The Riemannium

P. Leboeuf, A. G. Monastra, and O. Bohigas
Laboratoire de Physique Théorique et Modèles Statistiques, Bât. 100,
Université de Paris-Sud, 91405 Orsay Cedex, France

Abstract

The properties of a fictitious, fermionic, many-body system based on the complex zeros of the Riemann zeta function are studied. The imaginary part of the zeros are interpreted as mean-field single-particle energies, and one fills them up to a Fermi energy E_F. The distribution of the total energy is shown to be non-Gaussian, asymmetric, and independent of E_F in the limit $E_F \to \infty$. The moments of the limit distribution are computed analytically. The autocorrelation function, the finite energy corrections, and a comparison with random matrix theory are also discussed.

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According to the Riemann hypothesis, the complex zeros of the function $\zeta(s) = \sum n^{-s}$ (defined for $\text{Re}\, s > 1$, and by analytic continuation to the rest of the complex plane) are located on the critical line $s_\mu = 1/2 + i\, E_\mu$, with E_μ real. There is a natural connexion between this hypothesis and quantum mechanics that originates on a spectral interpretation of the complex zeros s_μ. Indeed, a general strategy to prove the hypothesis was suggested by Hilbert and Pólya, who proposed to find an hermitian operator whose eigenvalues are precisely the E_μ's. This operator may be viewed as the quantization of an hypothetical classical dynamical system, which determines the "Riemann dynamics". Although the classical Hamiltonian corresponding to the Riemann dynamics is not known, there are evidences for its existence. When the E_μ are interpreted as quantum eigenvalues, both their statistical properties [1–3] as well as a semiclassical interpretation of their density [4] indicate that the Riemann dynamics is fully chaotic and has no time-reversal symmetry. The semiclassical interpretation provides in fact much more detailed information (cf Eq.(2)), in particular concerning the classical periodic orbits. This leads to a somewhat paradoxical situation: although the "Riemann Hamiltonian" is not known, the detailed properties of its dynamics are. From the perspective of a dynamicist, the Riemann zeta thus offers the rare opportunity of a chaotic motion for which the relevant dynamical information is simple and explicitly known. Not to mention the impressive amount of existing numerical data on the complex zeros [2], as well as the number-theoretic background of $\zeta(s)$, also useful in the present context.

Many aspects of the Riemann dynamics have been investigated (see [5,6] for recent review articles). Here we explore new facets of the problem related to a different use of the complex zeros as suggested by physical analogies. We consider the location $E_\mu > 0$ of the complex zeros of $\zeta(s)$ as the single-particle levels of a fermionic many-body system. In the mean field approximation, the ground-state total energy is obtained by filling the single-particle levels from the lowest Riemann zero up to a "Fermi energy" E_F. We are interested in the properties of such a Fermi gas. Following nuclear physics terminology, we refer to this "element" as the Riemannium.

At zero temperature, the properties of the Riemannium are described by the grand potential

$$\Omega(E_F) = \int_0^{E_F} (E - E_F) \rho(E) dE = - \int_0^{E_F} \mathcal{N}(E) dE \ . \tag{1}$$

Here $\rho(E) = \sum_\mu \delta(E - E_\mu)$ is the spectral density of the complex Riemann zeros interpreted as quantum eigenvalues and $\mathcal{N}(E) = \int \rho(E) dE$ its counting function. $\Omega(E)$ corresponds to the sum of the imaginary part of the Riemann zeros satisfying $0 < E_\mu \leq E_F$, using E_F as reference energy [7]. It is therefore the total energy of the system. To calculate the grand potential we make use of the decomposition of the spectral density in smooth plus oscillatory parts, $\rho = \overline{\rho} + \widetilde{\rho}$. The former, as a Weyl series, has an explicit asymptotic expansion for large E. The oscillating term is an interference sum over the prime numbers $p = 2, 3, \ldots$ [8]

$$\widetilde{\rho}(E) = -\frac{1}{\pi} \text{Re} \sum_p \sum_{r=1}^{\infty} \frac{\log p}{p^{r/2}} \exp(i\, r E \log p) \ . \tag{2}$$

