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HCIZ integral and 2D Toda lattice hierarchy

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The expression of the large N Harish Chandra–Itzykson–Zuber (HCIZ) integral in terms of the moments of the two matrices is investigated using an auxiliary unitary two-matrix model, the associated biorthogonal polynomials and integrable hierarchy. We find that the large N HCIZ integral is governed by the dispersionless Toda lattice hierarchy and derive its string equation. We use this to obtain various exact results on its expansion in powers of the moments.

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1. Introduction and definitions

The Harish Chandra–Itzykson–Zuber integral [1,2] plays a key role in matrix models [2,3,4,5,6], in particular in relation to (discretized) two-dimensional quantum gravity. It also displays interesting connections [7,8] with free probability theory [9,10]. It is defined by

$$I_N(\mathrm{A},\mathrm{B}) = \int_{\Omega\in U(N)} \mathrm{d}\Omega\, \mathrm{e}^{N\mathrm{tr}\mathrm{A}\Omega\mathrm{B}\Omega^\dagger} \tag{1.1}$$

where A and B are $N\times N$ matrices and $\mathrm{d}\Omega$ is the normalized Haar measure on $U(N)$. There are many ways to evaluate this integral [1,2,4,11,12,13]; the result depends only on the eigenvalues a_i of A and b_i of B, and is given by

$$I_N(\mathrm{A},\mathrm{B}) = \frac{\det(\mathrm{e}^{Na_ib_j})}{\Delta(a_i)\Delta(b_j)} \tag{1.2}$$

Here $\Delta(\cdot)$ denotes the Vandermonde determinant: $\Delta(a_i)=\prod_{i<j}(a_i-a_j)$. In all equations, constants that only depend on N will be ignored; they can be restored when required.

For practical applications it is important to look at the large N limit of the HCIZ integral: it corresponds to the "planar" limit of matrix models. One considers a sequence of $N\times N$ matrices A and B, whose spectral distributions converge to some fixed distributions as N goes to infinity. It is known (and in fact, proved rigorously [14]) that the following quantity, the "large N free energy", is well-defined:

$$F[\theta_q,\tilde\theta_q] = \lim_{N\to\infty}\frac{\log I_N(\mathrm{A},\mathrm{B})}{N^2} \tag{1.3}$$

and depends on the spectral distributions of A and B only through their moments:

$$\theta_q = \lim_{N\to\infty}\frac{\mathrm{tr}}{N}\mathrm{A}^q \qquad \tilde\theta_q = \lim_{N\to\infty}\frac{\mathrm{tr}}{N}\mathrm{B}^q \qquad q\ge 1 \tag{1.4}$$

We also define for future use the generating series of these moments, that is the resolvents of A and B:

$$G(a) \equiv \lim_{N\to\infty}\frac{\mathrm{tr}}{N}\frac{1}{a-\mathrm{A}} = 1/a + \sum_{q\ge1}\theta_q/a^{q+1} \tag{1.5a}$$

$$\tilde G(b) \equiv \lim_{N\to\infty}\frac{\mathrm{tr}}{N}\frac{1}{b-\mathrm{B}} = 1/b + \sum_{q\ge1}\tilde\theta_q/b^{q+1} \tag{1.5b}$$

As a function of the θ_q and the $\tilde\theta_q$, F has a power series expansion around 0. We shall investigate here the determination of the coefficients of this expansion, as well as the more general question of the evolution of F as one varies the θ_q and $\tilde{\theta}_q$. We shall see that this evolution is governed by an infinite set of partial differential equations, a particular scaling of the Toda lattice hierarchy.

More precisely, we shall rewrite in section 2 the HCIZ integral as a unitary two-matrix model, then show in section 3 that the latter satisfies the Toda lattice hierarchy; in section 4, we shall take the large N limit which relates F to the dispersionless Toda lattice hierarchy; finally, we shall apply this technology to the expansion of F in section 5. Section 6 is devoted to a summary of results.

2. HCIZ integral as a unitary two-matrix model

We shall first use a trick to rewrite the HCIZ integral as an integral over two matrices. The derivation can be performed either by computing residues in the two-matrix integral, or by character expansion; we choose the latter. We thus start with the following identity, valid for any $N \times N$ matrix M:

$$e^{N \mathrm{tr} \mathrm{M}} = \sum_{R \geq 0} c_R d_R \chi_R(\mathrm{M}) \tag{2.1}$$

Let us explain the symbols used. R stands for an analytic irrep of $GL(N)$ with positive highest weights, or the associated Young diagram; more explicitly, it can be described by its "shifted" highest weights h_i, $1 \leq i \leq N$ which form a strictly decreasing sequence (they are related to the usual highest weights m_i by $h_i = m_i + N - i$). $\chi_R(\mathrm{M})$ is the character associated to the irrep R and the matrix M. c_R and d_R are coefficient of the expansion:

$$c_R = \frac{N^{|R|}}{\prod_{i=1}^{N} h_i!} \qquad d_R = \Delta(h_i) \tag{2.2}$$

where $|R| = \sum_i h_i - N(N-1)/2$ is the number of boxes of R. These explicit expressions of the coefficients play no role in what follows since the trick we use is quite general.[1]

Applying the identity (2.1) to the definition of the HCIZ integral (1.1) and using orthogonality relations for matrix elements of irreps yields

$$I_N(\mathrm{A}, \mathrm{B}) = \sum_{R \geq 0} c_R \chi_R(\mathrm{A}) \chi_R(\mathrm{B}) \tag{2.3}$$

[1] One can treat similarly any expression of the form $\frac{\det f(a_i, b_j)}{\Delta(a_i)\Delta(b_j)}$, cf Eq. (1.2). However here we do not use Eq (1.2) – in fact, we prove it along the way.

As a side remark, we note that using Weyl's formula for characters ($\chi_R(\mathrm{A}) = \det(a_i^{h_j})/\Delta(a_i)$), recombining the two resulting determinants into one single determinant and performing the summation, one recovers Eq. (1.2).

