

Monochromatic path crossing exponents and graph connectivity in 2D percolation

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We consider the fractal dimensions d_k of the k -connected part of percolation clusters in two dimensions, generalizing the cluster ($k = 1$) and backbone ($k = 2$) dimensions. The codimensions $\tilde{x}_k = 2 - d_k$ describe the asymptotic decay of the probabilities $P(r, R) \sim (r/R)^{\tilde{x}_k}$ that an annulus of radii $r \ll 1$ and $R \gg 1$ is traversed by k disjoint paths, all living on the percolation clusters. Using a transfer matrix approach, we obtain numerical results for \tilde{x}_k , $k \leq 6$. They are well fitted by the Ansatz $\tilde{x}_k = \frac{1}{12}k^2 + \frac{1}{48}k + (1 - k)C$, with $C = 0.0181 \pm 0.0006$.

Percolation is a classical model of statistical mechanics [1,2], and plays an important role in the study of disordered systems [3]. It is also one of the simplest models displaying a critical point. In two dimensions, exact values for a variety of critical exponents have been found over the last two decades, and quite recently many of them have been confirmed by rigorous probabilistic arguments [4]. Most of these exponents can be defined through the fractal dimensions of suitably defined sets at the percolation threshold.

In the present Letter we shall be concerned with an infinite family of critical exponents whose exact values remain unknown to this date. These exponents characterize the connectivity structure of the percolating cluster(s) at criticality.

Following Tutte [5], we define a graph to be k -connected (for $k \geq 1$) if no separation into disconnected subgraphs is possible by eliminating at most $k-1$ vertices along with their ingoing edges. It is easy to see that we may equivalently require any two vertices in the graph to be connected by (at least) k disjoint paths. 1-connected graphs are simply the percolation clusters, and to inquire into the connectivity structure of a given cluster, we may decompose it into its largest 2-connected components [5] (better known as 2-blocks, or “blobs”, in the percolation literature), and so on. Tutte has shown that the decomposition of a 2-connected graph into its largest 3-blocks is unique [5]. 3-blocks are relevant for applying Kirchhoff’s laws to resistor networks, and are useful for analyzing the performance of certain algorithms [6].

To better study the transport properties of percolation clusters, we henceforth consider critical percolation in a large square of linear size L , and we specialize to clusters that connect to the boundary of the system. In the limit $L \rightarrow \infty$, the boundary becomes the “point at infinity”, and the above definition states that a given vertex is k -connected if it is connected to infinity by (at least) k disjoint paths within a percolating cluster. Following Ref. [6], we shall call the set of k -connected vertices the k -

bone. The 2-bone is of course nothing but the (geometrical) backbone, i.e. the part of the percolation cluster that sustains a non-zero current, when a voltage difference is applied between its two terminal points. (As usual we disregard rare Wheatstone’s bridge-like arrangements.)

We shall here be interested in the fractal dimension d_k of the k -bone, assuming its mass to change with system size like L^{d_k} . The cluster dimension $d_1 = \frac{91}{48}$ has been known exactly for a long time [7–9]. The backbone dimension has recently been related to the solution of a partial differential equation [9], which however appears to be intractable, even numerically. Still, numerical estimates are available from Monte Carlo [10] and transfer matrix methods [11]: $d_2 = 1.6431 \pm 0.0006$. After the completion of this work, a first estimate for d_3 appeared: $d_3 = 1.2 \pm 0.1$ [6]. Actually, this result was obtained from block-decomposition of clusters and backbones, but for reasons of universality we expect it to apply to the 3-bone as well. Also, the above definitions are stated for site percolation, but the exponents should be the same for bond percolation, with the clusters being separated by cutting edges rather than vertices.

A useful alternative formulation of the k -bone problem is obtained by passing to an annular geometry, limited by two concentric circles of radii $r \ll 1$ and $R \gg 1$, by means of a conformal mapping. (This is permissible since percolation has been proved to be conformally invariant [4].) Interpreting the inner circle as the point which is a potential element of the k -bone, and the outer as the point at infinity, we see that a given percolating configuration in the annulus contributes to the k -bone if and only if the two circles are connected by k disjoint paths on the percolating cluster(s); see Fig. 1. The fractal dimension d_k of the k -bone is linked to the scaling of the path-crossing probability $P_k(r, R) \sim (r/R)^{\tilde{x}_k}$ through the codimension $\tilde{x}_k = 2 - d_k$ [12].