The comparison of this equation with the semiclassical Gutzwiller trace formula for the spectral density of a dynamical system [9], $\widetilde{\rho}(E) = \sum_{po} A_{po} \cos(S_{po}(E)/\hbar)$ where po are the classical periodic orbits, shows that each prime number labels an unstable periodic orbit of a fully chaotic system of action $S_p = E\,T_p$, period $T_p = r \log p$, Lyapounov stability $\lambda_p = 1$, repetitions labeled by r, and $\hbar = 1$ [4]. An unusual fact of the Riemann dynamics that plays an important role in what follows is the independence with respect to the energy E of the periods and Lyapounov exponents (notice also the minus sign in front of Eq.(2)).

The spectral density can be integrated twice to obtain the smooth and oscillatory contributions to the grand potential, $\overline{\Omega}(E_F) = -E_F^2 \log(E_F/2\pi)/4\pi + 3E_F^2/8\pi - 7E_F/8 - \log E_F/48\pi + cte$, and

$$\widetilde{\Omega}(E_F) = -\frac{1}{\pi}\mathrm{Re}\sum_p \sum_{r=1}^{\infty} \frac{\exp(i\,E_F\,r\log p)}{r^2\;p^{r/2}\log p}\ . \tag{3}$$

We are interested in the statistical properties of $\widetilde{\Omega}$ as a function of the Fermi energy E_F. From Eq.(3) we have $\langle\widetilde{\Omega}\rangle = 0$. The average is done over an energy window which is small compared to E_F but contains several oscillations of $\widetilde{\Omega}$ (the typical scale of oscillation will be given below).

As we will now see, the Riemannium has very peculiar properties. The most important ones demonstrated here are: i) the distribution of $\widetilde{\Omega}$, denoted $P(\widetilde{\Omega})$, is independent of energy as $E_F \to \infty$; ii) $P(\widetilde{\Omega})$ is non-Gaussian and asymmetric; iii) all the moments of $P(\widetilde{\Omega})$ may be computed with very good accuracy from Eq.(3), their value being dominated by the contribution of the lowest prime numbers; iv) the asymmetry of the distribution is due to an interference effect between repetitions of periodic orbits; v) the autocorrelation $\langle\widetilde{\Omega}(E_F)\widetilde{\Omega}(E_F+\epsilon)\rangle$ is a non-decaying irregular oscillatory function; vi) the finite energy corrections to $P(\widetilde{\Omega})$ are universal and well described by a circular unitary ensemble of random matrices (CUE).

Some of the statistical properties of $\widetilde{\Omega}$ were explored in the past. Selberg [10] computed the even moments of the distribution. Up to an additive constant and a global change of sign, $\widetilde{\Omega}(E)$ coincides with the function denoted $S_1(t)$ by him. More recently, Odlyzko [2] has numerically calculated the first four moments, but no comparison has been made with the analytical results of Selberg.

On Fig. 1 is displayed the distribution $P(\widetilde{\Omega})$ whose properties are now discussed. We begin by computing the second moment. Equation (3) allows to express it as an integral over the period of the orbits

$$\langle\widetilde{\Omega}^2\rangle = (1/2\pi^2)\int_0^{\infty} dT\,K(T)/T^4\ , \tag{4}$$
$$K(T) = \langle\sum_{i,j} A_i A_j \cos[E_F(r_i\log p_i - r_j\log p_j)]\delta(T-\bar{T})\rangle.$$

