} \frac{1}{N^3} \sum_{i,j,k=1}^N \sqrt{\frac{(r_i + r_j + r_k)^3}{r_i r_j r_k}}. \tag{149}$$

Of course, when all residues are equal,  $A = \pi\sqrt{3}/2$  as in (89).

When  $\Omega \rightarrow \infty$  from Eq. (143) one gets

$$\begin{aligned}
r_2(\Omega) &\rightarrow 1 + \langle r \rangle^2 \int_0^\infty d\alpha_1 \int_{-\infty}^0 d\alpha_2 e^{-\Omega \langle r \rangle (\pi(|\alpha_1| + |\alpha_2|) - \frac{i}{v^2}(\alpha_1 + \alpha_2))} + c.c. \\
&= 1 + \frac{2}{\Omega^2(\pi^2 + 1/v^2)},
\end{aligned} \tag{150}$$

which differs from Eq. (92) only by a suitable definition of the coupling constant.

Note that Eq. (149) is valid only for non-zero values of the residues. Otherwise, the pre-factor  $A$  formally diverges. This divergence is a consequence of a simple fact that when some  $r_j = 0$  there exist certain energy levels exactly equal unperturbed levels. Therefore the set of new energy levels consists of two parts. The first includes energy levels which are changed by the perturbation. Their correlation function is given by the formulas above where only non-zero residues are taken into account. The second part consists of energy levels which are not changed by the perturbation. Their correlation functions are the same as for the Poisson process and, in particular, they do not display level repulsion. As the cross-correlations between (a finite number) of old and new energy levels disappear when  $N \rightarrow \infty$ , the resulting statistics is a superposition of two independent distributions and, in general, it will not have level repulsion (i.e.  $R_2(\epsilon) \neq 0$  when  $\epsilon \rightarrow 0$ ).

The above case is realized e.g. when a  $\delta$ -function potential (11) is added to a rectangular billiard with the Dirichlet boundary conditions and the positions of the singular point ( $\vec{x}_0 = (x_0, y_0)$ ) are commensurable with the corresponding sides of the rectangular ( $a$  and  $b$ ). In this model unperturbed wave functions are determined by two integers,  $n$  and  $m$ ,

$$\psi_{\vec{n}} = \frac{2}{\sqrt{ab}} \sin\left(\frac{\pi}{a}nx\right) \sin\left(\frac{\pi}{b}my\right) \quad (151)$$

and the residues are

$$r_{\vec{n}} = \frac{4}{ab} \sin^2\left(\frac{\pi}{a}nx_0\right) \sin^2\left(\frac{\pi}{b}my_0\right). \quad (152)$$

If

$$\frac{x_0}{a} = \frac{p_1}{q_1}, \quad \frac{y_0}{b} = \frac{p_2}{q_2} \quad (153)$$

for co-prime integers  $p_i$  and  $q_i$  there exist only a finite number of different residues depending on values  $n \bmod q_1$  and  $m \bmod q_2$ . In particular, when  $n$  is divisible by  $q_1$  or  $m$  is divisible by  $q_2$ ,  $r_{\vec{n}} = 0$ . It means that for all these values of  $n$  and  $m$  wave functions and energy eigenvalues will not be changed by the perturbation and the resulting distribution (included all energy levels) will not describe level repulsion.

Another interesting case corresponds to a model when all residues are also independent random variables with a probability  $d\mu(r)$ . If  $r_j$  never take very small values (more precisely, the mean value of  $1/\sqrt{r}$  is finite) the only modification is that mean value over residues,  $\langle f(r) \rangle$ , should be taken over the given distribution i.e. instead of Eq. (144) one has to use

$$\langle f(r) \rangle = \int f(r) d\mu(r). \quad (154)$$

In particular the value of the pre-factor  $A$  is

$$A = \frac{\pi}{6} \int d\mu(r_1) d\mu(r_2) d\mu(r_3) \sqrt{\frac{(r_1 + r_2 + r_3)^3}{r_1 r_2 r_3}}. \quad (155)$$

But, if the probability of small values of residues is large, certain expansions should be modified. A natural example is e.g. the Seba billiard with the Dirichlet boundary conditions when ratios of the positions of the singularity to the corresponding sides (as in (153)) are non-commensurable irrational numbers. In this case  $r_n$  defined in (152) are equivalent to random variables

$$r_{\vec{\phi}} = \frac{4}{ab} \sin^2 \phi_1 \sin^2 \phi_2, \quad (156)$$

where angles  $\phi_1$  and  $\phi_2$  are uniformly distributed between 0 and  $\pi$ .

