

# Breakdown of superfluidity of an atom laser past an obstacle

Nicolas Pavloff

Laboratoire de Physique Théorique et Modèles Statistiques<sup>1</sup>,  
Université Paris Sud, bât. 100, F-91405 Orsay Cedex, France  
email: pavloff@ipno.in2p3.fr

## Abstract

The 1D flow of a continuous beam of Bose-Einstein condensed atoms in the presence of an obstacle is studied as a function of the beam velocity and of the type of perturbing potential (representing the interaction of the obstacle with the atoms of the beam). We identify the relevant regimes: stationary/time-dependent and superfluid/dissipative; the absence of drag is used as a criterion for superfluidity. There exists a critical velocity below which the flow is superfluid. For attractive obstacles, we show that this critical velocity can reach the value predicted by Landau's approach. For penetrable obstacles, it is shown that superfluidity is recovered at large beam velocity. Finally, enormous differences in drag occur when switching from repulsive to attractive potential.

PACS numbers:

03.75.-b Matter waves

05.60.Gg Quantum transport

42.65.Tg Optical solitons; nonlinear guided waves

67.40.Hf Hydrodynamics in specific geometries, flow in narrow channels

---

<sup>1</sup>Unité Mixte de Recherche de l'Université Paris XI et du CNRS (UMR 8626).

# 1 Introduction

The rapid progresses in the technology of guiding cold atoms (using hollow optical fibers [1], magnetic guides [2] and microchips [3]) opens up the prospect of similar studies of guided Bose-Einstein condensed beams, i.e., of guided continuous atom lasers. Indeed, important progress in this direction is presently being made, which uses the techniques developed for guiding cold atoms: wave guides have been designed for a Bose-Einstein condensate using a blue-detuned hollow laser beam [4]; Bose condensation has been obtained over a microchip [5, 6] and a Bose-Einstein wave-packet has been propagated in a microfabricated magnetic wave-guide [7]. Also, a continuous beam of cold atoms has been loaded into a magnetic guide, as a first step in order to perform evaporative cooling and condensation in the guide [8].

In the present work we address the question of superfluidity of a continuous (guided) atom laser, namely: what are the conditions for the flow to be dissipationless? A criterion for superfluidity has been proposed long ago by Landau [9] which states that dissipation doesn't occur if the velocity of the flow is lower than the critical value  $v_{crit} = \min \{E(q)/q\}$ , where  $E(q)$  is the energy of an excitation with momentum  $q$ .

Many experiments have been done in liquid helium II trying to test Landau's idea, and indeed one finds a critical velocity, but, in many instances it is much lower than Landau's expectation. As Feynman first suggested [10], this is linked to vortex formation, i.e., to non-linear perturbation of the fluid (and not to elementary excitations as implied by Landau [11]). The experiments done at M.I.T. for Bose-Einstein condensates confirm this view: in these systems there is also a critical velocity [12, 13], it is lower than Landau's expectation, and it is also linked to vortex formation [14].

In the following we are concerned with one-dimensional flows, which are relevant to atom laser physics. We devise an alternative phrasing of Landau's argument based on perturbation theory and identify its limit of validity. In the generic, non-perturbative case, we confirm that, for obstacles represented by *repulsive potentials*, dissipation begins at a velocity lower than the one expected on the basis of Landau's argument and corresponds to emission of solitons (which are the one-dimensional analogues of vortices). However, we show that Landau's critical velocity is always reached for *attractive potentials*. Furthermore, following a previous study [15], above Landau's critical velocity we identify a new (numerically stable) regime, stationary *and* dissipative. In this regime, the drag exerted on an obstacle can be computed with little numerical effort. Moreover, in this regime, we show that at large velocity the drag exerted on a penetrable obstacle goes to zero, i.e., superfluidity is recovered.

The paper is organized as follows: In Sec. 2 we set up the theoretical framework and notations. The natural criterion for the breakdown of superfluidity is the absence of drag, and in this section we show precisely how the drag can be computed. In Sec. 3 we determine the different types of flow and the corresponding drag for an obstacle represented by an external potential (a weak potential in Sec. 3.1, a delta-peak in Sec. 3.2, a square well in Sec. 3.3 and a Gaussian potential in Sec. 3.4). Finally we discuss our results in Sec. 4, where we emphasize the important differences between attractive and repulsive potentials in the non-linear regime.

## 2 A criterion for superfluidity

We work in a quasi one-dimensional regime, or more precisely we use an adiabatic approximation where the condensate wave-function  $\Psi(\vec{r}, t)$  can be cast in the form [16, 15]

$$\Psi(\vec{r}, t) = \psi(x, t) \phi(\vec{r}_\perp; n), \quad (1)$$

where  $\psi(x, t)$  describes the motion along the axis of the laser.  $\phi$  is the equilibrium wave function (normalized to unity) in the transverse ( $\vec{r}_\perp$ ) direction; it depends parametrically on the longitudinal density  $n(x, t) = \int d^2 r_\perp |\Psi|^2 = |\psi(x, t)|^2$ . The beam is confined in the transverse direction by a trapping potential  $V_\perp(\vec{r}_\perp)$ . The adiabatic approximation assumes that the transverse scale of variation of the profile is much smaller than the longitudinal one. The transverse degrees of freedom are not completely frozen, but adapt to the smooth longitudinal dynamics: this is the essence of the parametric dependence of  $\phi$  on  $n(x, t)$ . This represents a significant improvement to what is generally defined as quasi one-dimensional approach and results in a non-typical nonlinearity of the 1D reduction of the Gross-Pitaevskii equation below (Eqs. (2,3)).

