Projection on higher Landau levels and non Commutative Geometry

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Abstract

The projection of a two dimensional planar system on the higher Landau levels of an external magnetic field is formulated in the language of the non commutative plane and leads to a new class of star products.

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Introduction: Consider a two dimensional Hamiltonian of a particle in a scalar potential $V(z,\bar{z})$ coupled to a magnetic field B, in the symmetric gauge ($e=\hbar=m=1$)

$$H(z) = -2\partial\bar{\partial} + \omega_c(\bar{z}\bar{\partial} - z\partial) + \frac{1}{2}\omega_c^2 z\bar{z} + V(z,\bar{z}) \tag{1}$$

We assume without any loss of generality that $B \geq 0$ and denote by $\omega_c = +B/2$ half the cyclotron frequency. It is well known that by projecting (1) on the lowest Landau level (LLL) spanned by the states

$$\psi(z) = f(z)e^{-\frac{\omega_c}{2}z\bar{z}} \tag{2}$$

where $f(z)$ is analytic, one obtains an eigenvalue equation

$$: V(z, \frac{1}{\omega_c}\partial_z) : f(z) = (E - \omega_c)f(z) \tag{3}$$

The normal ordering $:\ :$ means that the differential operator $\frac{1}{\omega_c}\partial$ is put on the left of z. Equation (3) is a reformulation of the Peierls substitution [1] (which does not specify the correct ordering in general) and as such has been derived in [2] (which specifies the correct ordering).

Clearly, the usual two dimensional plane has been traded for a non commutative space

$$[\frac{1}{\omega_c}\partial_z, z] = \frac{1}{\omega_c} \tag{4}$$

In view of the above commutation relation, the non commutative space can be interpreted as the phase space corresponding to a one dimensional space. However this interpretation may not be entirely satisfactory because z is a complex coordinate. There is another natural interpretation in terms of a non commutative space which is a two dimensional plane with non commuting "real" coordinates (X,Y). It is well known that the algebra of operators depending on the two non commutative coordinates (X,Y) is equivalent to a deformation of the classical algebra of functions on the usual commutative plane with coordinates (x,y). This deformation is defined through a non commutative star product. Although its construction is well known we will review it below in a self contained way, giving, as a by product, a more systematic and more simple derivation of (3).

The main point of this letter is to show that non commutative geometry is by no means specific to the LLL projection but can be obtained as well by projecting the two dimensional system on any given higher Landau level. As an illustration the case of the first Landau level (1LL) will be analyzed in detail. We will generalise the Peierls substitution to the 1LL and reformulate it in a non commutative plane language. We will find that although the canonical commutation relation between the non commutative coordinates will be the same as for the LLL, a new non commutative star product will appear to be naturally connected with the 1LL. We will then give the general expression for the star product associated to any given Landau level, thereby defining a new class of star product [3].

It might be objected that projecting a system on a higher Landau level is counter intuitive: usually the LLL projection is regarded as physically justified when the cyclotron gap $\hbar\omega_c$ is sufficiently large -thus the LLL projection is associated to a strong magnetic field- compared to the temperature and/or to the potential so that the excited states above the LLL can be ignored. Clearly such an interpretation becomes meaningless in a higher Landau level. However, restricting the two dimensional Hilbert space to a given Landau level subspace is a well defined mathematical procedure, and will be considered as such in what follows.

Before starting let us remind that the Landau spectrum is made of degenerate Landau levels $(2n+1)\omega_c, n \geq 0$ with in the nth Landau level eigenstates labelled by the radial/orbital quantum numbers $n, l \geq 0$ (analytic) and $n+l, -n \leq l < 0$ (anti-analytic). There is, in a given landau level, an infinite number of analytic eigenstates

$$< z, \bar{z}|n, l >= z^l L_n^l(\omega_c z\bar{z}) e^{-\frac{1}{2}\omega_c z\bar{z}} \quad l \geq 0 \tag{5}$$

and a finite number of anti-analytic eigenstates

$$< z, \bar{z}|n+l, l >= \bar{z}^{-l} L_{n+l}^{-l}(\omega_c z\bar{z}) e^{-\frac{1}{2}\omega_c z\bar{z}} \quad -n \leq l < 0 \tag{6}$$

Projection on the LLL: In the lowest Landau level, $n = 0$, $l \geq 0$, the eigenstates are analytic

$$< z, \bar{z}|0, l >= (\frac{\omega_t^{\ell+1}}{\pi \ell!})^{\frac{1}{2}} z^\ell e^{-\frac{1}{2}\omega_t z\bar{z}}, \quad \ell \geq 0 \tag{7}$$