Now the 2-point correlation function at small  $\Omega$  will differ from Eq. (145). Indeed, formal calculation of the pre-factor (155) shows that it diverges at small  $r$  and its leading behavior is the following

$$A \rightarrow \frac{\pi}{2} \langle r \rangle \langle \frac{1}{\sqrt{r}} \rangle^2. \quad (157)$$

But for variable (156)

$$\langle r \rangle = \frac{4}{\pi^2 ab} \left( \int_0^\pi \sin^2 \phi d\phi \right)^2 = \frac{1}{ab}, \quad (158)$$

and

$$\left\langle \frac{1}{\sqrt{r}} \right\rangle = \frac{\sqrt{ab}}{2\pi^2} \left( \int_{\phi_0}^{\pi} \frac{d\phi}{\sin \phi} \right)^2 \approx \frac{\sqrt{ab}}{2\pi^2} \ln^2 \phi_0, \quad (159)$$

where  $\phi_0$  is a cut-off of the integration over  $\phi$ . With logarithmic accuracy  $\phi_0$  is proportional to  $\Omega$ ,  $\phi_0 \rightarrow \Omega/\Omega_0$ , and when  $\Omega \rightarrow 0$

$$r_2(\Omega) \rightarrow \frac{1}{8\pi^3} \Omega \ln^4(\Omega/\Omega_0). \quad (160)$$

The results of numerical calculations of 100000 levels of the Seba billiard with Dirichlet boundary conditions (with irrational ratios (153) and  $v' = 1$ ) are presented at Figs. 3 and 4. In Fig. 3 the cumulative nearest-neighbor distribution is plotted. The dashed and thin solid lines are the same as in Fig. 1, the dotted line corresponds to the Poisson distribution

$$N_P(s) = 1 - e^{-s}. \quad (161)$$

Note that the computed distribution is quite far from all standard examples. Though the resulting distribution is more close to the Poisson distribution than the one for the Seba billiard with periodic boundary conditions (see Fig. 1) one can check that this difference will be present for all non-zero values of the coupling constant (and, in particular, when  $v' \rightarrow \infty$ ).

The 2-point correlation function is shown in Fig. 4. The limiting behavior for small (see (160)) and large (see (150)) values of arguments are also indicated for comparison by thick solid lines. The value of  $\Omega_0$  in Eq. (160),  $\Omega_0 = 52.25$ , has been obtained by fitting expression (160) to numerical result for small  $\Omega$ .

## VIII. CONCLUSION

We have computed analytically the 2-point correlation function for zeros of random meromorphic functions with large number of poles when these poles are independent random variables. It is demonstrated that the statistics of these zeros corresponds to a distribution with level repulsion which differs from known examples. The resulting distribution is not universal but depends on residues.

A natural example, where energy levels obey such meromorphic equation, corresponds to a rectangular billiard with a short-range impurity and our results give the spectral statistics of these models. It is of interest that different boundary conditions give very different results. Even the asymptotic behavior for small energy difference is different (cf. Eqs. (80) and (160)).

We also proposed methods which permit to find the behavior of the 2-point correlation function at small and large arguments without the knowledge of the exact solution. The methods can be applied for cases where exact solutions are not available.