In the regime in which Eq. (1) holds [17], the equation governing the time evolution of  $\psi(x, t)$  reads [16, 15] (we set units such that  $\hbar = m = 1$ ):

$$-\frac{1}{2}\partial_{xx}^2\psi + \left\{ V_\parallel(x) + \epsilon[n(x, t)] \right\} \psi = i\partial_t\psi. \quad (2)$$

In (2),  $V_\parallel(x)$  represents the effect of the obstacle. We restrict ourselves to the case of a localized perturbation with  $\lim_{x \rightarrow \pm\infty} V(x) = 0$ . Such an obstacle can be realized by crossing the trajectory of the atom laser with a detuned optical laser beam whose waist is large compared with the perpendicular extension of the condensed beam. An other possibility is to bend the trajectory of the guided atom laser; this results in an attractive effective potential proportional to the square of the curvature (see [15]).

$\epsilon(n)$  is a non-linear term describing the mean-field interaction inside the beam, and the way it is affected by the transverse confinement. For a transverse confining harmonic potential of pulsation  $\omega_\perp$  one has (see [15]):

$$\begin{aligned} \epsilon(n) &= 2\omega_\perp na_{sc} && \text{in the low density regime} && na_{sc} \ll 1, \\ \epsilon(n) &= 2\omega_\perp \sqrt{na_{sc}} && \text{in the high density regime} && na_{sc} \gg 1, \end{aligned} \quad (3)$$

where  $a_{sc}$  denotes the s-wave scattering length of the two-body inter-atomic potential (we consider only the case  $a_{sc} > 0$ , i.e. a repulsive effective interaction). In the following we use a formalism allowing to treat both the high and the low density regime, since both will be of interest in future guided atom laser experiments.

We want to characterize the superfluidity of the flow past the obstacle. To this end we compute the drag  $F_d$  exerted by the atom laser on the obstacle: a finite drag implies dissipation, whereas  $F_d = 0$  corresponds to a superfluid flow.  $F_d$  is defined as

$$F_d(t) = \int_{-\infty}^{+\infty} dx n(x, t) \frac{dV_\parallel(x)}{dx}. \quad (4)$$

This definition is quite natural: the force exerted on the obstacle is the mean value of the operator  $dV_{\parallel}(x)/dx$  over the condensate wave-function. It is rigorously justified by the analysis below in term of stress tensor (Eq. (6)).

For analytical determination of the drag, we use the following procedure: the 1D version of the stress tensor of the fluid is [18]

$$T(x, t) = -\text{Im}(\psi^* \partial_t \psi) + \frac{1}{2} |\partial_x \psi|^2 - \varepsilon[n(x, t)] - V_{\parallel}(x) n(x, t) \quad \text{where} \quad \varepsilon(n) = \int_0^n \epsilon(\rho) d\rho, \quad (5)$$

and its impulsion density is  $J(x, t) = \text{Im}(\psi^* \partial_x \psi)$  (in our units, it is also the current density). By the conservation equation  $\partial_t J + \partial_x T + n \partial_x V_{\parallel} = 0$ , the total impulsion of the beam  $P(t) = \int_{-\infty}^{+\infty} dx J(x, t)$  is related to  $F_d$  by:

$$\frac{dP}{dt} = T(-\infty, t) - T(+\infty, t) - F_d(t). \quad (6)$$

The physical content of (6) is clear:  $dP/dt$  equals the total force exerted over the beam. One part of this force ( $-F_d$ ) is due to the potential, the other one is the stress on the boundaries of the beam (at left and right infinity). Hence Eq. (6) confirms the heuristic guess (4); besides, it allows to determine the drag in a simple fashion in the stationary regime where  $T$  and  $P$  are time independent:

$$F_d = T(-\infty) - T(+\infty) \quad \text{in the stationary regime.} \quad (7)$$

Hence, in the following, we devote particular attention to stationary solutions of (2). They are of the form  $\psi(x, t) = \exp\{-i\mu t\} A(x) \exp\{iS(x)\}$ , with  $A$  and  $S$  real functions. The density is  $n(x) = A^2(x)$ , the velocity  $v(x) = dS/dx$  and the current  $J(x) = n(x)v(x)$  is a constant that we note  $J_{\infty}$ . From Eq. (2), the amplitude  $A(x)$  obeys a Schrödinger-like equation

$$-\frac{1}{2} \frac{d^2 A}{dx^2} + \left\{ V_{\parallel}(x) + \varepsilon[n(x)] + \frac{J_{\infty}^2}{2n^2(x)} \right\} A(x) = \mu A(x). \quad (8)$$

As discussed in [15], the radiation condition requires that solutions of (8) have no wake far down-stream: long range perturbations of the beam only occur up-stream. Hence the boundary conditions have to be imposed down-stream: because of nonlinearity one cannot disentangle an incident and a reflected part in the perturbed up-stream flow. In all the following we consider a beam going in the negative  $x$  direction, with down-stream boundary conditions  $n(x \rightarrow -\infty) = n_{\infty}$  and  $v(x \rightarrow -\infty) = -v_{\infty}$  (with  $v_{\infty} > 0$ ). Then, the chemical potential is  $\mu = v_{\infty}^2/2 + \varepsilon(n_{\infty})$ . In the following we refer to  $v_{\infty}$  as the beam velocity and to  $c_{\infty} = (n_{\infty} d\varepsilon/dn|_{n_{\infty}})^{1/2}$  as the sound velocity (the proper denomination should be ‘‘sound velocity evaluated at constant density  $n_{\infty}$ ’’). We also express the lengths in units of the relaxation length  $\xi = [2\varepsilon(n_{\infty})]^{-1/2}$ .