We now want to compare expression (2.3) with the partition function of a matrix model of two $N \times N$ matrices U and V:

$$\tau_N[t_q, \tilde{t}_q] = \oint\oint \mathrm{dUdV} \mathrm{e}^{\sum_{q\geq 1} t_q \mathrm{trU}^q + \sum_{q\geq 1} \tilde{t}_q \mathrm{trV}^q + \mathrm{trU}^{-1}\mathrm{V}^{-1}} \tag{2.4}$$

Here the t_q are as yet unknown coefficients; they must however satisfy $\sup_q |t_q|^{1/q} < m < \infty$ and $\sup_q |\tilde{t}_q|^{1/q} < \tilde{m} < \infty$; this renders the summation convergent provided that the spectral radii of U and V are less or equal to $1/m$, $1/\tilde{m}$. Since the integrand is analytic in U and V the exact contours of integration are irrelevant; for the sake of definiteness, we choose U and V to be unitary up to a multiplicative constant: $\mathrm{UU}^\dagger = m^{-2}$, $\mathrm{VV}^\dagger = \tilde{m}^{-2}$. The measure of integration is simply the normalized Haar measures for $m\,\mathrm{U}$ and $\tilde{m}\,\mathrm{V}$. All these details can be safely ignored if one remembers that in the integral (2.4) only the contribution coming from poles at the origin should be picked up.

Note that we have defined (2.4) so that N does not appear anywhere explicitly in the action. It is however convenient to rescale the matrices to restore the usual N dependence of matrix models:

$$\tau_N[t_q, \tilde{t}_q] = \oint\oint \mathrm{dUdV} \mathrm{e}^{\sum_{q\geq 1} t_q N^{-q/2} \mathrm{trU}^q + \sum_{q\geq 1} \tilde{t}_q N^{-q/2} \mathrm{trV}^q + N\mathrm{trU}^{-1}\mathrm{V}^{-1}} \tag{2.5}$$

Next we apply one of the forms of the Cauchy identity:

$$\mathrm{e}^{\sum_{q\geq 1} t_q N^{-q/2} \mathrm{trU}^q} = \sum_{R\geq 0} s_R(t_q N^{-q/2}) \chi_R(\mathrm{U}) \tag{2.6}$$

where s_R is the Schur function associated to the Young diagram R.

Plugging twice (2.6) as well as the identity (2.1) into (2.5) results in:

$$\tau_N[t_q, \tilde{t}_q] = \sum_{R, R_1, R_2 \geq 0} s_{R_1}(t_q N^{-q/2}) s_{R_2}(\tilde{t}_q N^{-q/2}) c_R d_R \oint\oint \mathrm{dUdV} \chi_{R_1}(\mathrm{U}) \chi_{R_2}(\mathrm{V}) \chi_R(\mathrm{U}^{-1}\mathrm{V}^{-1}) \tag{2.7}$$

Again, orthogonality relations imply that

$$\tau_N[t_q, \tilde{t}_q] = \sum_{\substack{R\geq 0 \\ \#\mathrm{rows}\leq N}} c_R s_R(t_q N^{-q/2}) s_R(\tilde{t}_q N^{-q/2}) \tag{2.8}$$

Note that we have written explicitly the constraint that the Young diagram R should have no more than N rows, since contrary to prior expressions the summand in Eq. (2.8) is not necessarily zero if the number of rows exceeds N.

Comparing Eqs. (2.8) and (2.3), we see that the two expressions are equal (up to a numerical factor) iff

$$t_q N^{-q/2} = \frac{1}{q}\mathrm{trA}^q \qquad \tilde{t}_q N^{-q/2} = \frac{1}{q}\mathrm{trB}^q \tag{2.9}$$

or in terms of the normalized moments introduced before

$$t_q = N^{q/2+1}\frac{\theta_q}{q} \qquad \tilde{t}_q = N^{q/2+1}\frac{\tilde{\theta}_q}{q} \tag{2.10}$$

The constraint on the number of rows of R is automatically satisfied if the t_q are of the form (2.9). However, the expression (2.8) has the advantage that it is defined for arbitrary (independent) t_q at finite N, which is not the case of the original HCIZ integral (where there are only N independent t_q). Also note that the condition of finiteness of $\sup_{q\geq 1}|t_q|^{1/q}$ and $\sup_{q\geq 1}|\tilde{t}_q|^{1/q}$ is automatically satisfied at finite N if t_q is of the form (2.9), and remains true as $N\to\infty$ provided the spectra of A and B remain bounded.

3. Biorthogonal polynomials and Toda lattice hierarchy

We have found in the previous section that calculating the HCIZ integral is equivalent to solving the matrix model given by Eq. (2.4). A convenient way to do so, at least formally, is via biorthogonal polynomials. This is standard material, and is very similar to derivations found in the literature [15]. τ_N will then be identified with a tau function of the 2D Toda lattice hierarchy.

3.1. Setup

We start with the following determinant form of the partition function (2.4), obtained by diagonalizing U and V:

$$\tau_N = \det\left(\oint\oint \frac{\mathrm{d}u}{2\pi i u}\frac{\mathrm{d}v}{2\pi i v} u^j v^i \mathrm{e}^{\sum_{q\geq 1} t_q u^q + \sum_{q\geq 1}\tilde{t}_q v^q + u^{-1}v^{-1}}\right)_{0\leq i,j\leq N-1} \tag{3.1}$$

where once again the contours of integrations are sufficiently small circles around the origin.

This suggests to introduce a non-degenerate bilinear form on the space of polynomials by

$$\langle q|p\rangle = \oint\oint \frac{\mathrm{d}u}{2\pi i u}\frac{\mathrm{d}v}{2\pi i v} p(u)q(v)\mathrm{e}^{\sum_{q\geq 1} t_q u^q + \sum_{q\geq 1}\tilde{t}_q v^q + u^{-1}v^{-1}} \tag{3.2}$$

so that $\tau_N = \det M_N$, with $M_N = (m_{ij})$, $0 \le i,j \le N-1$ and $m_{ij} = \langle v^i | u^j \rangle$.

Next we define normalized biorthogonal polynomials $q_n(v)$ and $p_n(u)$ with respect to the bilinear form above on $\mathbb{C}[v] \times \mathbb{C}[u]$, that is $\langle q_n | p_m \rangle = \delta_{nm}$. We write:

$$p_n(u) = \sum_{k=0}^{n} p_{kn} u^k \qquad q_n(v) = \sum_{k=0}^{n} q_{kn} v^k \tag{3.3}$$

There is an arbitrariness in the normalization which is fixed by assuming $p_{nn} = q_{nn} \equiv h_n^{-1}$.