## IX. APPENDIX.

The purpose of this Appendix is to compute the main integral (52)

$$I(\alpha_1, \alpha_2) = \int_{-W}^W (1 - \exp(i\frac{\alpha_1}{E_1 - e} + i\frac{\alpha_2}{E_2 - e})) de. \quad (162)$$

Let us first derive a few useful relations.

$$(\frac{\partial}{\partial \alpha_1} - \frac{\partial}{\partial \alpha_2})J(\alpha_1, \alpha_2) = i\omega \int_{-\infty}^{\infty} \exp(i\frac{\alpha_1}{E_1 - e} + i\frac{\alpha_2}{E_2 - e}) \frac{de}{(E_1 - e)(E_2 - e)}, \quad (163)$$

where

$$\omega = E_1 - E_2. \quad (164)$$

Perform in this integral the change of variable

$$E_1 - e = \frac{\omega}{1+t}. \quad (165)$$

Now

$$E_2 - e = -\frac{t\omega}{1+t}, \quad (E_1 - e)(E_2 - e) = -\frac{t\omega^2}{(1+t)^2}, \quad de = \frac{\omega}{(1+t)^2} dt. \quad (166)$$

Hence

$$(\frac{\partial}{\partial \alpha_1} - \frac{\partial}{\partial \alpha_2})J(\alpha_1, \alpha_2) = -i \exp(i\frac{\alpha_1 - \alpha_2}{\omega}) \int_{-\infty}^{\infty} \exp(i\frac{\alpha_1}{\omega}(t - \frac{\alpha_2}{\alpha_1 t})) \frac{dt}{t}. \quad (167)$$

The following standard integrals will be useful for us (see [14]). When  $\text{Im}\mu > 0$  and  $\text{Im}\beta^2\mu < 0$

$$\int_0^{\infty} t^{\nu-1} \exp(i\frac{\mu}{2}(t - \frac{\beta^2}{t})) dt = 2\beta^\nu e^{i\nu\pi/2} K_{-\nu}(\beta\mu), \quad (168)$$

and when  $\text{Im}\mu > 0$  and  $\text{Im}\beta^2\mu > 0$

$$\int_0^{\infty} t^{\nu-1} \exp(i\frac{\mu}{2}(t + \frac{\beta^2}{t})) dt = i\pi\beta^\nu e^{-i\pi\nu/2} H_{-\nu}^{(1)}(\beta\mu). \quad (169)$$

Here  $K_\nu(x)$  and  $H_\nu^{(1)}(x) = J_\nu(x) + iY_\nu(x)$  are respectively the Macdonald and Hankel functions.

Note that  $K_0(x)$  is a real function, therefore

$$\begin{aligned} \int_{-\infty}^{\infty} \exp(i\frac{\alpha_1}{\omega}(t - \frac{\alpha_2}{\alpha_1 t})) \frac{dt}{t} &= \int_0^{\infty} \exp(i\frac{\alpha_1}{\omega}(t - \frac{\alpha_2}{\alpha_1 t})) \frac{dt}{t} - c.c. \\ &= \begin{cases} 0, & \text{if } \alpha_1\alpha_2 > 0 \\ 2\pi i J_0(-\frac{2}{\omega}\sqrt{\alpha_1\alpha_2}) \text{sgn}(\alpha_1), & \text{if } \alpha_1\alpha_2 < 0 \end{cases}, \end{aligned} \quad (170)$$

and  $J_0(x)$  is the Bessel function of zero order. Finally

$$(\frac{\partial}{\partial \alpha_1} - \frac{\partial}{\partial \alpha_2})J(\alpha_1, \alpha_2) = \exp(i\frac{\alpha_1 - \alpha_2}{\omega}) \Phi(\alpha_1, \alpha_2), \quad (171)$$

where



$$\Phi(\alpha_1, \alpha_2) = 2\pi J_0\left(\frac{2}{\omega}\sqrt{-\alpha_1\alpha_2}\right)\Theta(-\alpha_1\alpha_2)\text{sgn}(\alpha_1), \quad (172)$$

$\Theta(x) = 1$  if  $x > 0$  and  $\Theta(x) = 0$  if  $x < 0$ . Note that

$$\frac{\partial^2}{\partial\alpha_1\partial\alpha_2}\Phi(\alpha_1, \alpha_2) = \frac{1}{\omega^2}\Phi(\alpha_1, \alpha_2). \quad (173)$$

