Topological relaxation of entangled flux lattices: Single *vs* collective line dynamics

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A symbolic language allowing to solve statistical problems for the systems with nonabelian braid–like topology in 2+1 dimensions is developed. The approach is based on the similarity between growing braid and "heap of colored pieces". As an application, the problem of a vortex glass transition in high–T_c superconductors is re-examined on microscopic level.

Statistics of ensembles of uncrossible linear objects with topological constraints has very broad application area ranging from problems of self–diffusion of directed polymer chains in flows and nematic–like textures to dynamical and topological aspects of vortex glasses in high temperature superconductors [1]. In this letter we propose a microscopic approach to a diffusive dynamics of entangled uncrossible lines of arbitrary physical nature.

It is well known that the main difficulties in the statistical topology of linear uncrossible objects are due to two facts: (a) the topological constraints are non–local and (b) different entanglements do not commute. The point (a) can be overcomed by introducing the corresponding (abelian) Gauss–like topological invariant which properly counts windings of one chain around the other, while the circumstance (b) since now creates the major problem in constructive approach to topological theories beyond the abelian approximation. In order to have representative and physically clear image for the system of fluctuating lines with nonabelian topology we formulate the model in terms of entangled Brownian trajectories: such representation serves also as a geometrically clear image of Wilson loops in (2+1)D nonabelian field–theoretic path integral formalism.

We are aimed to develop a symbolic language which would permit us to construct the objects with a braid–like topology in 2+1 dimension and to solve the simplest statistical problems where the noncommutative character of topological constraints is properly taken into account. The results are applied to re-examination of the problem of a vortex glass transition in high–T_c superconductors [2]. Let us remind briefly that in $Cu0_2$–based high–T_c superconductors in fields less than H_{c2} there exists a region where the Abrikosov flux lattice is molten, but the sample of the supereconductor demonstrates the absence of the conductivity. This effect is explained by highly entangled state of flux lines due to their topological constraints [2].

The most attention in our investigation is paid to a quantitative estimation of a characteristic time of self–disentanglement of a particular "test" chain in a bunch of braided directed chains. We distinguish between two situations: (i) configurations of all lines in a bunch are quenched and form a lattice except one test line randomly entangled with the others, and (ii) no chain in a bunch of braided lines is fixed and any chain winds randomly like a test one. We compute the characteristic times of topological relaxations $\tau_{\rm si}$ and $\tau_{\rm co}$ in cases (i) and (ii) and demonstrate the *absence of qualitative difference* between $\tau_{\rm si}$ and $\tau_{\rm co}$, however the quantitative distinction between $\tau_{\rm si}$ and $\tau_{\rm co}$ is shown to be sufficiently strong. Our result contradicts in details with the statement of [2] on *qualitative difference* between $\tau_{\rm si}$ and $\tau_{\rm co}$ obtained in the frameworks of a scaling analysis. Nevertheless our result does not destroy the physical conclusions of the paper [2] about the possibility of topological glass transition in entangled flux state in high–T_c superconductors. According to the above mentioned cases (i) and (ii) we define respectively two discrete models I and II.

The *model I* is as follows. Take a square lattice in (xy)–plane with a spacing c and put in all vertices of this lattice the uncrossible obstacles. Consider a symmetric random walk with the step length c on a dual lattice shifted by $\frac{c}{2}$ in both x– and y–directions. We are interested in computing the probability $P_{\rm si}(N)$ of the fact that after N steps on the dual lattice the random path will be closed and unentangled with respect to the obstacles. It is clear that in the (2+1)–dimensional "space–time" $\mathbb{Z}^2(x,y)\times\mathbb{Z}^+(t)$ this model describes statistics of a "world lines" (time–ordered paths) of a single particle jumping on a square lattice in (xy)–projection, making each time a step toward t–axis and topologically interacting with the lattice of infinitely long straight lines. The (xy)–section of this model is shown in fig.1.

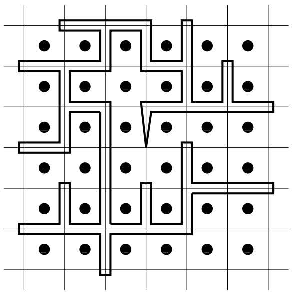

FIG. 1. The (xy)–projection of a path unentangled with the square lattice of topological obstacles.