Define upper triangular matrices $P_N = (p_{kn})$ and $Q_N = (q_{kn})$, $0 \le n,k \le N-1$; and $\tilde{P}_N = {Q_N^T}^{-1}$. Orthonormality of these polynomials is equivalent to writing

$$Q_N^T M_N P_N = 1 \tag{3.4}$$

or $M_N = \tilde{P}_N P_N^{-1}$, with P_N^{-1} upper triangular and $\tilde{P}_N$ lower triangular. In particular $\tau_N = \det M_N = \prod_{i=0}^{N-1} h_i^2$.

Next, introduce the semi-infinite matrices $M = M_\infty$, $P = P_\infty$, $Q = Q_\infty$, $\tilde{P} = \tilde{P}_\infty$ and the shift matrix Z: $Z_{ij} = \delta_{i\, j+1}$. Set

$$U = P^{-1} Z P \qquad V = Q^{-1} Z Q \tag{3.5}$$

U (resp. V) is the matrix of multiplication by u (resp. v) in the basis $(p_n(u))$ of $\mathbb{C}[u]$ (resp. $(q_n(v))$ of $\mathbb{C}[v]$). Also set $\tilde{U} \equiv V^T = \tilde{P}^{-1} Z^T \tilde{P}$. By definition of the m_{ij}, we have

$$\frac{\partial M}{\partial t_q} = M Z^q \qquad \frac{\partial M}{\partial \tilde{t}_q} = {Z^T}^q M \tag{3.6}$$

We then easily find, using $M = \tilde{P} P^{-1}$,

$$-P^{-1} \frac{\partial P}{\partial t_q} + \tilde{P}^{-1} \frac{\partial \tilde{P}}{\partial t_q} = U^q \qquad -P^{-1} \frac{\partial P}{\partial \tilde{t}_q} + \tilde{P}^{-1} \frac{\partial \tilde{P}}{\partial \tilde{t}_q} = \tilde{U}^q \tag{3.7}$$

$P^{-1} \frac{\partial P}{\partial t_q}$ is upper triangular, whereas $\tilde{P}^{-1} \frac{\partial \tilde{P}}{\partial t_q}$ is lower triangular, and they have opposite diagonal elements (and similarly for derivatives with respect to $\tilde{t}_q$); so that if one defines symbols $(\cdots)_+ = (\cdots)_{>0} + \frac{1}{2}(\cdots)_0$ and $(\cdots)_- = (\cdots)_{<0} + \frac{1}{2}(\cdots)_0$ for lower and upper diagonal parts respectively (it is convenient to include one half of the diagonal part, though this does not make them projections), we have

$$\frac{\partial P}{\partial t_q} = -P\,(U^q)_- \qquad \frac{\partial P}{\partial \tilde{t}_q} = -P\,(\tilde{U}^q)_- \tag{3.8a}$$

$$\frac{\partial \tilde{P}}{\partial t_q} = \tilde{P}\,(U^q)_+ \qquad \frac{\partial \tilde{P}}{\partial \tilde{t}_q} = \tilde{P}\,(\tilde{U}^q)_+ \tag{3.8b}$$

and finally

$$\frac{\partial U}{\partial t_q} = -[(U^q)_+, U] \qquad \frac{\partial U}{\partial \tilde{t}_q} = [(\tilde{U}^q)_-, U] \tag{3.9a}$$

$$\frac{\partial \tilde{U}}{\partial t_q} = -[(U^q)_+, \tilde{U}] \qquad \frac{\partial \tilde{U}}{\partial \tilde{t}_q} = [(\tilde{U}^q)_-, \tilde{U}] \tag{3.9b}$$

Equations (3.9) are equivalent to equations of Zakharov–Shabat type [16]

$$\frac{\partial}{\partial t_q}(U^r)_+ - \frac{\partial}{\partial t_r}(U^q)_+ + [(U^q)_+, (U^r)_+] = 0 \tag{3.10a}$$

$$\frac{\partial}{\partial \tilde{t}_q}(U^r)_+ + \frac{\partial}{\partial t_r}(\tilde{U}^q)_- - [(\tilde{U}^q)_-, (U^r)_+] = 0 \tag{3.10b}$$

$$\frac{\partial}{\partial \tilde{t}_q}(\tilde{U}^r)_- - \frac{\partial}{\partial \tilde{t}_r}(\tilde{U}^q)_- - [(\tilde{U}^q)_-, (\tilde{U}^r)_-] = 0 \tag{3.10c}$$

which ensure compatibility of Eqs. (3.8). If we expand U and $\tilde{U}$ in the following way:

$$U = r\, Z + \sum_{k=0}^{\infty} u_k\, Z^{T\,k} \tag{3.11a}$$

$$\tilde{U} = Z^T\, r + \sum_{k=0}^{\infty} Z^k\, v_k \tag{3.11b}$$

where $r = \mathrm{diag}(r_n)$, the $u_k = \mathrm{diag}(u_{k;n})$ and the $v_k = \mathrm{diag}(v_{k;n})$ are diagonal matrices, then Eqs. (3.10) are the (semi-infinite) Toda lattice hierarchy of differential equations for the coefficients of U and $\tilde{U}$.

3.2. Examples

The simplest equation, the usual Toda lattice equation, which is the case $q = r = 1$ in Eq. (3.10*b*), can be obtained directly from the determinant form (3.1) by applying Jacobi's determinant identity, which yields $\tau_{n+1}\tau_{n-1} = \tau_n \frac{\partial^2}{\partial t_1 \partial \tilde{t}_1}\tau_n - \frac{\partial}{\partial t_1}\tau_n \frac{\partial}{\partial \tilde{t}_1}\tau_n$, or

$$\frac{\tau_{n+1}\tau_{n-1}}{\tau_n^2} = \frac{\partial^2}{\partial t_1 \partial \tilde{t}_1} \log \tau_n \tag{3.12}$$

The left hand side is nothing but $(h_n/h_{n-1})^2 = r_n^2$. This implies

$$r_{n+1}^2 + r_{n-1}^2 - 2r_n^2 = \frac{\partial^2}{\partial t_1 \partial \tilde{t}_1} \log r_n^2 \tag{3.13}$$

which is the Toda lattice equation in terms of the $\log r_n^2$. In the next section we shall see how to use this equation in the large N limit.