Use has been made here of the definition of the form factor, $K(T) = 4\pi\int_0^{\infty} d\epsilon\cos(\epsilon T)\langle\tilde{\rho}(E_F)\tilde{\rho}(E_F+\epsilon)\rangle$, with the amplitudes defined as $A_i = \log p_i/p_i^{r_i/2}$, and $\bar{T} = (T_{p_i}+T_{p_j})/2$. Rigorous arguments valid for times $T_{min} \ll T \ll T_H$, where $T_{min} = \log 2$ is the shortest period of the system and $T_H = 2\pi\bar{\rho} = \log(E_F/2\pi)$ is the Heisenberg time at E_F, and heuristic for longer ones, show [1] that the form factor of the complex zeros of $\zeta(s)$ tends, as E_F goes to infinity, to the corresponding function K_{GUE} of Gaussian random matrices with unitary symmetry (GUE). The latter behaves as $K_{GUE}(T) = T$ for $T < T_H$, and $K_{GUE}(T) = T_H$ for $T > T_H$. However, the replacement $K = K_{GUE}$ in Eq.(4) and its extrapolation to short times is of no use to understand the behavior of $\widetilde{\Omega}$, because the integral diverges. In other words, the second moment is dominated by the (non-universal) short periodic orbits whose contribution has to be considered explicitly. We therefore keep as leading-order approximation the diagonal part ($i = j$) in the double sum (4). This gives the *convergent* sum

$$\langle \widetilde{\Omega}_0^2 \rangle = \frac{1}{2\pi^2} \sum_p \sum_{r=1}^{\infty} \frac{1}{r^4 p^r \log^2 p} \approx 7.9 \times 10^{-2} \ . \tag{5}$$

This equation exhibits two of the main properties of $P(\widetilde{\Omega})$. The first one is its asymptotic independence with respect to energy. This fact is not generic for dynamical systems, since for example for a Fermi gas in a chaotic cavity $\langle \widetilde{\Omega}^2 \rangle$ grows linearly with the Fermi energy [11]. It is due to the independence of the periods and Lyapounov exponents with respect to energy in the Riemann dynamics. The second property is its non-universality (the sum (5) depends on the particular properties of the short orbits). The longer, statistically universal, periodic orbits provide next-to-leading order corrections to Eq.(5). For times $T \gg T_{min}$ the form factor K_{GUE} can be used in Eq.(4), leading to the correction (independent of the prime numbers)

$$\langle \widetilde{\Omega}^2 \rangle = \langle \widetilde{\Omega}_0^2 \rangle - \frac{1}{12\pi^2 \log^2(E_F/2\pi)} \ . \tag{6}$$

We have numerically checked the validity of Eq.(6) down to values of E_F of the order of a few thousands.

The third basic feature of the distribution is its asymmetry, as revealed by the third moment $\langle \widetilde{\Omega}^3 \rangle$, whose computation is now sketched. The third power of $\widetilde{\Omega}$ computed from Eq.(3) involves products of three cosines containing as argument the action of three different prime numbers. This product may be expressed as a sum of cosines involving the sum and differences of the actions. The term involving the sum of actions has zero average. As before, the remaining sum is dominated by the smallest primes. Due to the non-commensurability of the periods of the different primes we restrict moreover to the approximation $p_i = p_j = p_k = p$. The oscillating factors in the sum now have the typical form $\cos[E(r_i + r_j - r_k)\log p]$. The only terms of this type having a non-zero average are those satisfying $r_k = r_i + r_j$. Since there are three different possibilities for choosing the backward repetition r_k, we finally obtain the convergent sum

$$\begin{aligned} \langle \widetilde{\Omega}_0^3 \rangle &= -\frac{3}{4\pi^3} \sum_p \sum_{r_i, r_j = 1}^{\infty} [r_i^2 r_j^2 (r_i + r_j)^2 p^{r_i + r_j} \log^3 p]^{-1} \\ &\approx -5.78 \times 10^{-3} \ . \end{aligned} \tag{7}$$

The asymmetry of $P(\widetilde{\Omega})$ is therefore related to a simple but non-trivial interference phenomenon occurring between two (different or not) repetitions of a given orbit compensated by a backward one. The extreme case where only $p = 2$ and its repetitions are taken into account in Eq.(7) yields already 90% of its value.