Consider the projector on the LLL Hilbert space $P_o = \sum_{l=0}^{\infty} |0, l >< 0, l|$

$$< z, \bar{z}|P_o|z', \bar{z}' >= \frac{\omega_c}{\pi} e^{-\frac{\omega_c}{2}(z\bar{z}+z'\bar{z}'-2z\bar{z}')} \tag{8}$$

A state of the LLL $|\psi >= \sum_{l=0}^{\infty} a_l |0, l >$, is analytic up to the Landau gaussian factor

$$< z|\psi >= f(z) e^{-\frac{\omega_c}{2} z\bar{z}} \tag{9}$$

with

$$f(z) = \sum_{l=0}^{\infty} a_l z^l \tag{10}$$

One can check that $|\psi >$ satisfies $P_o|\psi >= |\psi >$.

Then project the Hamiltonian (1) on the LLL

$$< z,\bar z|P_oHP_o|\psi >=< z,\bar z|P_oH|\psi >= \int dz'd\bar z'\frac{\omega_c}{\pi}e^{-\frac{\omega_c}{2}(z\bar z+z'\bar z'-2z\bar z')}H(z')f(z')e^{-\frac{\omega_c}{2}z'\bar z'} \quad (11)$$

Using the Bargman identity

$$\frac{\omega_c}{\pi}\int dz'd\bar z'e^{-\omega_c(z'\bar z'-z\bar z')}h(z') = h(z) \quad (12)$$

it is not difficult to see that the eigenvalue equation

$$< z,\bar z|P_oHP_o|\psi >= E < z,\bar z|\psi > \quad (13)$$

or

$$\int dz'd\bar z'\frac{\omega_c}{\pi}e^{-\omega_c(z'\bar z'-z\bar z')}\left(\omega_c + V(z',\bar z')\right)f(z') = Ef(z) \quad (14)$$

can be transformed into a differential equation which is precisely the Peierls substitution equation (3).

Projection on the 1LL: Consider now the projector on the 1LL Hilbert space $P_1 = \sum_{l=0}^{\infty}|1,l >< 1,l| + |0,-1 >< 0,-1|$

$$< z,\bar z|P_1|z',\bar z' >=< z,\bar z|P_0|z',\bar z' > [1-\omega_c(z'-z)(\bar z'-\bar z)] \quad (15)$$

Using $L_1^l(\omega_c z\bar z) = l+1-\omega_c z\bar z$ one obtains that a state of the 1LL $|\psi >= \sum_{l=0}^{\infty} a_l|1,l >$ $+a_{-1}|0,-1 >$ is of the form.

$$< z,\bar z|\psi >= \left(f(z) + \bar z g(z)\right)e^{-\frac{\omega_c}{2}z\bar z} \quad (16)$$

with $f(z)$ and $g(z)$ analytic,

$$f(z) = -\frac{1}{\omega_c}\partial g(z) \quad (17)$$

and

$$g(z) = -\omega_c\sum_{l=0}^{\infty}a_l z^{l+1} + a_{-1} \quad (18)$$

Another way to recover this result is to impose equivalently that $P_1|\psi >= |\psi >$, i.e.

$$\int dz'd\bar z' < z,\bar z|P_1|z',\bar z' >< z',\bar z'|\psi >=< z,\bar z|\psi > \quad (19)$$

One first infers that necessarily $< z,\bar z|\psi >= \left(f(z) + \bar z g(z)\right)e^{-\frac{\omega_c}{2}z\bar z}$. Equation (19) then becomes

$$\int dz'd\bar{z}'\frac{\omega_c}{\pi}e^{-\omega_c(z'\bar{z}'-z\bar{z}')}[1-\omega_c(z'-z)(\bar{z}'-\bar{z})]\left(f(z')+\bar{z}'g(z')\right)=f(z)+\bar{z}g(z) \tag{20}$$

and finally

$$-\frac{1}{\omega_c}\partial g(z)+\bar{z}g(z)=f(z)+\bar{z}g(z) \tag{21}$$

The relation (17) directly follows.