Another equation one can derive (case $q = 2$, $r = 3$ of Eq. (3.10a)) is the usual Kadomtsev–Petviashvili (KP) equation, which is given in terms of $\chi = 2\frac{\partial^2}{\partial t_1^2} \log \tau_N$:

$$3\frac{\partial^2 \chi}{\partial t_2^2} + \frac{\partial}{\partial t_1}\left(-4\frac{\partial \chi}{\partial t_3} + 6\chi\frac{\partial \chi}{\partial t_1} + \frac{\partial^3 \chi}{\partial t_1^3}\right) = 0 \tag{3.14}$$

3.3. String equation

So far we have not used the specific form of the interaction between our two matrices in (2.4). This occurs when one tries to determine the *string equation*, that is an additional relation which fixes a particular solution of the Toda hierarchy. Here it is very easy to derive the string equation by arguments similar to those used in the standard two-matrix model. We write that

$$0 = \oint\oint \frac{\mathrm{d}u}{2\pi i u}\frac{\mathrm{d}v}{2\pi i v} u \frac{\mathrm{d}}{\mathrm{d}u}\left[p_j(u) q_i(v) \mathrm{e}^{\sum_{q\geq 1} t_q u^q + \sum_{q\geq 1} \tilde{t}_q v^q + u^{-1}v^{-1}}\right] \tag{3.15}$$

expand and obtain

$$UD + \sum_{q\geq 1} q t_q U^q - \tilde{U}^{-1}U^{-1} = 0 \tag{3.16}$$

where D is the derivative with respect to u in the basis $p_n(u)$, and U^{-1} is a particular left inverse of U whose matrix elements are defined by Eq. (3.15), and similarly $\tilde{U}^{-1}$ is a particular right inverse of $\tilde{U}$. Using $[D, U] = 1$ and $U^{-1}U = 1$, we derive from Eq. (3.16) the string equation:

$$[U^{-1}, \tilde{U}^{-1}] = 1 \tag{3.17}$$

Let us note that Eq. (3.17) is *not* the usual string equation of the hermitean two-matrix model.

4. Large N limit

4.1. Classical/dispersionless limit

Next we want to consider the large N limit. As always in matrix models, this identifies with a *classical* limit for the operators involved, the small parameter being $\hbar \equiv 1/N$. Therefore in this limit, the various operators U, $\tilde{U}$, Z become classical variables u, v, z, up to some rescalings which will be detailed later.

What happens to the Toda equation? The large N limit corresponds to the dispersionless limit of the Toda hierarchy [17,18]. In this limit, operators of discrete difference in the index n (e.g. Eq. (3.13)) can be replaced with a differential operator $\partial/\partial\nu$ with $\nu \equiv n/N$. Furthermore, the logarithm of the tau-function τ_N is required to be of order N^2 (cf our Eq. (1.3)), with a smooth $1/N$ expansion,[2] which leads to some simplifications in the differential equations. We shall only write down a few examples of these dispersionless equations; we refer the reader to e.g. [17,18,20] for a complete and more rigorous description of the dispersionless hierarchy. Let us simply mention that, as mentioned above, we are dealing with a classical limit, that is the commutators become of order $\hbar \equiv 1/N$, and their leading term defines a Poisson bracket. Equations (3.9)–(3.17) can be rewritten in the large N limit, once rescaled, by simply replacing commutators with Poisson brackets. In particular, the string equation (3.17) becomes:

$$\{u^{-1}, v^{-1}\} = 1 \tag{4.1}$$

(where now u^{-1} and v^{-1} are ordinary inverses since in the large N limit all operators are invertible). It is worth noting that the same string equation occurs in the interior Dirichlet boundary problem [21]. Also, it is identical to the string equation of [22] in which the radius has the unphysical value $R = -1$.

4.2. Scaling. Some examples

In order to give some examples, we need to give some additional details on the particular scaling of the variables that is appropriate for our model. First, we have for our operators U, $\tilde{U}$, Z: (cf the rescaling from Eq. (2.4) to Eq. (2.5)):

$$U \sim N^{-1/2} u \qquad \tilde{U} \sim N^{-1/2} v \qquad Z \sim z \tag{4.2}$$

where u, v, z have a finite limit when $N \to \infty$ and are related by:

$$u = r\,z + \sum_{k=0}^{\infty} u_k\, z^{-k} \tag{4.3a}$$

$$v = r\,z^{-1} + \sum_{k=0}^{\infty} v_k\, z^{k} \tag{4.3b}$$

[2] The assumption of smoothness is less innocent than it seems; it is the equivalent of the "single cut" hypothesis in matrix models (cf a similar remark in [19] the context of 1D Toda). However in formal power series around $t_q = \tilde{t}_q = 0$ this hypothesis is certainly valid.

in terms of appropriately rescaled coefficients (in particular $r_N \sim rN^{-1/2}$). We have already derived the rescaling of the moments, cf Eq. (2.10).

In what follows, we shall freely change from original to rescaled variables, using the same letters by abuse of notation. In most circumstances this does not cause any confusion.

For example, let us see what happens to the KP equation (3.14). In the large N limit, performing all the rescalings one term drops out and we find

$$2\frac{\partial^2\chi}{\partial\theta_2^2}+\frac{\partial}{\partial\theta_1}\left(-2\frac{\partial\chi}{\partial\theta_3}+\chi\frac{\partial\chi}{\partial\theta_1}\right)=0 \tag{4.4}$$

which is just the dispersionless KP equation (or Khokhlov–Zabolotskaia equation).

Let now see what happens to the Toda lattice equation (3.12) in the large N limit. Here, it is important to notice that an extra ingredient is required if one considers equations in which variations with respect to n appear. It is our scaling hypothesis, based on the fact that we consider only the "planar" large N limit (first term in the $1/N$ expansion) and on the scaling of the t_q and $\tilde{t}_q$. Considering Eq. (1.3), we write that:

$$\tau_n = c_n\,\mathrm{e}^{n^2 F[n^{-q/2-1}qt_q,\, n^{-q/2-1}q\tilde{t}_q]+O(1)} \tag{4.5}$$

The constant c_n can be determined by setting $t_q=\tilde{t}_q=0$ in Eq. (3.1); one easily finds $c_n=(\prod_{i=0}^{n-1} i!)^{-1}$.