The higher moments of the distribution are computed likewise. In the limit $E_F \to \infty$ to leading order they are given by expressions similar to, though more complicated than, Eq.(7). Table I shows a comparison of the analytical results with numerical data up to the sixth moment. The agreement between both calculations is astonishing. Our numerical results also coincide for the lowest moments with those obtained in [2].

An additional characterization of the Riemannium comes from the autocorrelation of the total energy $\mathcal{C}_\Omega(\epsilon) = \langle\widetilde{\Omega}(E_F)\ \widetilde{\Omega}(E_F+\epsilon)\rangle/\langle\widetilde{\Omega}_0^2\rangle$. We evaluate this quantity to leading order by using the expansion (3) and making a diagonal approximation. This leads to

$$\mathcal{C}_\Omega(\epsilon) = \frac{1}{2\pi^2\langle\widetilde{\Omega}_0^2\rangle}\sum_p\sum_{r=1}^{\infty}\frac{\cos(\epsilon\ r\log p)}{r^4\ p^r\log^2 p}\ . \tag{8}$$

Due to the dominance of the shortest orbits in the sum (8) $\mathcal{C}_\Omega$ is an irregularly fluctuating non-decaying function, as illustrated on Fig. 2. The fundamental period $2\pi/\log 2 \approx 9.06$ associated to the shortest orbit can be clearly observed.

It is interesting to make a comparison of the previous results with random matrix theory. As already emphasized, the GUE is not really appropriate because no short-time scale equivalent to T_{min} is built-in in the theory (and this produces a divergence of the moments). We therefore concentrate on the CUE which, contrary to the Gaussian one, has an inherent short-time scale (although more appropriate measures, not relevant for the present discussion, have been recently proposed [12]). The following analysis reveals also some striking similarities existing between the Riemann zeros and eigenvalues of circular ensembles. Consider an $N\times N$ unitary matrix U describing the time-periodic dynamical evolution of a quantum system. We fix for simplicity the periodicity to one. The Floquet spectrum of U is given by the eigenvalue equation $U\psi_\alpha = \exp(i\theta_\alpha)\psi_\alpha$, $\alpha = 1,\ldots,N$. ψ_α are the (stroboscopic) eigenstates and θ_α the eigenphases. The spectral density on the unit circle is $\rho_{CUE}(\theta) = \sum_\alpha \delta(\theta-\theta_\alpha)$. The 2π–periodicity of this function leads to a decomposition in smooth plus oscillatory parts, $\rho_{CUE}{=}N/2\pi + \widetilde{\rho}_{CUE}$, with

$$\widetilde{\rho}_{CUE}(\theta) = \frac{1}{\pi}\mathrm{Re}\sum_{k=1}^{\infty}\mathrm{Tr}U^k\exp(-ik\theta)\ . \tag{9}$$

As was done for Eq.(2), a direct comparison of Eq.(9) with Gutzwiller trace formula allows to make a "naïve" semiclassical interpretation of this equation. We look at it as a sum over the periodic orbits of a classical map labelled by the index $k = 1, 2, \ldots$, having period $T_k = k$, action $S_k = k\ \theta$, (local) energy θ, stability amplitude $A_k = \mathrm{Tr}U^k$, and $\hbar = 1$ (repetitions are degenerated with the fundamental periods). The interpretation of θ as the energy satisfies, as it should, the classical relation $T_k = \partial S_k/\partial\theta$. We recover in this analogy one of the basic properties of the Riemann dynamics: the independence of the periods with respect to energy. Short orbits correspond to $T_k \approx T_{min} = 1$, while long ones contributing to the density at the scale of the mean level spacing have period $T_k \approx T_H = 2\pi\bar{\rho} = N$. If now an ensemble of unitary matrices is considered, Eq.(9) acquires a statistical meaning since the prefactors $\mathrm{Tr}U^k$ have now a distribution. It is well known through, e.g., the prime number theorem and the Hardy-Littlewood conjecture, that asymptotically the statistical properties of large prime numbers are those required to mimic random matrix statistics [1,6]. So for long times Eqs.(2) and (9) are very similar from a statistical point of view. This is not the case for times $T \approx T_{min}$. To have a quantitative description we look at the CUE distribution of the grand potential, obtained by integrating $\widetilde{\rho}_{CUE}$ twice with respect to θ,