It is well known that there exists a bijection between a path in the lattice of obstacles with coordinational number z and a path on a Cayley tree with z branches [3]. Different topological states of a given path coincide with the elements of the homotopy group of the multi–punctured plane Γ_∞, generated by a countable set of elements. The translational invariance allows to consider in a local basis the group $\Gamma_\infty/\mathbb{Z}^2 = \Gamma_{z/2}$, where $\Gamma_{z/2}$ is a free group with $z/2$ generators i.e. a z–branching tree. Any topological state of a path in the lattice of obstacles is encoded by an element of the group $\Gamma_{z/2}$, and uniquely corresponds to some irreducible word written in terms of generators of the group $\Gamma_{z/2}$. The length of this irreducible word coincides with the geodesic distance along the Cayley graph of the group $\Gamma_{z/2}$. The average "degree of entanglement" of an N–step path in the lattice of obstacles with coordinational number z for $N \gg 1$ is characterized by the averaged geodesic distance [3] $\langle L_{\rm si}(N)\rangle = \frac{z-2}{z}N$ on the Cayley tree with z branches. Hence, the normalized "complexity" $\langle l_{\rm si}\rangle$ of a typical topological state of a path can be defined as:

$$\langle l_{\rm si}\rangle \equiv \lim_{N\to\infty} \frac{\langle L_{\rm si}(N)\rangle}{N} = \frac{z-2}{z} \tag{1}$$

For diffusion on the Cayley tree, $P_{\rm si}(k,N)$ deftermines the probability for the N–step symmetric random walk to have a distance between ends in k steps along the tree. This probability has been computed many times—see, for example, [4]. Thus we reproduce the final result for the return probability $P_{\rm si}(k=0,N)$ as $N \to \infty$ on the Cayley tree:

$$P_{\rm si}(N) \equiv P_{\rm si}(k=0,N) = \frac{2\sqrt{2}p}{\sqrt{\pi}(1-4pq)} \frac{\alpha^N}{N^{3/2}} \tag{2}$$

where $q = \frac{1}{z}$ and $p \equiv 1-q = \frac{z-1}{z}$ are the probabilities of steps "towards" and "backwards" the origin of the Cayley tree; $\alpha = 2\sqrt{pq}$. Define Λ, a span of the N–step random walk in (xy)–projection. In physical terms of the original "vortex problem", $\sqrt{\Lambda^2}$ is the averaged size of thermal fluctuations of the vortex line. Hence, we can set $N = \frac{\Lambda^2}{c^2}$ and estimate the time of topological relaxation $\tau_{\rm si} = \frac{1}{P_{\rm si}(N)}$ of a single vortex line in an ensemble of immobile uncrossible lines for $z=4$ as follows

$$\tau_{\rm si} \sim \left(\frac{\Lambda^2}{c^2}\right)^{3/2} \left(\frac{2}{\sqrt{3}}\right)^{\frac{\Lambda^2}{c^2}} \tag{3}$$

In contrast to the model I, the *model II* describes collective dynamics of the world lines and ultimately leads to the consideration of the (2+1)–dimensional ("surface") *braid group* B_{n+1}^{2D}. Consider the two–dimensional lattice $\mathbb{Z}^2(x,y)$ and take distinct points $P_1, P_2, ...P_{(n+1)^2}, \in \mathbb{Z}^2$. A (2+1)–braid of $(n+1)^2$ strings on $\mathbb{Z}^2$ based at $\{P_1,...,P_{(n+1)^2}\}$ is an $(n+1)^2$–tuple $b=(b_1,\dots,b_{(n+1)^2})$ of paths, such that: (1) $b_i(1) = P_i$ and $b_1(1) \in \{P_1,...,P_{n^2}\}$ $\forall i \in \{1,...,(n+1)^2\}$; (2) $b_i(t) \neq b_j(t)$ $\forall\{i,j\} \in \{1,...,(n+1)^2\}$, $(i\neq j,\ t\in[1,N])$. The braid group B_{n+1}^{2D} on $\mathbb{Z}^2$ based at $\{P_1,...,P_{(n+1)^2}\}$ is the group of homotopy classes of braids based at $\{P_1,...,P_{(n+1)^2}\}$.