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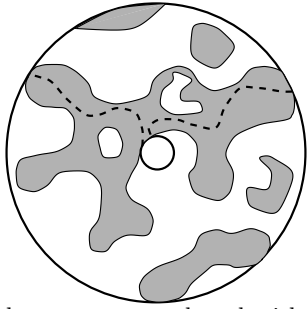


FIG. 1. Annular geometry endowed with critical percolation (here shown in the continuum limit). The existence of two disjoint traversing paths on the clusters implies that this configuration contributes to the 2-bone.

A more general class of path-crossing exponents can be defined from traversing configurations where some of the paths live on the percolating clusters (black paths), and the rest on the *dual* clusters (white paths). Interestingly, the corresponding critical exponents x_k only depends on the number of paths, $k \geq 2$, and not on their colors, provided that both are represented [12]. Their values

$$x_k = \frac{1}{12}(k^2 - 1) \quad (1)$$

are known rigorously [12] and differ from those of the monochromatic exponents \tilde{x}_k defined above.

Some information on the \tilde{x}_k is provided by the inequalities

$$\tilde{x}_k < x_{k+1}. \quad (2)$$

This inequality is valid since a configuration contributing to x_{k+1} can be taken to have k black paths and one white path. Clearly, it then also contributes to \tilde{x}_k . Furthermore, as $k \rightarrow \infty$, the effect of a single extra path should be small and we expect it to be of the same order if one changes the color of one of the k existing paths; thus, $x_{k+1} - \tilde{x}_k \approx x_{k+1} - x_k = O(k)$.

In view of Eq. (1) we find the asymptotic result $\tilde{x}_k = \frac{1}{12}k^2 + O(k)$ as $k \rightarrow \infty$. In analogy with Eq. (1) it thus seems natural to conjecture that the spectrum \tilde{x}_k is quadratic in k . From $\tilde{x}_1 = \frac{5}{48}$ we then get

$$\tilde{x}_k = \frac{1}{12}k^2 + \frac{1}{48}k + (1 - k)C, \quad (3)$$

with $C = 0.0181 \pm 0.0006$ from the numerical result on \tilde{x}_2 [11].

To check the conjectured form of the spectrum, Eq. (3), we now turn to our numerical results. But first we must briefly describe our transfer matrix algorithm; it is a natural generalization of the one used in [11].

First we consider the annulus of Fig. 1 as a cylinder with circular space and radial time. We have tried several choices of lattices to discretize it. For practical applications, it turns out to be best (see [11]) to use a square lattice with a “light-cone” orientation, that is such that the periodic direction forms a 45 degrees’ angle with the

two axes of the lattice. We then define the discrete time slices such that they intersect the lattice at vertices only: we call L the number of such vertices (in units of the lattice spacing the period is then $L\sqrt{2}$).

Next we define the basis of states on which our transfer matrix acts. A basis state is a *collection* of path configurations; a path configuration is the data of the positions of our k paths at a given time, with possible additional “arches” to allow backtracking of paths, see Fig. 2. Note that the encoding of states can be easily implemented as follows: basis states are encoded as sorted lists of path configurations, and path configurations are represented by words of length L made out of the four letters $\{\textit{opening}, \textit{closing}, \textit{path}, \textit{empty}\}$ and which contain k letters *path*. The letters *opening* and *closing* define the backtracking arches.

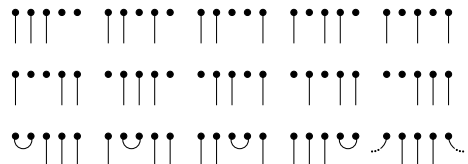


FIG. 2. Path configurations for $k = 3$, $L = 5$.