For stationary flows, the stress tensor (5) reads

$$T(x) = \frac{1}{2} \left( \frac{dA}{dx} \right)^2 + W[n(x)] - V_{\parallel}(x) n(x) \quad \text{where} \quad W(n) = -\varepsilon(n) + \mu n + \frac{J_{\infty}^2}{2n}. \quad (9)$$

In regions where the spatial variations of  $V_{\parallel}(x)$  are negligible,  $T(x)$  is a constant (as easily seen from (8)).

### 3 Determination of the drag

#### 3.1 Perturbative solution

Let us first evaluate the drag experienced by the obstacle when the effects of the potential  $V_{\parallel}(x)$  on the flow can be treated perturbatively. In this case, by adiabatically branching the potential, one can always find a stationary solution with  $A(x) = \sqrt{n_{\infty}} + \delta A(x)$  having the correct boundary condition (i.e., verifying the radiation condition). Introducing the notation  $\kappa = 2|v_{\infty}^2 - c_{\infty}^2|^{1/2}$  one finds (see [15]) :

$$\delta A(x) = \begin{cases} -\frac{\sqrt{n_{\infty}}}{\kappa} \int_{-\infty}^{+\infty} V_{\parallel}(y) \exp\{-\kappa|x-y|\} dy & \text{when } v_{\infty} < c_{\infty} , \\ \frac{2\sqrt{n_{\infty}}}{\kappa} \int_{-\infty}^x V_{\parallel}(y) \sin\{\kappa(x-y)\} dy & \text{when } v_{\infty} > c_{\infty} . \end{cases} \quad (10)$$

The asymptotic evaluation of (10) when  $x \rightarrow \pm\infty$  allows to compute the drag through (7). From (9) and (10),  $T(-\infty)$  and  $T(+\infty)$  differ only in the supersonic case where

$$T(-\infty) = W(n_{\infty}) \quad \text{and} \quad T(x \rightarrow +\infty) = T(-\infty) + \frac{1}{2} \left( \frac{d\delta A(x)}{dx} \right)^2 + \frac{\kappa^2}{2} \delta A^2(x) , \quad (11)$$

with (always in the super-sonic regime)

$$\delta A(x) \xrightarrow{x \rightarrow +\infty} \frac{2\sqrt{n_{\infty}}}{\kappa} \text{Im} \left\{ e^{i\kappa x} \hat{V}_{\parallel}(\kappa) \right\} , \quad (12)$$

and  $\hat{V}_{\parallel}(\kappa) = \int_{-\infty}^{+\infty} dx \exp(-i\kappa x) V_{\parallel}(x)$  is the Fourier transform of  $V_{\parallel}(x)$ . One thus obtains

$$F_d = 0 \quad \text{when } v_{\infty} < c_{\infty} , \quad F_d = -2n_{\infty} |\hat{V}_{\parallel}(\kappa)|^2 \quad \text{when } v_{\infty} > c_{\infty} . \quad (13)$$

The gross behavior characterized by (13) is general: at low velocity the flow is superfluid, whereas at high velocity dissipation occurs. This corresponds to Landau's criterion which determines a critical velocity below which the flow is dissipationless:  $v_{crit} = \min \{E(q)/q\}$ , where  $E(q)$  is the energy of an excitation with momentum  $q$ . For our system  $E(q)$  is given by the Bogoliubov dispersion relation  $E(q) = q(c_{\infty}^2 + q^2/4)^{1/2}$  (see, e.g., [19]) and the Landau critical velocity is then  $v_{crit} = c_{\infty}$ . Hence the present perturbative approach is identical to Landau's criterion since both give the same value of velocity for the onset of dissipation and have the same physical content: excitation of small, non-localized, perturbations is allowed only above  $v_{crit}$ .

However Landau's criterion, as well as the reasoning leading to Eq. (13) are, by essence, perturbative. We show below that, as discussed in the introduction, non-linear effects alter these simple perturbative views.

#### 3.2 delta peak potential

A first hint of the failure of the perturbative approach can be obtained if studying the effect on the flow of a delta-potential  $V_{\parallel}(x) = \lambda \delta(x)$ . In that case a stationary solution is obtained by matching two free propagation modes of the laser (i.e., solutions of (8) in the absence of potential) at  $x = 0$

with the condition  $A'(0^+) - A'(0^-) = 2\lambda A(0)$ . The upstream and downstream stress tensors are constant, they are related by  $T_{up} = T_{down} + \lambda A(0)[A'(0^-) + A'(0^+)]$ . Besides, at velocity  $v_\infty > c_\infty$ , the radiation condition imposes that the down-stream ( $x < 0$ ) density is constant: this gives  $n(0) = n_\infty$  and  $A'(0^-) = 0$ . Hence, for beam velocity  $v_\infty$  larger than  $c_\infty$ , in the stationary regime (7) yields:

$$F_d = -2n_\infty \lambda^2 . \quad (14)$$

The same formula would have been obtained by using the perturbative result (13). Hence it may seem that the perturbative approach is valid for all range of delta potentials and velocities. However, one would expect perturbation to fail for strong potentials and near  $v_\infty = c_\infty$ , since it is meaningful only if  $|\delta A(x)| \ll \sqrt{n_\infty}$ , i.e., if  $|\lambda| \ll \kappa = 2|v_\infty^2 - c_\infty^2|^{1/2}$  (see Eq. (10)). Indeed, whereas a naive perturbative approach (leading to (10)) predicts that a stationary solution exists for all values of  $v_\infty$ , an exact solution in presence of a delta-potential can be found only in some precise range of velocity and potentials. This has been studied in Ref. [15] and is illustrated on Figure 1. Roughly speaking, stationary regimes only exist for low and high values of  $v_\infty$  (in the shaded zone of Fig. 1). In the low velocity case, the stationary profile is a trough (or a bump, depending on the sign of the potential) localized on the perturbation [20], such flows are dissipationless. At high velocity the stationary profile has a (non-linear) wake extending to infinity in the upstream direction and this corresponds to dissipation (by Eq. (7)). In between, the flow is time-dependent and there is no reason why, in this case, the drag should be given by (14).