Using the same method one can prove the following set of equalities

$$\frac{\partial^2}{\partial\alpha_1\partial\alpha_2}J(\alpha_1, \alpha_2) = -\frac{i}{\omega}\exp\left(i\frac{\alpha_1 - \alpha_2}{\omega}\right)\Phi(\alpha_1, \alpha_2). \quad (174)$$

$$\frac{\partial^2}{\partial\alpha_1^2}J(\alpha_1, \alpha_2) = \exp\left(i\frac{\alpha_1 - \alpha_2}{\omega}\right)\frac{\partial}{\partial\alpha_1}\Phi(\alpha_1, \alpha_2). \quad (175)$$

$$\frac{\partial^2}{\partial\alpha_2^2}J(\alpha_1, \alpha_2) = -\exp\left(i\frac{\alpha_1 - \alpha_2}{\omega}\right)\frac{\partial}{\partial\alpha_2}\Phi(\alpha_1, \alpha_2). \quad (176)$$

$$\frac{\partial^4}{\partial\alpha_1^2\partial\alpha_2^2}J(\alpha_1, \alpha_2) = -\frac{1}{\omega^2}\left(\frac{\partial}{\partial\alpha_1} - \frac{\partial}{\partial\alpha_2}\right)\left[\exp\left(i\frac{\alpha_1 - \alpha_2}{\omega}\right)\Phi(\alpha_1, \alpha_2)\right]. \quad (177)$$

Note that the differentiation of function  $\Phi$  produces the  $\delta$ -functions coming from the factor

$$\Theta(-\alpha_1\alpha_2)\text{sgn}(\alpha_1) = \frac{1}{2}(\text{sgn}(\alpha_1) - \text{sgn}(\alpha_2)). \quad (178)$$

From Eq. (175) it follows that the second derivative

$$\frac{\partial^2}{\partial\alpha_1^2}J(\alpha_1, \alpha_2) = \phi_1(\alpha_1, \alpha_2) \quad (179)$$

is known. Therefore

$$J(\alpha_1, \alpha_2) = J(0, \alpha_2) + \frac{\partial J(0, \alpha_2)}{\partial\alpha_1} + \int_0^{\alpha_1} (\alpha_1 - y)\phi_1(y, \alpha_2)dy, \quad (180)$$

where  $J(0, \alpha_2)$  and  $\partial J(0, \alpha_2)/\partial\alpha_1$  are values of  $J$  and its first derivative at  $\alpha_1 = 0$ .

In Section III it was demonstrated that

$$J(0, \alpha_2) = \pi|\alpha_2|$$

and symmetrically

$$J(\alpha_1, 0) = \pi|\alpha_1|.$$

As the second derivatives equal zero when  $\alpha_1\alpha_2 > 0$  the expression of  $J$  in these regions which is continuous when crossing the  $\alpha_1$  and  $\alpha_2$  axis is

$$J(\alpha_1, \alpha_2) = \pi(\alpha_1 + \alpha_2)\text{sgn}(\alpha_2). \quad (181)$$

It is clear that the function  $J(\alpha_1, \alpha_2)$  is a continuous function but with discontinuous first derivatives. The values of these discontinuities follow from the above discontinuity of the function  $\Phi$ .

Therefore in the region

$$\alpha_1 > 0, \alpha_2 < 0, \alpha_1 + \alpha_2 < 0$$

(which is the continuation of the lower left square  $\alpha_1 < 0, \alpha_2 < 0$  through the negative  $\alpha_2$  axis) the function  $J(\alpha_1, \alpha_2)$  should take the form

$$J(\alpha_1, \alpha_2) = -\pi(\alpha_2 + \alpha_1) + 2\pi\alpha_1 \exp(-i\alpha_2/\omega) + \int_0^{\alpha_1} (\alpha_1 - y)\phi_0(y, \alpha_2)dy, \quad (182)$$

where

$$\phi_0(\alpha_1, \alpha_2) = \exp(i\frac{\alpha_1 - \alpha_2}{\omega}) \frac{\partial}{\partial \alpha_1} \phi(\alpha_1, \alpha_2), \quad (183)$$

and

$$\phi(\alpha_1, \alpha_2) = 2\pi J_0(\frac{2}{\omega}\sqrt{-\alpha_1\alpha_2}) \quad (184)$$

coincides with the function  $\Phi$  but without the discontinuous factor.