Now we project the Hamiltonian (1) on the 1LL,

$$< z,\bar{z}|P_1HP_1|\psi >=\int dz'd\bar{z}' < z,\bar{z}|P_1|z',\bar{z}' >< z',\bar{z}'|H|\psi > \tag{22}$$

The eigenvalue equation

$$< z,\bar{z}|P_1HP_1|\psi >= E < z,\bar{z}|\psi > \tag{23}$$

becomes

$$\begin{aligned}\int dz'd\bar{z}'\frac{\omega_c}{\pi}e^{-\omega_c(z'\bar{z}'-z\bar{z}')}[1-\omega_c(z'-z)(\bar{z}'-\bar{z})]\ V\ (z',\bar{z}')(f(z')+\bar{z}'g(z'))\\ =(E-3\omega_c)(f(z)+\bar{z}g(z))\end{aligned} \tag{24}$$

Using again (12) one transforms (24) into the differential equation

$$\begin{aligned}-\frac{1}{\omega_c}\partial\left(:V+\frac{1}{\omega_c}\partial\bar{\partial}V:g(z)\right)+\bar{z}:V+\frac{1}{\omega_c}\partial\bar{\partial}V:g(z)\\ =(E-3\omega_c)(f(z)+\bar{z}g(z))\end{aligned} \tag{25}$$

where it is again understood that $\bar{z}$ is replaced by $\frac{1}{\omega_c}\partial$ both in V and in $\partial\bar{\partial}V$, and then the normal ordering is taken.

As an example, consider $V(z,\bar{z})=\omega_c^2 z\bar{z}/2$. We obtain from (25)

$$\begin{aligned}\frac{\omega_c}{2}(3+z\partial)g&=(E-3\omega_c)g(z)\\ \frac{\omega_c}{2}(4+z\partial)f&=(E-3\omega_c)f(z)\end{aligned} \tag{26}$$

It is easy to see that the couple of equations lead to the spectrum $E-3\omega_c=\omega_c(3+l)/2, l\geq 0$. For a general $V(z,\bar{z})$ this conclusion still holds true: the 1LL projection induces an eigenvalue equation

$$:V(z,\frac{1}{\omega_c}\partial)+\frac{1}{\omega_c}\bar{\partial}\partial V(z,\frac{1}{\omega_c}\partial):g(z)=(E-3\omega_c)g(z) \tag{27}$$

where $g(z)$ is analytic. The equation for $f(z)$ is obtained by differentiating (27) with respect to z. Equation (27) is, in the 1LL, the analogous of the LLL Peierls substitution equation (3).

Non commutative plane and $\star$ products: It is well-known that the LLL quantum mechanics can be equivalently reformulated in a non commutative geometry setting. It is tempting to generalize this construction to higher Landau levels, with as a result a class of non commutative $*_n$ products associated to the higher Landau level projection. Let us first start by a reminder of the LLL non commutative formulation.

Lowest Landau Level: Consider the non-commutative guiding center coordinates

$$X = \frac{1}{2}(z + \frac{1}{\omega_c}\partial), Y = \frac{1}{2i}(z - \frac{1}{\omega_c}\partial) \tag{28}$$

They satisfy the commutation relation

$$[X, Y] = \frac{1}{2i\omega_c} \tag{29}$$

We have seen in (3) that the LLL operator associated to the classical potential $V(x, y)$ is

$$: V(z, \frac{1}{\omega_c}\partial z) : \tag{30}$$

Introducing the Fourier transform of the classical potential

$$V(x, y) = \int dkdl e^{i(kx+ly)}\tilde{V}(k, l) = \int dkdl e^{i(\frac{k}{2}(z+\bar{z})+\frac{l}{2i}(z-\bar{z}))}\tilde{V}(k, l) \tag{31}$$

and using (28) the operator (30) can be written as

$$\hat{V}^{(0)}(X, Y) = \int dkdl : e^{ikX+ilY} : \tilde{V}(k, l) \tag{32}$$

inducing a mapping between a classical potential and an operator. Note that (32) is different from the Weyl ordering which is often used in the field theory context [4] and corresponds to drop the normal order double colons in (32). Note also for further use that, thanks to the identity

$$: e^{ikX+ilY} := e^{-\frac{1}{8\omega_c}(k^2+l^2)} e^{ikX+ilY} \tag{33}$$

one has

$$\hat{V}^{(0)}(X, Y) = \int dkdl e^{-\frac{1}{8\omega_c}(k^2+l^2)} e^{ikX+ilY}\tilde{V}(k, l) \tag{34}$$