The group B_{n+1}^{2D} has $2n^2+2n$ generators $\sigma_{ij}^{(x)}$, $i \in \{1,\dots,n+1\}, j \in \{1,\dots,n\}$; $\sigma_{ij}^{(y)}$, $i \in \{1,\dots,n\}, j \in \{1,\dots,n+1\}$ and their inverses with the standard relations [5]. The geometric representation of generatoes of B_{n+1}^{2D} is shown in fig.2a. An element of the braid group B_{n+1}^{2D} is set by a word in the alphabet $\{\sigma_{11}^{(x)}, \sigma_{11}^{(y)}, \dots\}$. By the *length* N of a braid we call a length of a word in a given record of the braid, and by the irreducible length (or *primitive length*)—the minimal length of a word, in which the braid can be written. The irreducible length can be also viewed as a distance from the unity on the graph of the group. Graphically the braid is represented by a set of strings, going upwards in accordance with a growth of a braid length—see fig.3.

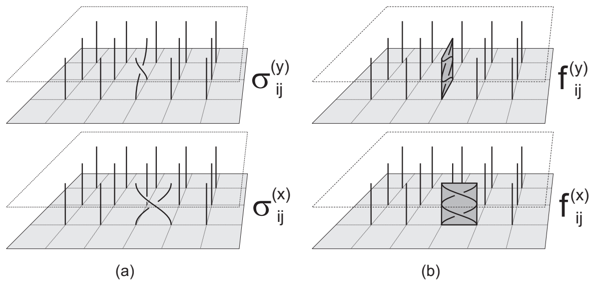

FIG. 2. The generators of the surface groups B_n^{2D} (a) and $\mathcal{LF}_{n+1}^{2D}$ (b).

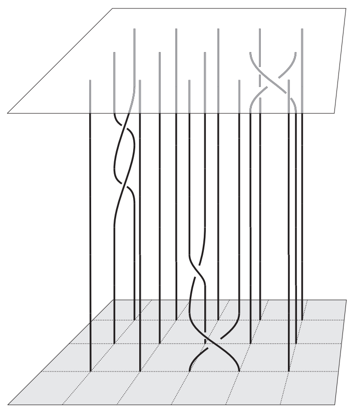

FIG. 3. The (2+1)–dimensional braid.

Define now a symmetric random walk on a set of generators ("letters") $\{\sigma_{11}^{(x)}, \sigma_{11}^{(y)}, \dots\}$ with the transitional probability $\frac{1}{2n^2+2n}$. Namely, we raise recursively an N–letter random word W adding step-by-step the letters (say, from the right–hand side) to a growing word. The probability that the N–letter word is completely contractible (i.e. has a zero's primitive length) defines the probability to have topologically trivial braid of the record length N.

Our main tool in the investigation of the braid group is the so-called *locally free group* [8]. The (2+1)D ("surface") locally free group $\mathcal{LF}_n^{2D}$ is obtained from the braid group B^{2D}_{n+1} by omitting the braiding relations. The group $\mathcal{LF}^{2D}_n$ has $2n^2+2n$ generators $f^{(x)}_{ij}$, $i\in\{1,\dots,n+1\}, j\in\{1,\dots,n\}$; $f^{(y)}_{ij}$, $i\in\{1,\dots,n\}, j\in\{1,\dots,n+1\}$ and their inverses with the relations (see also fig.2b):

$$\left\{\begin{array}{l} f^{(x)}_{i_1,j_1} f^{(x)}_{i_2,j_2} = f^{(x)}_{i_2,j_2} f^{(x)}_{i_1,j_1} \;\; (|j_1-j_2|>0 \text{ or } |i_1-i_2|>1) \\ f^{(x)}_{i_1,j_1} f^{(y)}_{i_2,j_2} = f^{(y)}_{i_2,j_2} f^{(x)}_{i_1,j_1} \;\; (i_2-i_1 \text{ or } j_1-j_2)\neq\{0,1\}) \\ f^{(x)}_{i,j}\left(f^{(x)}_{i,j}\right)^{-1} = f^{(y)}_{i,j}\left(f^{(y)}_{i,j}\right)^{-1} = e \end{array}\right.$$

There is a bijection between words in locally free group and *colored heaps*, whose elements are either "white" $f^{(x,y)}_{ij}$ or "black" $\left(f^{(x,y)}_{ij}\right)^{-1}$. That is, any word written in terms of letters–generators of the group $\mathcal{LF}^{2D}_n$ represents a configuration of a colored heap (see fig.4) in a box of the base of $n\times n$ cells and any such heap uniquely defines some word in the group $\mathcal{LF}^{2D}_n$. The configuration of the heap with a "black" block following immediately after a "white" one in the same column is forbidden.