After integration by part and certain transformations one obtains that in all regions

$$\begin{aligned} J(\alpha_1, \alpha_2) &= \pi(\alpha_1 + \alpha_2)\text{sgn}(\alpha_2) - \pi[i(\alpha_1 + \alpha_2)G(-\frac{\alpha_2}{\omega}, \frac{\alpha_1}{\omega}) \\ &+ (\alpha_2 J_0(\xi) + i\sqrt{-\alpha_1\alpha_2}J_1(\xi)) \exp(i\frac{\alpha_1 - \alpha_2}{\omega})](\text{sgn}(\alpha_1) - \text{sgn}(\alpha_2)), \end{aligned} \quad (185)$$

where  $\xi = \frac{2}{\omega}\sqrt{-\alpha_1\alpha_2}$  and

$$G(x, y) = e^{iy} \int_x^\infty J_0(2\sqrt{yt})e^{it} dt. \quad (186)$$

The function  $G(x, y)$  obeys the following relations

$$\frac{\partial G(x, y)}{\partial x} = -J_0(\xi)e^{ix+iy}, \quad (187)$$

$$\frac{\partial G(x, y)}{\partial y} = -i\sqrt{\frac{x}{y}}J_1(\xi)e^{ix+iy}, \quad (188)$$

where  $\xi = 2\sqrt{xy}$ . To prove the second identity note that

$$\frac{\partial G(x, y)}{\partial y} = e^{iy} \int_x^\infty (iJ_0 + J_0' \sqrt{\frac{t}{y}})e^{it} dt. \quad (189)$$

But the integrand equals

$$-i\frac{\partial}{\partial t}(\sqrt{t/y}J_0'(2\sqrt{ty})e^{it}), \quad (190)$$

and as  $J_0' = -J_1$  one gets the above relation.

The functional equation

$$G(x, y) + G(y, x) = i + iJ_0(\xi)e^{ix+iy} \quad (191)$$

can be proved by comparing the derivatives of both sides and noting that the integral ([14], p.50)

$$\int_0^{\infty} J_0(at)e^{-\gamma t^2} dt = \frac{1}{(2\gamma)} \exp\left(-\frac{a^2}{4\gamma}\right) \quad (192)$$

requires that  $G(0, y) = i$ .

When  $y < x$  the above integral can be indefinitely taken by parts which leads to expansion

$$G(x, y) = ie^{ix+iy} \sum_{n=0}^{\infty} (-i\eta)^n J_n(\xi), \quad (193)$$

where  $\eta = \sqrt{y/x}$  and  $\xi = 2\sqrt{xy}$ .

The above collection of formulae permits to find the expansion of  $G(x, y)$  for all values of its arguments.

Another useful representation of our integral is

$$\begin{aligned} J(\alpha_1, \alpha_2) &= \pi(\alpha_1 + \alpha_2)\text{sgn}(\alpha_2) \\ &\quad - \pi\omega \left[ i(y-x)G(x, y) + \left( x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y} \right) G(x, y) \right] (\text{sgn}(x) + \text{sgn}(y)), \end{aligned} \quad (194)$$

where  $x = -\alpha_2/\omega, y = \alpha_1/\omega$ .