More explicitly, if one introduces the dispersionless tau function $F[\nu,\theta_q,\tilde{\theta}_q] = \lim_{N\to\infty}(\frac{1}{N^2}\log\tau_n+\frac{1}{2}\nu^2\log N-\frac{3}{4})$ with $\nu=n/N$, then we have the scaling

$$F[\nu,\theta_q,\tilde{\theta}_q]=-\frac{1}{2}\nu^2\log\nu+\frac{3}{4}(\nu^2-1)+\nu^2F[\theta_q\nu^{-q/2-1},\tilde{\theta}_q\nu^{-q/2-1}] \tag{4.6}$$

where our old function $F[\theta_q,\tilde{\theta}_q]$ is recovered by setting $\nu=1$.

We now rewrite the Toda equation (3.12):

$$\mathrm{e}^{\frac{\partial^2}{\partial\nu^2}F}=\frac{\partial^2}{\partial\theta_1\partial\tilde{\theta}_1}F \tag{4.7}$$

which is just the dispersionless Toda equation (integrated twice); then apply the scaling (4.6). The final result is the following partial differential equation for F:

$$\begin{aligned}&\exp\Big(2F+\sum_{q\geq 1}(\tfrac{q}{2}+1)(\tfrac{q}{2}-2)(\theta_q\partial_q F+\tilde{\theta}_q\tilde{\partial}_q F)\\&+\sum_{q,r\geq 1}(\tfrac{q}{2}+1)(\tfrac{r}{2}+1)(\theta_q\theta_r\partial_q\partial_r F+2\theta_q\tilde{\theta}_r\partial_q\tilde{\partial}_r F+\tilde{\theta}_q\tilde{\theta}_r\tilde{\partial}_q\tilde{\partial}_r F)\Big)=\partial_1\tilde{\partial}_1 F\end{aligned} \tag{4.8}$$

with $\partial_q \equiv \partial/\partial\theta_q$ and $\tilde\partial_q \equiv \partial/\partial\tilde\theta_q$. This is in general a fairly complicated equation.

Application. Assume that all θ_q and $\tilde\theta_q$ are zero except θ_1 and $\tilde\theta_1$. This greatly simplifies Eq. (4.8). Note furthermore that by obvious homogeneity property, F only depends on the product $x \equiv \theta_1\tilde\theta_1$. Rewriting Eq. (4.8) for $F(x)$ leads to an ordinary differential equation:

$$\mathrm{e}^{2F + 9x^2 F''} = F' + xF'' \tag{4.9}$$

with the initial condition $F(0) = 0$, which can be solved: if $\xi = x\,\mathrm{d}F/\mathrm{d}x$, then ξ is the solution of the third degree equation

$$16\xi^3 + 8\xi^2 + (1 - 36x)\xi + x(27x - 1) = 0 \tag{4.10}$$

that vanishes at $x = 0$. We have the following expansion:

$$\xi = \sum_{n=0}^{\infty} x^{n+1} \frac{(3n)!\,2^n}{(n+1)!\,(2n+1)!} \tag{4.11}$$

Curiously, the number $\frac{(3n)!2^n}{(n+1)!(2n+1)!}$ also appears in the context of the enumeration of planar maps;[3] it counts trivalent rooted maps with $2n$ nodes [23]. This suggests that a direct combinatorial proof of this result might be possible. As a corollary, the HCIZ integral with θ_1, $\tilde\theta_1 \neq 0$ is in the universality class of pure 2D quantum gravity: the singularity of $\xi(x)$ closest to the origin is of the form $\xi = (x - x_c)^{3/2} + \cdots$ (with $x_c = 2/27$, $\xi_c = 1/12$).

More generally, if one turns on a *finite* number of θ_q and $\tilde\theta_q$, one can show using the formalism developed in this section that F can be expressed in terms of the solution of an algebraic equation. This is because the string equation implies that $a = 1/u$ and $b = 1/v$ are polynomials in z^{-1}, z:

$$a = \sum_{q=1}^{\tilde\ell+1} \alpha_q z^{-q} \qquad b = \sum_{q=1}^{\ell+1} \beta_q z^q \tag{4.12}$$

where $\ell = \max\{q|\theta_q \neq 0\}$, $\tilde\ell = \max\{q|\tilde\theta_q \neq 0\}$, and as will be shown explicitly in the sections below (cf Eq. (4.18)), the knowledge of a and b allows to compute derivatives of the free energy.[4]

[3] The author would like to thank J.-B. Zuber for pointing this out to him.

[4] Note that since a, b have a rational parameterization, the curve (a, b) has genus zero.

For example, if only θ_1, θ_2, $\cdots$, θ_ℓ, $\tilde\theta_1$ are non-zero, by homogeneity one can set $\tilde\theta_1 = 1$; then, $\chi = 2\frac{\partial^2 F}{\partial\theta_1^2} \equiv 2\psi^4$, and $r^2 = \frac{\partial^2 F}{\partial\theta_1\partial\tilde\theta_1} \equiv \psi$ are given by the degree $2\ell+1$ equation

$$-1+\psi+\sum_{q=1}^{\ell}(-1)^q\frac{(2q)!}{(q!)^2}\theta_q\psi^{2q+1} = -1+\psi-2\theta_1\psi^3+6\theta_2\psi^5-20\theta_3\psi^7+\cdots=0 \quad (4.13)$$

with the initial condition $\psi = 1$ for all θ_q zero. One can check that if $\ell \geq 3$, $\chi = 2\psi^4$ does satisfy the partial differential equation (4.4).

It is expected that by turning on an arbitrary number of θ_q and $\tilde\theta_q$, one should be able to recover the critical behavior (i.e. singularities of F and its derivatives) of all $c < 1$ minimal models coupled to gravity.