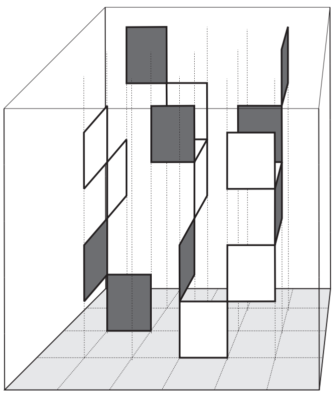

FIG. 4. The (2+1)–dimensional colored heap.

For the uniform Markov dynamics on the set of generators we can compute the average primitive length $\langle L_{\rm co}(N)\rangle$ of the N–letter word which characterizes the degree of entanglement of threads in a braid (compare to the definition of $\langle L_{\rm si}(N)\rangle$). Let us take into account that $\mathcal{LF}^{2D}_n$ is a subgroup of B^{2D}_{n+1}, while B^{2D}_{n+1} is a factor–group of $\mathcal{LF}^{2D}_n$. This is manifested in the two following facts. (1) By definition the locally free group $\mathcal{LF}^{2D}_n$ has less relations than the braid group B^{2D}_{n+1}. Hence, the number of distinct words of primitive length $L_{\rm co}$ in braid group *is bounded from above* by the number of distinct words of the same primitive length $L_{\rm co}$ in the locally free group. (2) By construction (compare figs.2a and 2b) $f^{(x,y)}_{i,j} = \left(\sigma^{(x,y)}_{i,j}\right)^2$ $(\{i,j\}\in[1,n])$. Thus, the number of distinct words of length $2L_{\rm co}$ in the braid group *is bounded from below* by the number of distinct words of the length $L_{\rm co}$ in the locally free group [8]. The facts (1)–(2) allow us to get the bilateral estimation for the average primitive length $\left\langle L_{\rm co}(N|B^{2D}_{n+1})\right\rangle$ of the N–step random walk on the surface braid group B^{2D}_{n+1}:

$$\frac{1}{2}\left\langle L_{\rm co}(N|\mathcal{LF}^{2D}_n)\right\rangle \le \left\langle L_{\rm co}(N|B^{2D}_{n+1})\right\rangle \le \left\langle L_{\rm co}(N|\mathcal{LF}^{2D}_n)\right\rangle \tag{4}$$

The computation of $\left\langle L_{\rm co}(N|\mathcal{LF}^{2D}_n)\right\rangle$ involves the concept of the *roof of the heap*. In physical terms a roof of a heap consists of a set of "most top" blocks which can be removed from the heap without changing the rest of it. The projection of a roof to the (xy)–plane for some particular configuration of the most top blocks is shown in fig.5. Let us stress that local hights (measured from the bottom of the box) of different roof's blocks might be different.

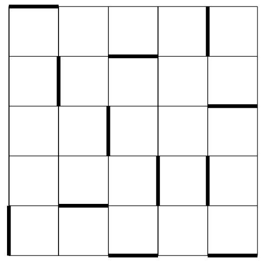

FIG. 5. A particular configuration of a roof of (2+1)D heap is shown by bold segments.