The transfer matrix itself acts on a basis states by “evolving” all the path configurations it contains with a single configuration of bonds (i.e. percolating/non-percolating state of each bond) and recombining the result into a single state, then summing over all configurations of bonds. Evolving a path configuration means considering all possible continuations of the paths from one time slice to the next, including possible appearances of new arches, or existing arches connecting to paths or other arches. We use sparse matrix factorization techniques to build up the entire transfer matrix; dihedral symmetries are quotiented over once a time slice has been completed.

As in [11], it is convenient to allow “perpendicular tangencies” for paths even though they should be in principle excluded; that is to allow two paths to touch at one vertex in the configuration whenever the tangent of either path is perpendicular to the transfer direction (we still exclude “parallel tangencies”). It is expected, and we have numerically verified, that inclusion of either or both types of tangencies and/or changing the lattice orientation does not alter the first finite-size correction of the eigenvalue of the transfer matrix, which yields \tilde{x}_k . However, the particular choice of allowing perpendicular tangencies on the light-cone oriented lattice has the advantage of greatly decreasing the number of states generated (see below) and improving the convergence properties of the finite-size data.

Finally, the following procedure is used to compute the matrix elements of the transfer matrix: Starting with an arbitrary basis state (e.g. the one consisting of the single path configuration $\textit{path}^k \textit{empty}^{L-k}$), one acts on it with the transfer matrix, stores the corresponding matrix ele-

ments, then considers all new basis states generated and iterates the procedure until no further new states are generated [13]. What we build this way is a submatrix of the transfer matrix corresponding to a stable subspace; this submatrix is in fact much smaller than the full transfer matrix, which is essential for practical applications. One then extracts its largest eigenvalue $\lambda_k(L)$, which yields the codimension \tilde{x}_k via the formula

$$\frac{1}{2^{2L}}\lambda_k(L) = 1 - \frac{\pi\tilde{x}_k}{L} + o(L^{-1}) \quad (4)$$

We show in Tab. I the results for $2 \leq k \leq 6$. Since the memory and time requirements presumably grow factorially with L , we cannot push the calculation very far in L . It is however sufficient to estimate \tilde{x}_k , which is also given in Tab. I together with approximate error bars. The data for $k = 2$ are in fact taken from [11], where the state space was reduced by exploiting that the case of two paths can be treated as an extra backtracking arch.

For $k > 2$, all the exponents \tilde{x}_k are consistent with, but less precise than, the conjectured spectrum (3) with C given by \tilde{x}_2 . We also note that the inequality $x_k < \tilde{x}_k$ seems to be satisfied, and it would be interesting if one could prove this. Another open question is the possible rationality of the \tilde{x}_k ; in this respect, the failure of computing these exponents by conformal field theory is particularly intriguing.

For $k \geq 5$, the dimensions $d_k = 2 - \tilde{x}_k$ are negative. Physically this means that k -bones with $k \geq 5$ become increasingly rare as the system size increases. Of course, the lattice model does not support a k -block when k exceeds the coordination number of the lattice. However, the exponents \tilde{x}_k characterize the continuum limit, and are thus believed to be independent of the microscopic details; an appropriate lattice definition of the k -bone for high k is obtained by demanding that k independent paths connect any small neighborhood to the point at infinity.

Finally, we remark that the k -bone problem extends to the Kasteleyn-Fortuin representation [14] of the q -state Potts model (bond percolation being the limit

$q \rightarrow 1$). Our numerical algorithm can straightforwardly be adapted to this case as well.

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TABLE I. Eigenvalues $\lambda_k(L)$ of the transfer matrix and estimate of \tilde{x}_k for $2 \leq k \leq 6$.

k	2	3	4	5	6
L					
4	0.718747415570	0.413598206498	0.121093750000		
5	0.775012703547	0.526618869796	0.257122218539	0.061523437500	
6	0.812529692986	0.603476157424	0.362299981029	0.153371684616	0.031005859375
7	0.839330907375	0.658986646726	0.443031423565	0.237989873966	0.088905009155
8	0.859432882632	0.700919030179	0.506272495802	0.310651059489	0.150977532764
9	0.875067710677		0.556925756584	0.372225212770	0.210050339522
10					0.263993624780
\tilde{x}_k	0.3569 ± 0.0006	0.77 ± 0.02	1.33 ± 0.03	2.1 ± 0.2	3.0 ± 0.3