4.3. Derivatives of F with respect to the t_q

Our goal is now to express the first few derivatives of F with respect to the t_q and the $\tilde t_q$. Let us define the differential operator

$$\nabla(a) = \sum_{q\geq 1}\frac{a^q}{q}\frac{\partial}{\partial t_q} \quad (4.14)$$

In the bosonic language, this is the positive modes of the full bosonic field $\phi(a) = \sum_{q\geq 1}\frac{a^q}{q}\frac{\partial}{\partial t_q} + \log a - \sum_{q\geq 1} t_q a^{-q}$. One could define similarly $\tilde\nabla(b)$.

Our first task is to compute $\nabla(a)F$ (see also [24,7] for a "saddle point" approach to this question). We note that in terms of the matrix model (2.4), $\nabla(a)F = -\lim_{N\to\infty}\langle \mathrm{tr}\log(1-a\mathrm{U})\rangle$, or

$$\frac{\mathrm{d}}{\mathrm{d}a}\nabla(a)F = u^2 D(u) - u \quad (4.15)$$

where $D(u) \equiv \lim_{N\to\infty}\left\langle \mathrm{tr}\frac{1}{u-\mathrm{U}}\right\rangle$ is the resolvent of U, and with the identification $u \equiv 1/a$ (we recall that appropriate scaling as $N \to \infty$ is implied). Next we use the following standard identity:

$$p_N(u) = h_N^{-1}\langle \det(u-\mathrm{U})\rangle \quad (4.16)$$

where the average is again over $N \times N$ matrices with the measure given by (2.4). This expression of $p_N(u)$ can be proved by checking that $\langle q_n | p_N\rangle = \delta_{nN}$ using it. Considering the logarithmic derivative with respect to u of Eq. (4.16) in the large N limit, we see that $D(u)$ is simply the operation of derivative with respect to $u = 1/a$ in the basis of the $p_n(u)$.

This operator was already introduced in the discussion of the string equation; Eq. (3.16) becomes, in the large N limit,

$$D(u) = u^{-2}(v^{-1} - G(u) + u) \tag{4.17}$$

where we recall that $G(u) = u + \sum_{q\geq 1} q t_q u^{q+1}$ is the resolvent of the original matrix A. Combining Eqs. (4.15) and (4.17) leads to

$$\frac{\mathrm{d}}{\mathrm{d}a}\nabla(a)F = b(a) - G(a) \tag{4.18}$$

where $b = 1/v$, $a = 1/u$ and u and v related by Eqs. (4.3), i.e. $b(a)$ is obtained by composing the series $b(z) = 1/v(z)$ and the inverse series $z(a)$ of $a(z)$. Note that $b(a) - G(a)$ is simply the non-negative part of the Laurent expansion of $b(a) = \frac{\mathrm{d}}{\mathrm{d}a}\phi(a)$ in powers of a.

Note that to derive the identity (4.18), we had to use the string equation under the form (3.16). In contrast, the second derivative $\nabla(a_1)\nabla(a_2)F$ is "universal" i.e. in our context it means that it is true for any solution of the dispersionless Toda hierarchy regardless of the string equation. Derivation of $\nabla(a_1)\nabla(a_2)F$ requires two identities. We shall prove the first one, which is an expression for z. We note that since Z is the shift operator, one should have in the large N limit $z = P_{n+1}(u)/P_n(u)$, that is taking the logarithm and using Eq. (4.16),

$$\log z = \log u - \log r - \sum_{q\geq 1} \frac{u^{-q}}{q} \frac{\partial^2}{\partial\nu\partial t_q} F \tag{4.19}$$

where it is recalled that the dependence on ν of F is given by Eq. (4.6).

The second relation we use the dispersionless limit of the Fay identity [18], which we shall not prove here and write under the form (see [25]):

$$(u_1 - u_2)\mathrm{e}^{\sum_{q,r\geq 1} \frac{u_1^{-q}}{q}\frac{u_2^{-r}}{r}\frac{\partial^2}{\partial t_q \partial t_r}F} = u_1\,\mathrm{e}^{-\sum_{q\geq 1} u_1^{-q}\frac{\partial^2}{\partial\nu\partial t_q}F} - u_2\,\mathrm{e}^{-\sum_{q\geq 1} u_2^{-q}\frac{\partial^2}{\partial\nu\partial t_q}F} \tag{4.20}$$

Combining Eqs. (4.19) and (4.20), using the variables $a_1 = 1/u_1$, $a_2 = 1/u_2$ and the power series inverse $z(a)$ of $a(z)$ leads to

$$\nabla(a_1)\nabla(a_2)F = \log\frac{z(a_1) - z(a_2)}{1/a_1 - 1/a_2} + \log r \tag{4.21}$$

(cf also a similar formula in [3]).

One can prove a formula for mixed derivatives: $\nabla(a)\tilde{\nabla}(b)F = -\log(1 - z(b)/z(a))$, but it will not be needed here.

Formulae for higher numbers of derivatives become more and more complicated, (see in particular "residue formulae" for three derivatives [20]), and we shall not write them down here.

5. Expansion around 0 of the HCIZ integral

We now come to the main goal of this paper: to apply the equations above to the expansion of the large N HCIZ free energy $F[\theta_q, \tilde{\theta}_q]$ in powers of θ_q and $\tilde{\theta}_q$. We shall consider the problem in an asymmetric way: we shall apply the operator $\nabla(a)$ defined in the previous section to the free energy, that is consider derivatives with respect to the θ_q, and then set $\theta_q = 0$ and see how the result depends on the $\tilde{\theta}_q$. The identities obtained above involve the functions $b(a) = 1/v(a = 1/u)$ and $z(a)$; let us see how to compute these functions when all t_q are zero.

If we set $t_q = 0$ in def. (3.2) of the bilinear form, we see that no positive powers of u are generated, so that $m_{ij} = \left\langle v^i \middle| u^j \right\rangle$ satisfies

$$m_{ij} = \begin{cases} 0 & j < i \\ \frac{1}{j!}\, \tilde{s}_{j-i} & j \geq i \end{cases} \tag{5.1}$$

where we have defined $\tilde{s}_q$ by $\varphi(v) \equiv \exp\left(\sum_{q\geq 1} \tilde{t}_q v^q\right) = \sum_{q\geq 0} \tilde{s}_q v^q$; they are Schur functions corresponding to single row Young diagrams.