The process of growth of a heap (i.e. the random walk on the surface locally free group $\mathcal{LF}^{2D}_n$) consists in adding step-by-step new "black" or "white" blocks to the roof. Hence the dynamics of a heap is controlled by the dynamics of a roof. For a particular configuration of a roof we define the "size" of a roof $\#T$ (i.e. the number of bold segments in fig.5) and the number of empty segments n_i having i bold neighbors (apparently, $n_i=0\,\forall\, i\ge 3$). In fig.5 one has $n=5$, $n_0=6$, $n_1=25$, $n_2=17$; $\#T=12$. For the values $\#T$, n_i the following conditions hold:

$$\left\{\begin{array}{l} n_0+n_1+n_2+\#T = 2n^2+2n \\ 6\#T-8(n+1)\le n_1+2n_2\le 6\#T \end{array}\right. \tag{5}$$

For a given configuration the local dynamics of a size of a roof reads

$$\left\{\begin{array}{ll} \Delta\#T=1 & \text{with probability } \frac{n_0}{2n^2+2n} \\ \Delta\#T=0 & \text{with probability } \frac{n_1+\#T}{2n^2+2n} \\ \Delta\#T=-1 & \text{with probability } \frac{n_2}{2n^2+2n} \end{array}\right.$$

which together with (5) allows to estimate $\#T$ as follows

$$1-\langle\Delta\#T\rangle \le \frac{7\#T}{2n^2+2n} \le 1+\frac{8(n+1)}{2n^2+2n}-\langle\Delta\#T\rangle$$

In a stationary case we have $\langle\Delta\#T\rangle=0$, what permits to get the asymptotic value $\langle\#T\rangle$ of the average number of block in the roof for $n\gg 1$:

$$\langle\#T\rangle = \frac{2n^2}{7} \tag{6}$$

Return to the random walk on $\mathcal{LF}_n^{2D}$. The conditional change of the primitive length $L_{\rm co}(N|\mathcal{LF}_n^{2D})$ for one step of the random walk is

$$\left\{\begin{array}{ll} \Delta L_{\rm co}(N|\mathcal{LF}_n^{2D}) = 1 & \text{with probability } 1-\frac{\#T}{2(2n^2+2n)} \\ \Delta L_{\rm co}(N|\mathcal{LF}_n^{2D}) = -1 & \text{with probability } \frac{\#T}{2(2n^2+2n)} \end{array}\right.$$

Taking into account (6) we get in a stationary state (when $N \gg 1$ and $n \gg 1$) the following asymptotic value for the average primitive length $\left\langle L_{\rm co}(N|\mathcal{LF}_n^{2D})\right\rangle$:

$$\left\langle L_{\rm co}(N|\mathcal{LF}_n^{2D})\right\rangle = \frac{6}{7}N$$

Thus, according to (4) we arrive at the bilateral estimation for the average length of the primitive word for the N–step random walk on the surface braid group

$$\frac{3}{7} \le \lim_{\substack{N\to\infty \\ n\to\infty}} \frac{\left\langle L_{\rm co}(N|B_n^{2D})\right\rangle}{N} \le \frac{6}{7} \qquad (7)$$

The quantity $\langle l_{\rm co}\rangle = \lim_{\substack{N\to\infty \\ n\to\infty}} \frac{\left\langle L_{\rm co}(N|B_n^{2D})\right\rangle}{N}$ characterizes the "complexity" of entangled state. Comparing (7) to (1) we can conclude that the averaged topological state of a braid $\left\langle L_{\rm co}(N|B_n^{2D})\right\rangle$ obtained in course of a collective motion of lines has the same asymptotics in N as the one of a single line motion and interpolates between entangled states in effective lattices of obstacles with coordinational numbers $z_{\rm eff}$ ranging in the interval $\left[\frac{7}{2}, 14\right]$.

Let us utilize now the "heap concept" and eq.(6) for estimating *from above* the time of topological relaxation in a bunch of vortex lines, considered as a braid of directed random walks. The random growth of a braid of $2n^2+2n$ lines of length N each, can be interpreted as an N–step random walk on a surface braid group B_{n+1}^{2D}. The topological state of a braid is uniquely characterized by a primitive word W_N in terms of generators of B_{n+1}^{2D}. The disentangled state of two neighboring trajectories means (in terms of the group B_{n+1}^{2D}) that the primitive word W_N does not contain the generators $\sigma_{i,j}^{(x)}, \sigma_{i,j}^{(y)}, \left(\sigma_{i,j}^{(x)}\right)^{-1}, \left(\sigma_{i,j}^{(x)}\right)^{-1}$ for some $1 \le \{i,j\} \le n$.