$$\widetilde{\Omega}_{CUE} = (1/\pi)\mathrm{Re}\sum_k (\mathrm{Tr}U^k/k^2) exp(-ik\theta) \ . \tag{10}$$

The second moment is

$$\langle \widetilde{\Omega}^2_{CUE} \rangle = (1/2\pi^2)\sum_k \langle |\mathrm{Tr}U^k|^2 \rangle / k^4 \ . \tag{11}$$

The average is done over the ensemble of matrices. This expression is the CUE analog of Eq.(4) ($K_{CUE} = \langle |\mathrm{Tr}U^k|^2 \rangle$). Using that $\langle |\mathrm{Tr}U^k|^2 \rangle = k$ if $k \leq N$ and $\langle |\mathrm{Tr}U^k|^2 \rangle = N$ if $k > N$ [13], we get

$$\langle \widetilde{\Omega}^2_{CUE} \rangle = \frac{\zeta(3)}{2\pi^2} - \frac{1}{12\pi^2 N^2} + \mathcal{O}(1/N^4) \ , \tag{12}$$

to be compared to Eq.(6). As for the Riemannium, the leading term $\zeta(3)/2\pi^2 \approx 6.1 \times 10^{-2}$ is controlled by short times (small values of k) in the sum (11). Because the short-time structure differ in both cases, the constants do not coincide. But the independence of the leading terms with respect to N (or energy) is not a coincidence and comes from the structural similarities between Eqs.(2) and (9). Moreover, the next-to-leading order corrections are the same if one identifies [14] the corresponding Heisenberg times $N = \log(E_F/2\pi)$. Both corrections come from times of order T_H, where the statistical properties agree. This situation is in contrast with the results obtained, for instance, for the distribution of $\log\zeta(1/2+iE)$, which has been shown to agree to leading order with the (finite N) CUE distribution, followed by non-universal corrections [14].

The qualitative agreement between $\langle \widetilde{\Omega}^2 \rangle$ and $\langle \widetilde{\Omega}^2_{CUE} \rangle$ is lost when considering odd powers of $\widetilde{\Omega}_{CUE}$. The latter can be shown to vanish in the limit $N \to \infty$. More generally, and contrary to the Riemann case, in that limit the distribution of $\widetilde{\Omega}_{CUE}$ tends to a Gaussian. This can be seen from Eq.(10). On the one hand we have demonstrated that $\widetilde{\Omega}_{CUE}$ is asymptotically dominated by the lowest contributions in the sum. But $\mathrm{Tr}U^k$ is known to be, for k finite and $N \to \infty$, Gaussian independent distributed [13]. Therefore $\widetilde{\Omega}$ has also a limiting normal distribution.