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# FIGURES

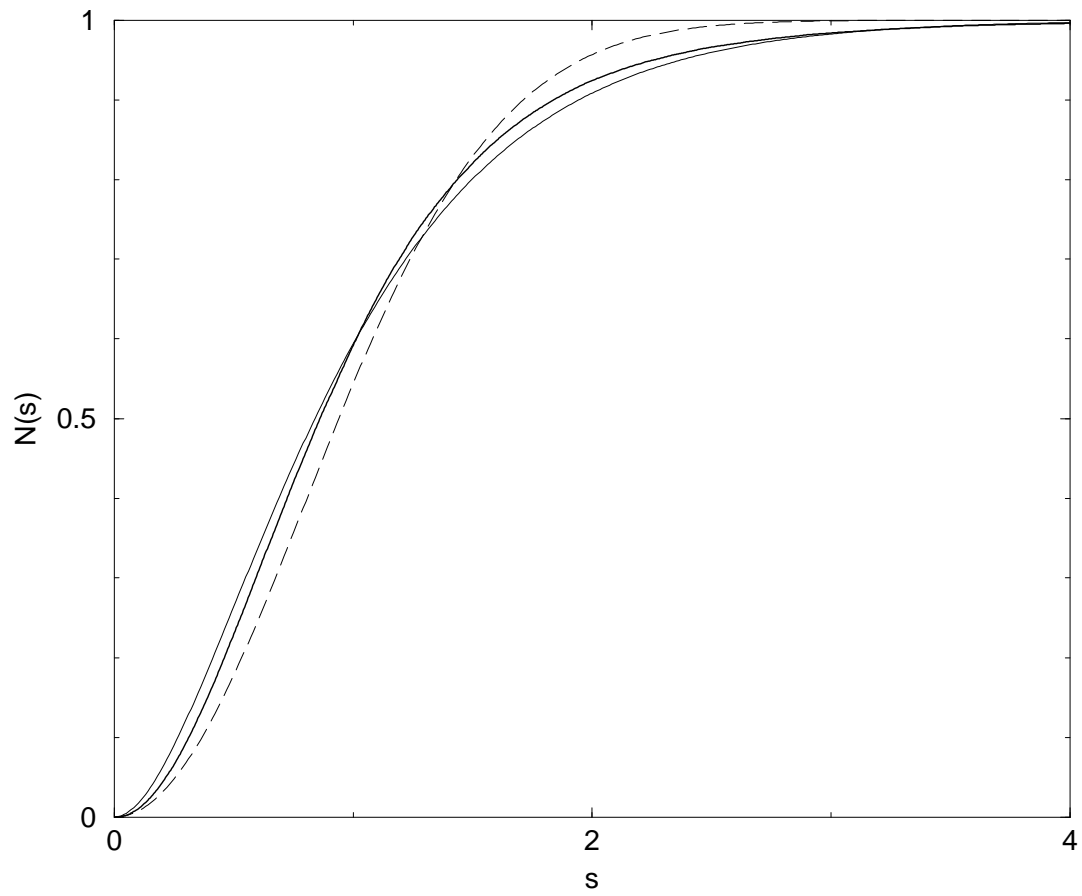


FIG. 1. Nearest-neighbor distribution for the Seba billiard with the periodic boundary conditions. Dashed line is the GOE result. Thin line is the semi-Poisson curve.