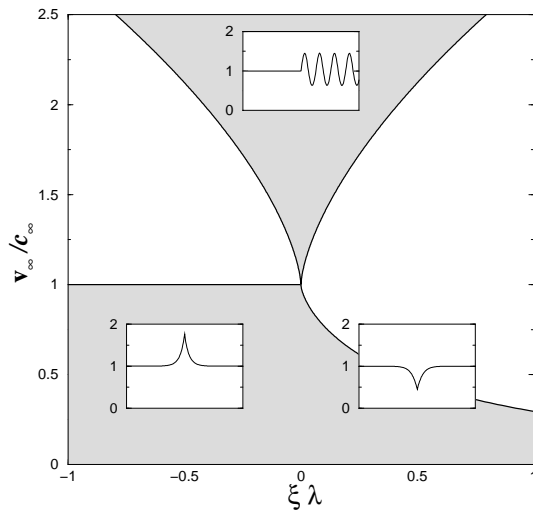


Figure 1: The shaded zone of the plane  $(v_\infty, \lambda)$  is the domain of existence of stationary solutions occurring for a potential  $V_{||}(x) = \lambda \delta(x)$ . The axis are labelled in dimensionless units. The insets represent density profiles  $n(x)/n_\infty$  typical for the different flows (the condensed beam is incident from the right). Each inset is located at values of  $v_\infty$  and  $\lambda$  typical for the flow it displays. The left (right) lower one is a superfluid flow across an attractive (repulsive) potential. The upper one is a dissipative flow.

We note here an important dissymmetry between repulsive ( $\lambda > 0$ ) and attractive ( $\lambda < 0$ ) potentials: the domain of superfluid flow extends up to  $v_\infty = c_\infty$  for  $\lambda < 0$ , i.e., the critical velocity

has the same value as the one predicted by Landau's approach. On the other hand, for  $\lambda > 0$  the critical velocity is potential-dependent and decreases significantly. This feature is very general (it occurs for all the other potentials we have studied), we will comment on it more thoroughly in Sec. 3.4 and in the Conclusion.

### 3.3 Repulsive square well

An other case with analytical stationary solutions is the repulsive square well:  $V_{\parallel}(x)$  is zero, except for  $0 < x < \sigma$  where it takes the constant and positive value  $V_0$ . In this case (4) yields:

$$F_d(t) = V_0 [n(0, t) - n(\sigma, t)] . \quad (15)$$

In the stationary subsonic regime, one can show that  $n(\sigma) = n(0)$  (see [15]) and the drag is zero : this characterizes a superfluid flow. For  $v_{\infty} > c_{\infty}$  instead, (15) yields a finite drag. In this case, when the flow is stationary,  $n(0) = n_{\infty}$  and  $n(\sigma)$  in (15) can be computed by quadrature, as a solution of

$$\sqrt{2} \left| \sigma - L_0 \operatorname{nint} \left( \frac{\sigma}{L_0} \right) \right| = \int_{\sqrt{n_{\infty}}}^{\sqrt{n(\sigma)}} \frac{dA}{[T_0 - W(n) + V_0 n]^{1/2}} , \quad (16)$$

where  $\operatorname{nint}$  denotes the nearest integer,  $L_0$  is the period of density oscillations inside the well (which is again expressible as an integral) and  $T_0 = W(n_{\infty}) - V_0 n_{\infty}$  is the constant value assumed by  $T(x)$  inside the well. Of course, (16) is only valid for a hypersonic stationary solution. In the regime where no stationary solution exists, the drag is time-dependent and should be computed numerically (as done in Section 3.4 below).

Figure 2 displays the evolution of the drag as a function of the beam velocity  $v_{\infty}$ . The drag has been computed only in the stationary regimes. In the subsonic stationary regime it is exactly zero [20]. In the super-sonic stationary regime the exact result (15) (solid line) has been computed in two manners: through the numerical solution of (16), and also by numerical integration of (8). Both methods agree within 4 digits. The exact expression is compared in Figure 2 with the perturbative result (13) (where the Fourier transform of the potential gives here  $|\hat{V}_{\parallel}(\kappa)|^2 = (2 V_0/\kappa)^2 \sin^2(\kappa\sigma/2)$ ). For large velocities, the perturbative approach becomes more and more accurate. This is expected from (10) since, when  $\kappa\sigma \gg 1$  (i.e., at large velocities) the perturbative solution is accurate if  $V_0 \ll \kappa^2$ .

One can remark in Figure 2 that, even in the supersonic regime, the drag happens to be exactly zero for some specific values of  $v_{\infty}$  [21]. As illustrated in the inset, this occurs when the period of the density oscillations inside the well is an exact divisor of its width  $\sigma$ . In this case, the density is unperturbed outside the well. The modification of density inside the well is minute in the case represented in the Figure (4%). However, for stronger potentials it can be quite substantial: for  $\sigma = 2\xi$  and  $V_0\xi^2 = 5$ , the maximum density inside the well reaches  $2n_{\infty}$  while the density remains unperturbed (i.e., equal to  $n_{\infty}$ ) outside the well.