The techniques employed here to analyze the distribution of the grand potential are not specific to the Riemann dynamics and can be applied in a wider context to the theory of Fermi gases. Our results indicate that in general the grand potential of a Fermi gas, as well as other physical quantities derived from it, are controlled by the shortest non-universal classical orbits. This dominance leads, via the interference mechanisms illustrated here for the grand potential of the Riemannium, to non-Gaussian asymmetric distributions for the associated quantity. Some related results illustrating this general phenomenon were obtained by Tsang [15], who demonstrated non-Gaussian asymmetric distributions for the error term in the mean square formula of $|\zeta(1/2+i\,E)|$, and for the fluctuations of the number of lattice points inside a curve, namely a circle (see also [16]) and an hyperbola (known as the circle problem and Dirichlet's divisor problem, respectively). The last two problems have a dynamical interpretation in terms of the fluctuations of the spectral counting function of an associated integrable system. The mechanism leading to the asymmetry in [15] is very similar to the one found here. In contrast, the fluctuations of the counting function in the Riemann case are known to be asymptotically Gaussian [10] and are not dominated by the short orbits. This difference in the behavior of the counting function can be traced back to the different short-time behavior of the form factor for integrable and chaotic systems, that produces a dominance of the short orbits in the former. We find however that integrals of the counting function (the grand potential being the first one, cf Eq.(1)) are *always* dominated by the short orbits, irrespective of the (chaotic or integrable) nature of the underlying classical dynamics. On the mathematical side, the present results can be extended in a straightforward manner to general Dirichlet's L-functions.

In summary we have considered, guided by physical analogies, new properties of the Riemann zeros that reveal new aspects of their dynamical interpretation as quantum eigenvalues of a classically chaotic system. Our results are relevant in the theory of Fermi systems, as well as in the general context of quantum chaotic motion. Concerning the Riemann hypothesis and the search of the Hilbert–Pólya Hamiltonian, the present study suggests that time–periodic dynamical evolutions have to be considered as serious candidates.

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FIGURES

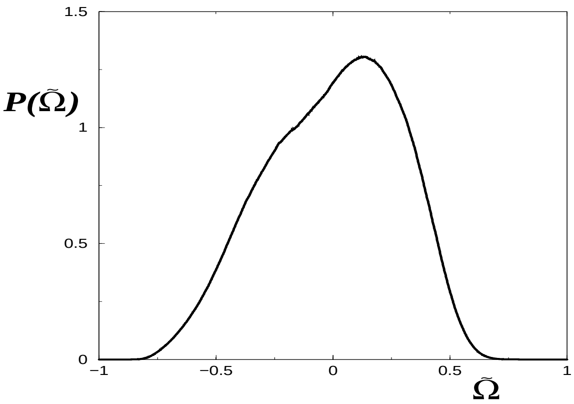

FIG. 1. Distribution of $\widetilde{\Omega}$ computed numerically at $E_F \approx 1.44 \times 10^{20}$ (results based on data from A. Odlyzko).

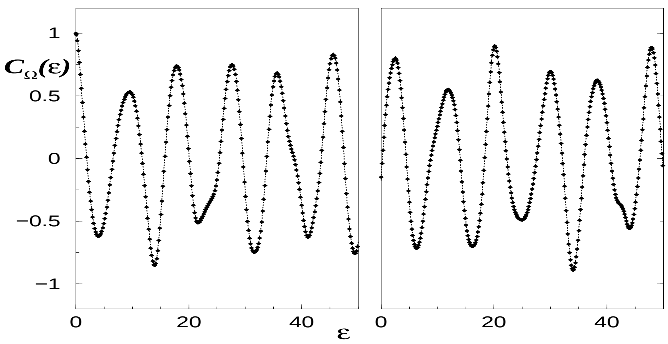

FIG. 2. The autocorrelation Eq.(8) (dotted line) compared to numerical data (dots). E_F as in Fig. 1. On the right part a constant $\epsilon_0 \approx 1.29 \times 10^{20}$ should be added to the abscissa.

TABLES

Moment	Semiclassics	Numerics
2	7.9290×10^{-2}	7.928×10^{-2}
3	-5.7822×10^{-3}	-5.785×10^{-3}
4	1.4814×10^{-2}	1.481×10^{-2}
5	-2.7787×10^{-3}	-2.776×10^{-3}
6	4.0007×10^{-3}	4.001×10^{-3}

TABLE I. Moments of the distribution $P(\widetilde{\Omega})$. Numerical values are computed for the distribution in Fig. 1.