Let $L_\sigma(W_N)$ be the number of generators $\sigma = \left\{\sigma_{i,j}^{(x)}, \left(\sigma_{i,j}^{(x)}\right)^{-1}\right\}$ in the primitive word W_N. Then the function $P_{\rm co}\{x,N\} \equiv P\{L_\sigma(W_N)=x\}$ defines the characteristic time $\tau_{\rm co} = \frac{1}{P_{\rm co}(0,N)}$ of disentanglement of a particular line in course of collective Brownian motion of all $2n^2+2n$ lines in a braid. Suppose that after N random steps we arrive at the word W_N. Then at the step $N+1$ we have $W_{N+1} = W_N\, W_{NN+1}$. The probability that the word W_{NN+1} contains a generator from the set σ (with a prescribed sign) is $P\{\sigma \in W_{NN+1}\} = \frac{1}{4}\times\frac{1}{8} = \frac{1}{32}$. Consider the probability $q_N^b = P\{L_\sigma(W_{N+1}) = L_\sigma(W_{N+1})-1\}$ of reducing the element σ by 1 at the step $N+1$. The probability q_N^b we can estimate from below, replacing the word W_{N+1} by the word $\tilde{W}_{N+1}$ which itself is obtained by replacing the generators of the group B_{n+1}^{2D} by the ones of the group $\mathcal{LF}_n^{2D}$. The word $\tilde{W}_{N+1}$ is also primitive by construction and $q_N^l = P\{L_f(\tilde{W}_{N+1}) = L_f(\tilde{W}_{N+1}-1)\} \le q_N^b$, where $f \in \left\{f_{i,j}^{(x)}, \left(f_{i,j}^{(x)}\right)^{-1}\right\}$. Denote by Q_N^l the probability that after N steps the roof contains the generators $f_{ij}^{(x)}$ or $f_{ij}^{(y)}$. In the case $n >> 1$ we can neglect the 'boundary effects' caused by marginal generators $f_{n+1,j}^{(x)}$, $f_{1,j}^{(x)}$, $j \in \{1,\dots,n\}$ and $f_{i,n+1}^{(y)}$, $f_{i,1}^{(y)}$, $i \in \{1,\dots,n\}$ so that the probability Q_N^l is the same for all $f \in \left\{f_{i,j}^{(x)}, \left(f_{i,j}^{(x)}\right)^{-1}\right\}$. Thus we have the obvious expression for the average number of blocks in the roof in the stationary case $N \gg 1$: $\langle\#T\rangle = 2n^2 Q^l$; which gives $Q^l = \frac{1}{7}$. Then the probability of cancellation of the generator f at the step $N+1$ is $q_N^l = \frac{1}{32}Q_N^l = \frac{1}{224} \le q^b$ where q^b is the stationary probability of cancellation of generator σ at $N \to \infty$. Substituting q^b for q in (2) we get the estimate from above for the time $\tau_{\rm co}^{\rm up}$:

$$\tau_{\rm co}^{\rm up} \sim \left(\frac{\Lambda^2}{c^2}\right)^{3/2} \left(\frac{112}{\sqrt{223}}\right)^{\frac{\Lambda^2}{c^2}} \qquad (8)$$

The numerical analysis of statistics of braid and locally free groups [6] enables us to conjecture that the estimate $\tau_{\rm co}^{\rm up}$ for the group $\mathcal{LF}_n^{2D}$ is close to the value $\tau_{\rm co}$ for the group B_n^{2D}.

Comparing (8) to (3) we conclude that the scaling dependences of the characteristic times $\tau_{\rm si}$ and $\tau_{\rm co}$ on $\frac{\Lambda}{c}$ are the same, however numerically $\tau_{\rm co}$ is much larger than $\tau_{\rm si}$, namely $\lim_{\frac{\Lambda}{c}\to\infty} \frac{c^2}{\Lambda^2}\ln\frac{\tau_{\rm co}}{\tau_{\rm si}} \approx 6.5$.

Summarizing the said above, let us emphasize that the random braiding can be analyzed within the frameworks of a symbolic dynamics on the locally free group describing the growth of a heap of colored pieces. The probability to have disentangled state of two vortex lines can be estimated from above by the probability to have no pieces (blocks) in a given column of a heap. This model seems to be a natural discretization of a standard ballistic growth process of Kardar–Parisi–Zhang type [9].

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