Let us first use the fact that M is upper triangular. Since $M = \tilde{P} P^{-1}$ with P upper triangular and $\tilde{P}$ lower triangular, $\tilde{P}$ is diagonal i.e. $q_n(v) = \sqrt{n!}\, v^n$. The normalization $h_n = (n!)^{-1/2}$ implies that $r_n = n^{-1/2}$, that is after rescaling $r = 1$. Finally, we conclude from the definition (3.5) of V that all the coefficients v_k in its expansion (or of its transpose $\tilde{U}$, Eq. (3.11)) are zero, or in the large N limit

$$v = z^{-1} \tag{5.2}$$

(note that this is consistent with Eq. (4.12)). We have now reduced the problem to a single function $b(a) \equiv 1/v(a) = z(a)$; its calculation was performed in [7] using a saddle point method, but it is particularly simple to rederive it in our framework. We start again from Eq. (5.1) and notice that one can easily invert this matrix, which yields the coefficients of $P = M^{-1}\tilde{P}$. If we introduce the $\tilde{e}_q$ defined by $\varphi(v)^{-1} \equiv \exp\left(-\sum_{q\geq 1} \tilde{t}_q v^q\right) = \sum_{q\geq 0} \tilde{e}_q v^q$ (they are $(-1)^q$ times the Schur functions corresponding to one-column Young diagrams), then expansion of the identity $\varphi(v)\varphi(v)^{-1} = 1$ leads to

$$p_{ij} = \begin{cases} 0 & j < i \\ \frac{i!}{\sqrt{j!}}\, \tilde{e}_{j-i} & j \geq i \end{cases} \tag{5.3}$$

We now define the matrix $A = (a_{ij})$ by $A = P^{-1} Z^T P$, which according to defs. (3.5) is a left inverse of U. We have

$$
\begin{aligned}
a_{ik} &= \sqrt{\frac{i!}{k!}} \sum_{j,\, i \le j \le k-1} \frac{1}{j!} \tilde{s}_{j-i} (j+1)! \tilde{e}_{k-1-j} \\
&= \sqrt{\frac{i!}{k!}} \sum_{q=0}^{k-1-i} (q+i+1) \tilde{s}_q \tilde{e}_{k-1-i-q}
\end{aligned} \tag{5.4}
$$

This time we recognize the expansion of $\varphi'(v)\varphi(v)^{-1} = \sum_{n\ge 0} v^{n-1} \sum_{q=0}^{n} q \tilde{s}_q \tilde{e}_{n-q}$; but we also have $\varphi'(v)\varphi(v)^{-1} = \sum_{n\ge 1} n \tilde{t}_n v^{n-1}$, so that

$$
a_{ik} = \begin{cases} 0 & k < i+1 \\ \sqrt{k} & k = i+1 \\ \sqrt{\frac{i!}{k!}} (k-i-1) \tilde{t}_{k-i-1} & k > i+1 \end{cases} \tag{5.5}
$$

This is an exact expression at finite N. In the large N limit this can be simply rewritten, performing the appropriate rescalings and remembering that $z = b$,

$$
a = \frac{1}{b} + \sum_{q \ge 1} \frac{\tilde{\theta}_q}{b^{q+1}} \tag{5.6}
$$

We recognize the resolvent of the original matrix B (Eq. (1.5)): $a(b) = \tilde{G}(b)$ (again this is consistent with the "dual" equation of (4.18), $\frac{\mathrm{d}}{\mathrm{d}b} \tilde{\nabla}(b) F = a(b) - \tilde{G}(b)$). Finally $b(a = 1/u)$ is the functional inverse of $a(b)$; it is well-known that $b(a)$ is nothing but the generating series of *free cumulants* $\tilde{m}_q$ of B [10] (cf also the expression of *connected* correlation functions in matrix models, as in Eq. (31) of [26]):

$$
b(a) = \frac{1}{a} + \sum_{q=1}^{\infty} \tilde{m}_q a^{q-1} \tag{5.7a}
$$

$$
\tilde{m}_q = - \sum_{\substack{\alpha_1, \dots, \alpha_q \ge 0 \\ \sum_i i\alpha_i = q}} \frac{(q + \sum_i \alpha_i - 2)!}{(q-1)!} \prod_i \frac{(-\tilde{\theta}_i)^{\alpha_i}}{\alpha_i!} \tag{5.7b}
$$

We can now apply the formulae obtained in previous section. We recall that $\nabla(a) \equiv \sum_{q \ge 1} \frac{a^q}{q} \frac{\partial}{\partial t_q}$. First we rewrite Eq. (4.18) for $t_q = 0$:

$$
\frac{\mathrm{d}}{\mathrm{d}a} \nabla(a) F = b(a) - \frac{1}{a} = \sum_{q=1}^{\infty} \tilde{m}_q a^{q-1} \tag{5.8}
$$

This equation can be found in [7], but it can already be extracted from [2]; see also [8] for a combinatorial proof.

The second derivatives are given by Eq. (4.21) with $z(a) = b(a)$:

$$\nabla(a_1)\nabla(a_2)F = \log \frac{b(a_1) - b(a_2)}{1/a_1 - 1/a_2}$$
$$= \log \left(1 - \sum_{q=2}^{\infty} \tilde{m}_q \sum_{k=1}^{q-1} a_1^k a_2^{q-k} \right) \tag{5.9}$$

Since the expression is in fact known for all t_q, one can differentiate once more and obtain the expression for the third derivatives at $t_q = 0$ with little extra work. First one computes the expression of $b(z)$ at first order in t_q via Eq. (3.9b); one obtains after inversion

$$r\,z = b + \sum_{q\geq 1} t_q \sum_{k\geq 0} c_{qk} b^{k+1} + \cdots \qquad t_q \to 0 \tag{5.10}$$

where the coefficients c_{qk} are best expressed using their generating function

$$\sum_{q\geq 1} \frac{a^q}{q} c_{qk} = \frac{\mathrm{d}b(a)}{\mathrm{d}a} b(a)^{-k-2} \tag{5.11}$$

Secondly one uses Eqs. (4.18) and (4.21) to deduce

$$\nabla(a_2)b(a_1) = \frac{\partial}{\partial a_1} \log(z(a_1) - z(a_2)) \tag{5.12}$$

Acting with $\nabla(a_3)$ on Eq. (4.21), setting $t_q = 0$, using identities (5.10)–(5.12) and performing simple manipulations results in

$$\nabla(a_1)\nabla(a_2)\nabla(a_3)F = \frac{\mathrm{d}b(a_1)/\mathrm{d}a_1}{(b(a_1) - b(a_2))(b(a_1) - b(a_3))} + \frac{\mathrm{d}b(a_2)/\mathrm{d}a_2}{(b(a_2) - b(a_1))(b(a_2) - b(a_3))}$$
$$+ \frac{\mathrm{d}b(a_3)/\mathrm{d}a_3}{(b(a_3) - b(a_1))(b(a_3) - b(a_2))} + 1 \tag{5.13}$$

One can go to higher derivatives by further differentiation, but most likely, the expressions become more and more cumbersome.