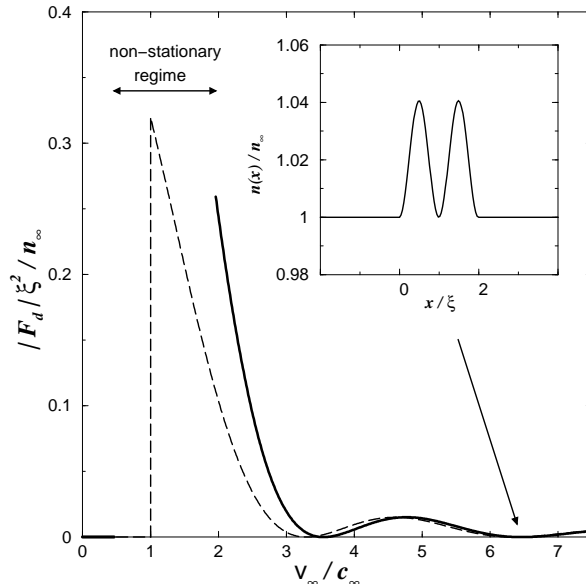


Figure 2: Drag exerted by a high density beam on a repulsive square well ( $V_0 \xi^2 = 0.2$ ,  $\sigma = 2\xi$ ), as a function of the beam velocity ( $F_d$  and  $v_\infty$  are expressed in dimensionless units). The dashed line is the perturbative result (13). The solid line is the exact drag (15). It is evaluated here only in the stationary regime. The inset represents the beam density in the supersonic regime, at a value of velocity (indicated by the arrow) where the drag is exactly zero.

### 3.4 Gaussian potential

The generality of the above deductions, based on the study of model potentials (a delta peak and a square well) can be tested numerically on more realistic potentials. We now consider the case  $V_{||}(x) = V_0 \exp\{-x^2/\sigma^2\}$  (with  $V_0 > 0$ ).

As in the previous cases, stationary solutions exist only if the beam velocity  $v_\infty$  is not too close to the sound velocity  $c_\infty$ . In the subsonic stationary regime the density is perturbed only in vicinity of the potential and the flow is superfluid (by Eq. (7)). In the supersonic stationary regime, the density oscillations extend upstream to infinity, and this corresponds to dissipation. At velocities where a stationary regime is possible, we have determined the drag indifferently using (4) or (7) (after having solved (8) numerically), whereas in the non-stationary case we used (4) after having solved the time dependent equation (2). The results are presented in Fig. 3.

The behavior of the drag in the stationary regime confirms what is expected from the results of the previous sections. In particular, the critical velocity for the onset of dissipation is lower than  $c_\infty$ . The reason for this is that, in the region of the repulsive obstacle, the density decreases and conservation of flux requires that the local fluid velocity increases. As a result, it may happen that Landau's criterion is verified (although  $v_\infty < c_\infty$ ) because the *local* fluid velocity reaches the sound velocity. For slowly varying potentials (i.e., in the regime  $\sigma \gg \xi$ ) this argument was put on a firm



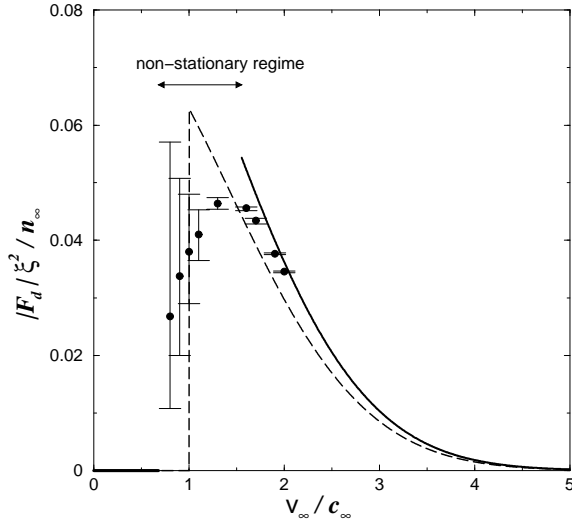


Figure 3: Drag exerted by a low density beam on a Gaussian potential ( $V_0 \xi^2 = 0.2$ ,  $\sigma = 0.5 \xi$ ), as a function of the beam velocity. The conventions are the same as in Fig. 2. The dashed line is the perturbative result (13). The solid line is the exact drag evaluated in the stationary regime. The circles correspond to the drag evaluated in the time-dependent regime. The error bars correspond to the extremal values of the time dependent  $F_d(t)$  (see the text).

mathematical basis by Hakim [22].

From this, one can infer that for *attractive potentials* instead, the critical velocity should be equal to  $c_\infty$  since for such potentials the density increases in the region of the obstacle and the local velocity decreases accordingly [23]. This has already been observed in the case of a delta-peak potential (see Fig. 1) and also for an attractive square well (see Ref. [15]). We have performed numerical checks showing that the same occurs for an attractive Gaussian potential. We will return in the final section to the difference between attractive and repulsive potentials.

Let us now come to the discussion of the time dependent data (the circles of Fig. 3). They are drawn with error bars: this does not correspond to a numerical uncertainty, but reflects the fact that the drag depends on time in the non-stationary regime. The “error bars” correspond to the extremal values of the time dependent function  $F_d(t)$  (see Fig. 4). The initial condition  $\psi(x, t = 0)$  was taken as the stationary superfluid solution in the presence of the well for the sub-sonic case  $v_\infty^{init} = 0.5 c_\infty$  to which a Galilean boost  $\exp\{i(v_\infty^{init} - v_\infty)x\}$  was applied at  $t = 0$  in order to reach the desired value of velocity. Accordingly, in the computation, the drag  $F_d(t)$  starts from 0 and after a set-up time reaches a regime where it oscillates around a mean value (see Fig. 4) which is represented by the circles in Fig. 3. The oscillations of  $F_d(t)$  around its mean value are of interest because they reflect the cause of time-dependence of the flow: the numerics indicate that whereas the up-stream flow reaches a quasi-stationary pattern, the down-stream density is perturbed by solitons, periodically emitted from the obstacle, that propagate in the same direction as the flow with a smaller velocity (such a behavior has already been observed by Hakim [22]). As illustrated in Figure 4, the drag decreases when a soliton has just been emitted (this was already