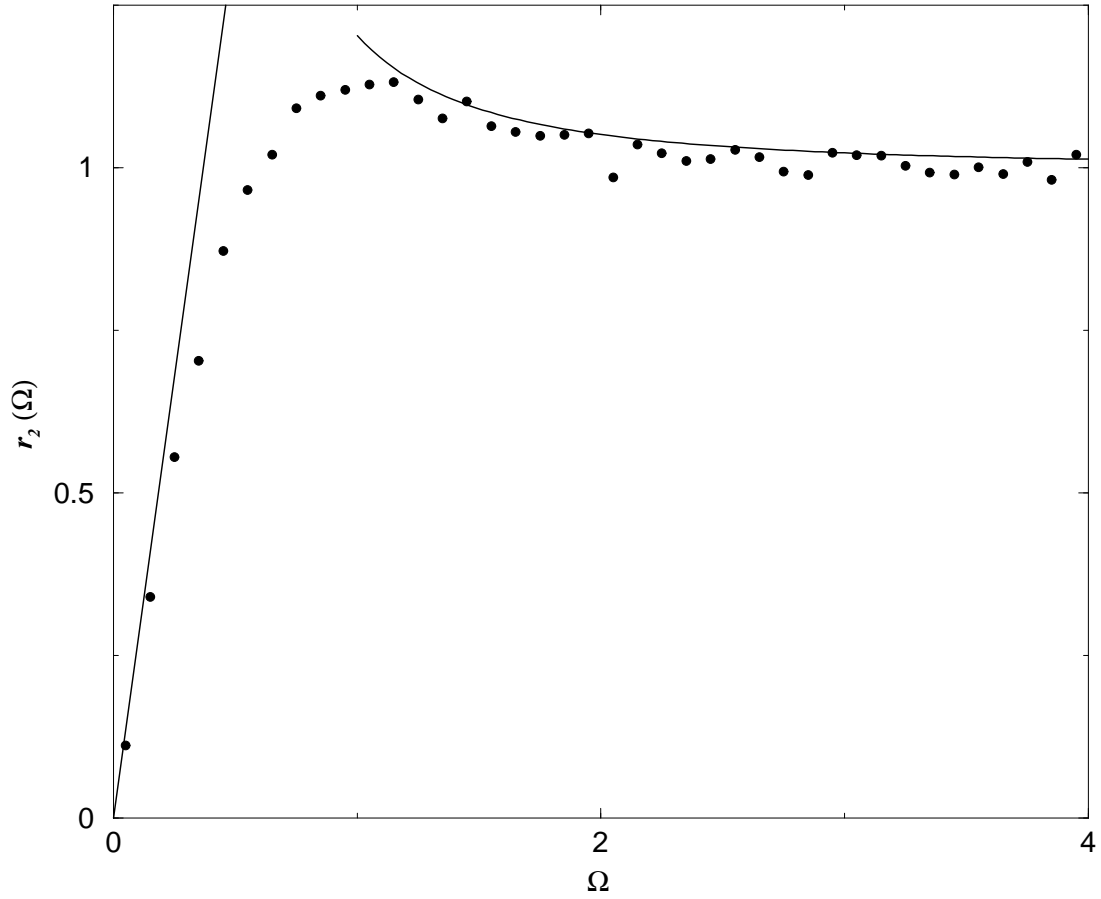


FIG. 2. The 2-point correlation function for the Seba billiard with the periodic boundary conditions. Solid lines correspond to the asymptotics (80) and (92) for small and large values of energy difference.