6. Summary of results

In this paper we have investigated the expansion of the large N HCIZ integral free energy in terms of powers of the moments of the matrices. We have shown that this free energy is the dispersionless tau function for the dispersionless 2D Toda hierarchy of partial differential equations, so that coefficients of the expansion satisfy an infinite set of non-linear relations. Also, we have seen that the solution of the Toda hierarchy is selected by a non-trivial string equation $\{u^{-1}, v^{-1}\} = 1$. The following explicit results have been found:

$\diamond$ First, one can compute some infinite series of coefficients of the expansion using these differential equations. The example worked out in section 4 is the set of coefficients of the form $\theta_1^k \tilde{\theta}_1^k$; if $x = \theta_1 \tilde{\theta}_1$, then $\xi = x\,\mathrm{d}F/\mathrm{d}x$ was found to satisfy the equation $16\xi^3 + 8\xi^2 + (1-36x)\xi + x(27x-1) = 0$, that is

$$F = \theta_1\tilde{\theta}_1 + \frac{1}{2}(\theta_1\tilde{\theta}_1)^2 + \frac{4}{3}(\theta_1\tilde{\theta}_1)^3 + 6(\theta_1\tilde{\theta}_1)^4 + \frac{176}{5}(\theta_1\tilde{\theta}_1)^5 + \cdots \tag{6.1}$$

More generally, one can compute generating series of coefficients involving only a finite number of distinct θ_q and $\tilde{\theta}_q$ as solutions of algebraic equations; a more complicated example was given without proof, cf Eq. (4.13). In principle, this allows to compute any given coefficient required, although it does not give a closed expression for the whole function of the infinite set of θ_q and $\tilde{\theta}_q$.

$\diamond$ Secondly, one can obtain *all* the coefficients (as functions of the $\tilde{\theta}_q$) of the first few orders of the expansion in products of θ_q: we rewrite their generating functions here (Eqs. (5.8), (5.9) and (5.13)):

$$\frac{\mathrm{d}}{\mathrm{d}a}\nabla(a)F = b(a) - \frac{1}{a} \tag{6.2.1}$$

$$\nabla(a_1)\nabla(a_2)F = \log\frac{b(a_1)-b(a_2)}{1/a_1 - 1/a_2} \tag{6.2.2}$$

$$\begin{aligned}\nabla(a_1)\nabla(a_2)\nabla(a_3)F = &\frac{\mathrm{d}b(a_1)/\mathrm{d}a_1}{(b(a_1)-b(a_2))(b(a_1)-b(a_3))} + \frac{\mathrm{d}b(a_2)/\mathrm{d}a_2}{(b(a_2)-b(a_1))(b(a_2)-b(a_3))}\\ &+ \frac{\mathrm{d}b(a_3)/\mathrm{d}a_3}{(b(a_3)-b(a_1))(b(a_3)-b(a_2))} + 1\end{aligned} \tag{6.2.3}$$

where in the rescaled variables, $\nabla(a) = \sum_{q\geq 1} a^q \frac{\partial}{\partial\theta_q}$. The expansion is given in terms of the function $b(a) = 1/a + \sum_{q\geq 1} \tilde{m}_q a^{q-1}$. The free cumulants $\tilde{m}_q$ can be reexpanded themselves in powers of the $\tilde{\theta}_q$: $\tilde{m}_q = \tilde{\theta}_q - \frac{q}{2!}\sum_{\substack{i,j\geq 1\\ i+j=q}} \tilde{\theta}_i\tilde{\theta}_j + \frac{q(q+1)}{3!}\sum_{\substack{i,j,k\geq 1\\ i+j+k=q}} \tilde{\theta}_i\tilde{\theta}_j\tilde{\theta}_k + \cdots$, so that we find for example

$$
\begin{aligned}
F = &\sum_{q\geq 1} \frac{1}{q}\theta_q\tilde{\theta}_q \\
&- \frac{1}{2} \sum_{\substack{q,r_1,r_2\geq 1\\ r_1+r_2=q}} \theta_q\tilde{\theta}_{r_1}\tilde{\theta}_{r_2} \\
&- \frac{1}{2} \sum_{\substack{r,q_1,q_2\geq 1\\ q_1+q_2=r}} \theta_{q_1}\theta_{q_2}\tilde{\theta}_r \\
&+ \frac{1}{4} \sum_{\substack{q_1,q_2,r_1,r_2\geq 1\\ q_1+q_2=r_1+r_2}} (q_1+q_2-\min(q_1,q_2,r_1,r_2)+1)\theta_{q_1}\theta_{q_2}\tilde{\theta}_{r_1}\tilde{\theta}_{r_2} \\
&+ \frac{1}{6} \sum_{\substack{q,r_1,r_2,r_3\geq 1\\ q=r_1+r_2+r_3}} (q+1)\theta_q\tilde{\theta}_{r_1}\tilde{\theta}_{r_2}\tilde{\theta}_{r_3} \\
&+ \frac{1}{6} \sum_{\substack{r,q_1,q_2,q_3\geq 1\\ r=q_1+q_2+q_3}} (r+1)\theta_{q_1}\theta_{q_2}\theta_{q_3}\tilde{\theta}_r \\
&+ \cdots
\end{aligned}
\tag{6.3}
$$

with the appropriate symmetry of interchange of the θ_q and $\tilde{\theta}_q$.

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