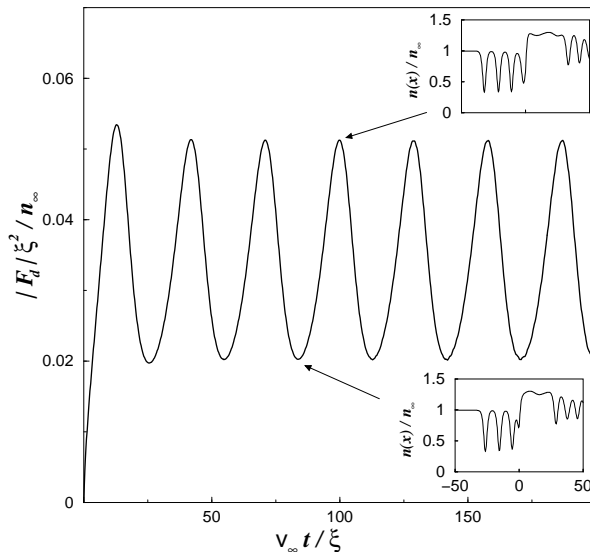


Figure 4: Time evolution of the drag exerted by a low density beam (of velocity  $v_\infty = 0.9 c_\infty$ ) on a Gaussian potential ( $V_0 \xi^2 = 0.2$ ,  $\sigma = 0.5 \xi$ ).  $F_d$  and  $t$  are expressed in dimensionless units. The upper (lower) inset represents a density profile observed when the drag is maximum (minimum).

noted in two-dimensional flows by Winiecki *et al.* [24], where vortex pairs are emitted from a moving obstacle).

In the super-sonic regime, it is also important to discuss the discrepancy between the results for the drag computed for stationary flows and for time dependent ones (solid line and circles, respectively, in Fig. 3). This discrepancy by no means implies that the stationary profile is unstable or that the asymptotic time-dependent flow is not stationary. On the contrary, numerics indicate that time-dependent flows reach an asymptotic stationary regime for velocities at which such a regime exists. Moreover the asymptotic density profile is of the expected type (flat down-stream and oscillating up-stream). The point is that our specific initial condition  $\psi(x, 0)$  does not asymptotically lead (when  $t \rightarrow \infty$ ) exactly to down-stream density and velocity which have the same value  $n_\infty$  and  $v_\infty$  as the initial flow. This is illustrated in Figure 5. On the Figure one sees that the asymptotic down-stream density is about 6% lower than  $n_\infty$  and a simple numerical check shows as well that the asymptotic velocity differs from  $v_\infty$  (is roughly 3% higher). This artifact becomes less and less important for increasing velocities: since the perturbative approach is more and more accurate, it is clear that down-stream modifications of the solution become minor and that the asymptotic solution is the expected one. As a result, when  $v_\infty$  increases, the black circles get closer to the solid line in Figure 3.

## 4 Discussion and conclusion

The above presented results illustrate the well known fact that non-linear effects alter the simple perturbative (Landau's) approach for the determination of the critical velocity at which dissipation

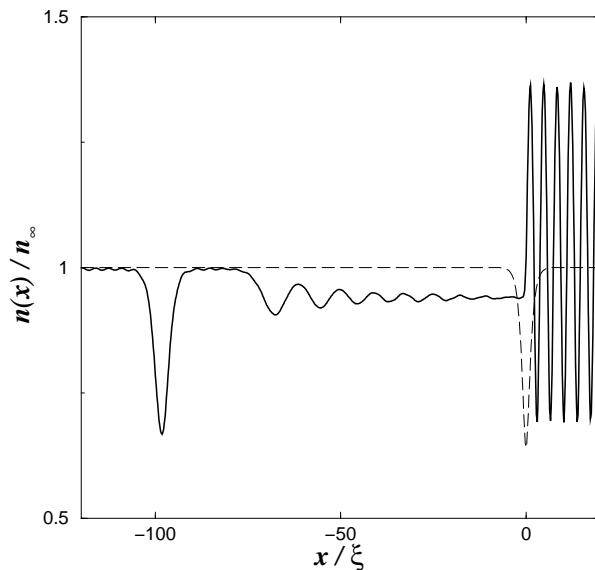


Figure 5: Density profile in the case  $v_\infty = 1.6 c_\infty$ , in the ‘presence of a Gaussian potential ( $V_0 \xi^2 = 0.2$ ,  $\sigma = 0.5 \xi$ ). The plot represents the profile (solid line) after evolution of the initial  $n(x, t = 0)$  (dashed line) during a time  $t$  with  $v_\infty t/\xi = 203.6$ . The density trough initially located at  $x = 0$  has moved to the left and is now at  $x \simeq -100 \xi$ . It will asymptotically move to left infinity. It is followed by a density slightly depressed compared to the initial value  $n_\infty$ . This depressed density is the asymptotic down-stream density.

occurs (we saw, however, that the perturbative approach could reach a regime of accuracy at large velocity). For repulsive potentials the critical beam velocity is smaller than the velocity of sound  $c_\infty$  (which is here the critical velocity from Landau’s criterion) because, in the region of the potential, the local fluid velocity can reach values higher than the sound velocity. The onset of dissipation corresponds to non-stationary flows with a wake asymptotically extending upstream to infinity, and down-stream periodic emission of solitons. In fact, another way of explaining the lower stability of the dissipationless flow in the presence of a repulsive potential is to remark that, for sub-sonic flow over such an obstacle, the density decreases in the region of the potential, allowing easier nucleation of solitons.