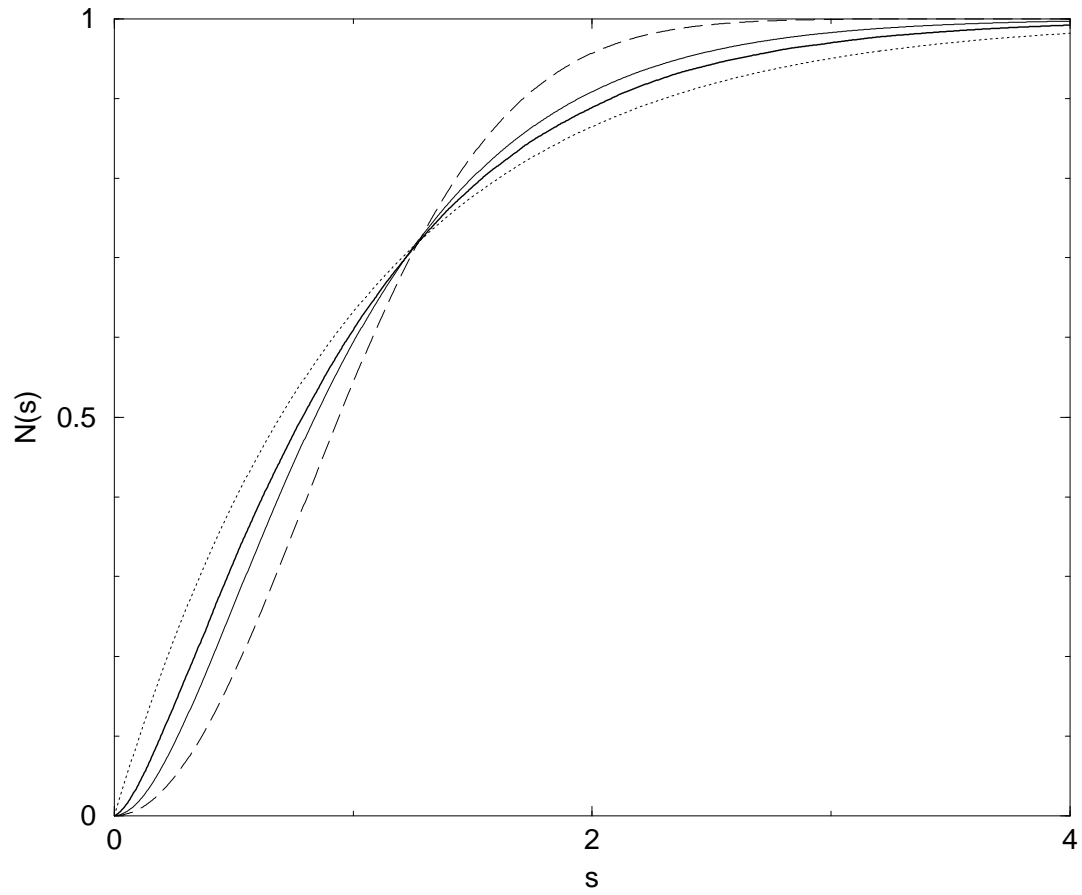


FIG. 3. Nearest-neighbor distribution for the Seba billiard with the Dirichlet boundary conditions. Dashed line is the GOE result. Thin line is the semi-Poisson curve. Dotted line is the Poisson prediction.

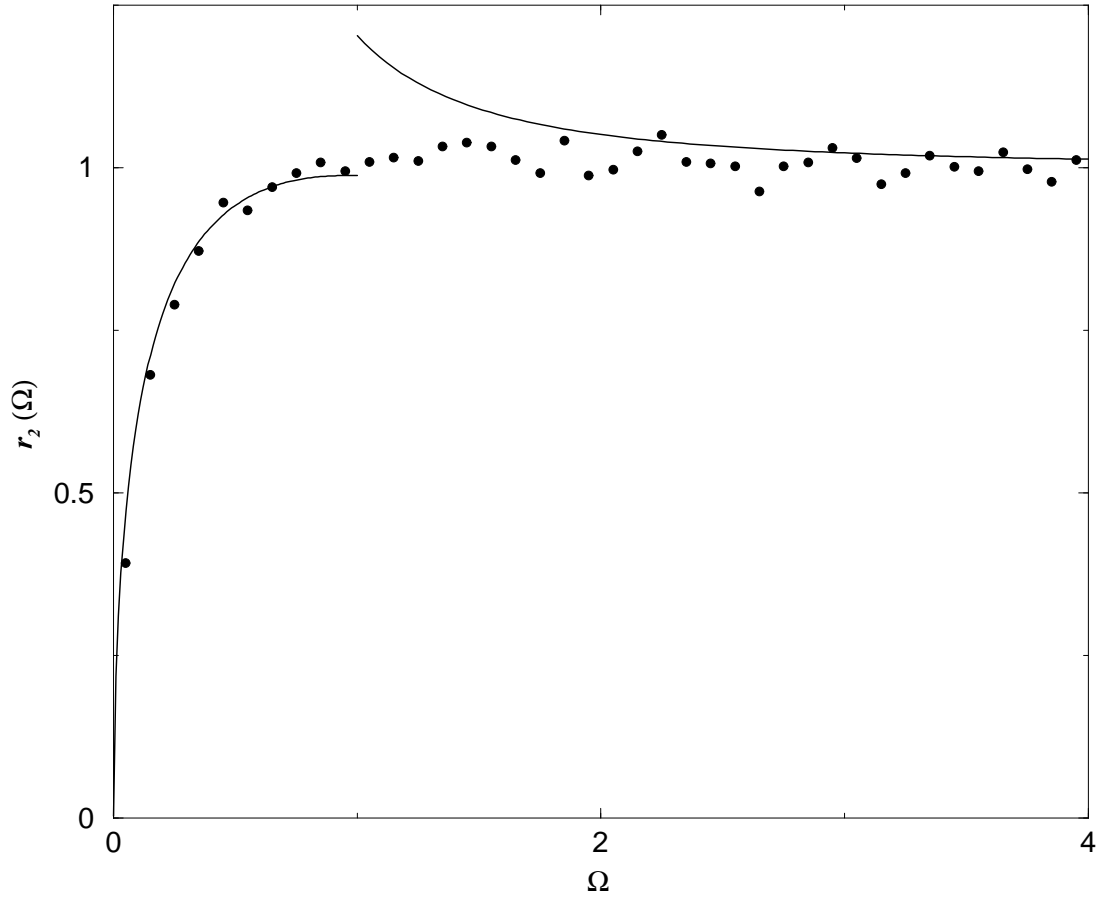


FIG. 4. The 2-point correlation function for the Seba billiard with the Dirichlet boundary conditions. Solid lines correspond to the asymptotics (160) and (150) for small and large values of energy difference.