On the other hand, we have shown that, for attractive potentials, stationary dissipationless solutions exist up to  $v_\infty = c_\infty$ : Bose-Einstein condensates appear to be excellent supports for reaching Landau critical velocity, more appropriate than superfluid helium because, in atomic vapors, it is simpler to construct obstacles described by purely attractive potentials.

We also showed, using analytical and numerical examples, that stationary dissipative profiles exist in hyper-sonic beams, provided the beam velocity is large enough. From the numerical study, these solutions seem stable, and moreover time-dependent flows tend asymptotically to such solutions (when they exist). It is interesting to note that dissipation is drastically reduced at very high velocity: i.e., superfluidity is recovered. Such an effect should also exist in higher

dimensions for penetrable potentials. It can be understood perturbatively: at high velocity, when the perturbative approach becomes valid, the relevant wave-vector (denoted  $\kappa$  in Sec. 3.1) is large and Bogoliubov dispersion relation becomes exactly quadratic. Hence, in this regime one has a matter-wave described by the linear Schrödinger equation. The drag in this regime can be shown to be proportional to the reflection coefficient which, as well known, decreases at high energy (in any dimension).

Note that, for the sake of clarity, we have always illustrated our conclusions using rather weak perturbing potentials. The reasons for this are twofold: (1) for stronger potentials the super-sonic stationary regime that we wanted to emphasize is rejected to higher velocity; (2) the numerical effort necessary in the study of the non-stationary regime is decreased for weak potentials since the domain of time-dependence is reduced.

Although for stronger potentials the qualitative results remain the same, very interesting new quantitative phenomena occur. In particular, enormous differences in drag can occur when switching from repulsive to attractive potentials (provided the potential is strong enough). This is illustrated in Fig. 6.

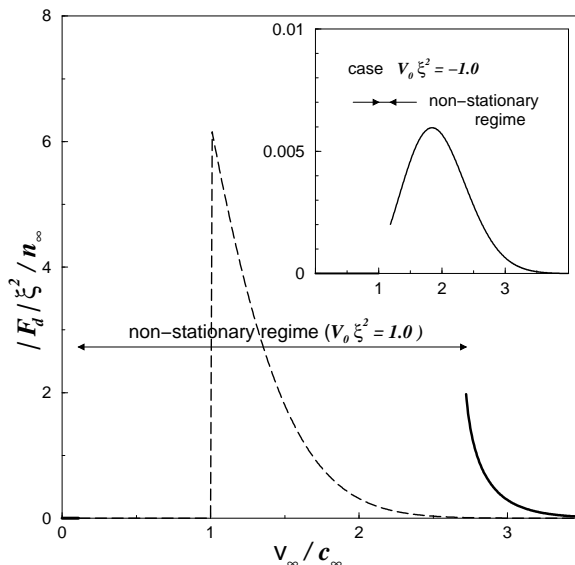


Figure 6: Drag exerted by a low density beam on a Gaussian potential ( $V_0 \xi^2 = \pm 1.0$ ,  $\sigma/\xi = 1.0$ ), as a function of the beam velocity. The conventions are the same as in Fig. 2. The main figure displays the drag for a repulsive potential (solid line) together with the perturbative result (13) (dashed line) which is not affected by the sign of the potential. The inset displays a blow-up of the main figure allowing to see the (very small) drag for an attractive potential.

One sees in the Figure that the domain of existence of stationary solutions is markedly different between attractive and repulsive potentials and that the values of the drag differ by more than two orders of magnitude ! The physical explanation for this phenomenon is subtle. At high velocity, contrarily to intuition, the density decreases (increases) in an attractive (repulsive) potential [25].

Hence, an attractive potential creates a density trough which, being the super-sonic analog of a gray soliton, does not create large perturbations in its wake. This results in a small drag. This feature is clearly missed by the perturbative approach which gives a drag insensitive to the sign of the potential (see Eq. (13)).

From the result of Fig. 6 it would be very interesting to redo with an *attractive* perturbing potential the experiments which have been done at M.I.T. with a repulsive potential [12, 13]. Although the present computations are valid in a quasi one-dimensional regime (whereas the experiments were done in truly three dimensional systems) the present results leaves no doubt that the critical velocity for the onset of dissipation should increase and that the energy transfer rate from the obstacle to the fluid (i.e.,  $F_d v_\infty$ ) should drastically decrease in case of an attractive obstacle [26].

Finally, we note that the present discussion sheds some light on the theory of wave resistance. The wave resistance is the part of the drag experienced by a body moving in a medium which is caused by excitation of waves in this medium (typically surface waves in the case of boats). In a superfluid a  $T = 0$  (i.e., in the present work), this is the only source of drag (if one broadens its definition in order to include non-linear effects such as vortex or soliton formation). Recent experiments of moving spheres in silicone oil [27] have shown that the wave resistance at Landau threshold has a smooth behavior as a function of the velocity, contrarily to the expectation of perturbation theory [28]. The same behavior was found here : non-linear effects smoothen out the unphysical step in drag predicted by perturbation theory. This smooth behavior was already observed in the experiments done at M.I.T. [12, 13] and in numerical simulations by Frisch *et al.* [29] and Winiecki *et al.* [24] when moving an impenetrable sphere in a superfluid. Hence, Bose condensed systems offer an interesting testing ground for ubiquitous non-linear hydrodynamical effects, in a particularly simple theoretical framework (the Gross-Pitaevskii equation).

## Acknowledgments

It is a pleasure to thank P. Leboeuf and C. Schmit for fruitful discussions.

## References

- [1] M. J. Renn *et al.*, Phys. Rev. A **55**, 3684 (1997); D. Müller, E. A. Cornell, D. Z. Anderson, and E. R. I. Abraham, Phys. Rev. A **61**, 033411 (2000).
- [2] J. Denschlag, D. Cassettari and J. Schmiedmayer, Phys. Rev. Lett. **82**, 2014 (1999); M. Key *et al.*, Phys. Rev. Lett. **84**, 1371 (2000); J. A. Richmond *et al.*, Phys. Rev. A **65**, 033422 (2002).
- [3] N. H. Dekker *et al.*, Phys. Rev. Lett. **84**, 1124 (2000); W. Hänsel, J. Reichel, P. Hommelhoff, and T. W. Hänsch, Phys. Rev. Lett. **86**, 608 (2001).
- [4] K. Bongs *et al.*, Phys. Rev. A **63**, 031602 (2001).
- [5] W. Hänsel, P. Hommelhoff, T. W. Hänsch, and J. Reichel, Nature **413**, 498 (2001).
- [6] H. Ott *et al.*, Phys. Rev. Lett. **87**, 230401 (2001).

- [7] A. E. Leanhardt *et al.*, cond-mat/0203214.
- [8] P. Cren *et al.*, cond-mat/0203618, to appear in Eur. Phys. J. D.
- [9] L. D. Landau, J. Phys. Moscow **5**, 71 (1941), reprinted in I. M. Khalatnikov, *An Introduction to the Theory of Superfluidity*, (Perseus Publishing, Cambridge, 2000).
- [10] R. P. Feynman in *Progress in Low Temperature Physics*, vol. I, p. 17, C. J. Gorter ed., (North-Holland, Amsterdam, 1957).
- [11] However, in liquid helium II, Landau's limit can be reached in some specific cases, see, e.g., P. V. E. McClintock and R. M. Bowley, *Progress in Low Temperature Physics*, vol. XIV, p. 2, W. P. Halperin ed., (Elsevier, Amsterdam, 1995).
- [12] C. Raman *et al.*, Phys. Rev. Lett. **83**, 2502 (1999).
- [13] R. Onofrio *et al.*, Phys. Rev. Lett. **85**, 2228 (2000).
- [14] S. Inouye *et al.*, Phys. Rev. Lett. **87**, 080402 (2001).
- [15] P. Leboeuf and N. Pavloff, Phys. Rev. A **64**, 033602 (2001).
- [16] A. D. Jackson, G. M. Kavoulakis, and C. J. Pethick, Phys. Rev. A **58**, 2417 (1998).
- [17] Note that we consider a quasi 1D system with infinite coherence length, and discard from the present study quasi-condensates such as observed by S. Dettmer *et al.*, Phys. Rev. Lett. **87**, 160406 (2001). In fact, we can also describe quasi-condensates, the assumption made is that the temperature is low enough so that the coherence length is large compared to the size of the structure we describe (typically, a few times the relaxation length).
- [18]  $T$  is constructed from the Lagrangian density  $\mathcal{L}[\psi, \psi^*, x]$  from which (2) is derived:  $T = -\partial_x \psi \partial \mathcal{L} / \partial [\partial_x \psi] - \partial_x \psi^* \partial \mathcal{L} / \partial [\partial_x \psi^*] + \mathcal{L}$ , see, e.g., G. Wentzel, *Quantum Theory of Fields*, (Interscience, New-York, 1949).
- [19] F. Dalfovo, S. Giorgini, L. P. Pitaevskii, and S. Stringari, Rev. Mod. Phys. **71**, 463 (1999).
- [20] Even in the subsonic regime one finds stationary solutions with a wake corresponding to dissipation. However, these solutions are presumably unstable, whereas the solutions localized on the perturbation are found to be numerically stable. Besides, when varying  $v_\infty/c_\infty$  and the strength of the potential, one finds that the dissipative sub-sonic solutions have a smaller domain of existence than the superfluid ones: i.e., on the edge of the domain of superfluidity, the localized dissipationless solutions are the only existing ones (see Ref. [15]).
- [21] Superfluid behavior at supersonic speed has already been found for other specific potentials: C. K. Law, C. M. Chan, P. T. Leung and M.-C. Chu, Phys. Rev. Lett. **85**, 1598 (2000).
- [22] V. Hakim, Phys. Rev. E **55**, 2835 (1997).

- [23] Taking into account modifications of the Bogoliubov spectrum which go beyond our adiabatic ansatz (1) slightly alters this oversimplified view and leads to a critical velocity slightly smaller than  $c_\infty$ , even for attractive potentials, see P. O. Fedichev and G. V. Shlyapnikov, Phys. Rev. A **63**, 045601 (2001).
- [24] T. Winiecki, J.F. McCann and C.S. Adams, Phys. Rev. Lett. **82**, 5186 (1999).
- [25] This is most clearly seen in the perturbative regime  $\kappa\sigma \gg 1$  and  $V_0 \ll \kappa^2$ , where (10) leads to:  $\delta A(x) = 2 \operatorname{sgn}(v_\infty - c_\infty) \kappa^{-2} \sqrt{n_\infty} V_{\parallel}(x)$ .
- [26] Moreover, the experiments can be done in the same quasi 1D regime as the present theory, by taking as an obstacle a laser whose waist is much larger than the transverse size of the condensate.
- [27] T. Burghelea and V. Steinberg, Phys. Rev. Lett. **86**, 2557 (2001).
- [28] E. Raphaël and P.-G. de Gennes, Phys. Rev. E **53**, 3448 (1996).
- [29] T. Frisch, Y. Pomeau and S. Rica, Phys. Rev. Lett. **69**, 1644 (1992).