Equation (32) induces a non commutative product $f \star_0 g$ between any classical functions $f(x, y)$ and $g(x, y)$ such that

$$\hat{f}^{(0)}(X, Y)\hat{g}^{(0)}(X, Y) = \widehat{(f \star_0 g)}^{(0)}(X, Y) \tag{35}$$

A straightforward computation leads to

$$(f \star_0 g)(x,y) = e^{-\frac{1}{4\omega_c}(\partial'_x + i\partial'_y)(\partial_x - i\partial_y)} f(x,y) g(x',y')|_{x=x',y=y'} \tag{36}$$

and

$$(\widetilde{f \star_0 g})(k,l) = \int dk'dl' \tilde{f}(k-k',l-l')\tilde{g}(k',l') e^{\frac{i}{4\omega_t}(kl'-lk')} e^{-\frac{1}{4\omega_c}(k'^2+l'^2-kk'-ll')} \tag{37}$$

The Fourier transform (37) of $f \star_0 g$ is a "deformation" of the usual convolution product of the Fourier transforms of f and g. The canonical commutation relation may be expressed as

$$x \star_0 y - y \star_0 x = \frac{1}{2i\omega_c} \tag{38}$$

In terms of the non commutative coordinates (X,Y) the eigenvalue equation (3) acting on $f(z)$ rewrites

$$\hat{V}^{(0)}(X,Y) f(X+iY) = (E-\omega_c) f(X+iY) \tag{39}$$

Looking at $f(X+iY) \equiv \hat{f}^{(0)}$ as an operator and using (34) one finds that (39) is equivalent to

$$\widehat{(V \star_0 f)}^{(0)}(X,Y) = (E-\omega_c) f(X+iY) \tag{40}$$

In other words the eigenvalues of (3) and of the operator equation (40) are identical. Note finally that the analyticity of f implies $V \star_0 f = Vf$, and therefore $\widehat{(Vf)}^{(0)}(X,Y) = (E-\omega_c) f(X+iY)$ [1].

First Landau Level: As seen in (27), the 1LL operator associated to the potential $V(x,y)$ is

$$: V(z, \frac{1}{\omega_c}\partial) + \frac{1}{\omega_c}\bar{\partial}\partial V(z, \frac{1}{\omega_c}\partial) : \tag{41}$$

This can be reexpressed as

$$\hat{V}^{(1)}(X,Y) = \int dkdl : e^{ikX+ilY} : (1 - \frac{1}{4\omega_c}(k^2+l^2))\tilde{V}(k,l) \tag{42}$$

Using the commutation relation (29) one can check that

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[1] Of course it does not imply that $Vf(x+iy) = (E-\omega_c)f(x+iy)$.

$$(k^2+l^2)e^{ikX+ilY} = [X,[X,e^{ikX+ilY}]+[Y,[Y,e^{ikX+ilY}] \tag{43}$$

Thus from (33) and (43) we find

$$\hat{V}^{(1)}(X,Y) = \hat{V}^{(0)}(X,Y) + \frac{1}{4\omega_c}\hat{\Delta}\hat{V}^{(0)}(X,Y) \tag{44}$$

where in (44)

$$\hat{\Delta} = [X,[.,X] + [Y,[.,Y] \tag{45}$$

and

$$(\widehat{\Delta V})^{(0)}(X,Y) = \hat{\Delta}\hat{V}^{(0)}(X,Y) \tag{46}$$

is understood. Equation (43) suggests that the operator (45) is the non commutative version of the Laplacian [5].

The 1LL mapping (42) between a classical function and an operator induces a new star product $\star_1$ such that for any classical functions $f(x,y)$ and $g(x,y)$

$$\hat{f}^{(1)}(X,Y)\hat{g}^{(1)}(X,Y) = (\widehat{f\star_1 g})^{(1)}(X,Y) \tag{47}$$

A differential expression for the $\star_1$ product can easily be found using (46). Indeed,

$$\begin{aligned}\hat{f}^{(1)}\hat{g}^{(1)} &= (\hat{f}^{(0)} + \frac{1}{4\omega_c}\hat{\Delta}\hat{f}^{(0)})(\hat{g}^{(0)} + \frac{1}{4\omega_c}\hat{\Delta}\hat{g}^{(0)}) \\ &= (\hat{f}^{(0)} + \frac{1}{4\omega_c}\widehat{\Delta f}^{(0)})(\hat{g}^{(0)} + \frac{1}{4\omega_c}\widehat{\Delta g}^{(0)})\end{aligned} \tag{48}$$

so that

$$f\star_1 g = (1+\frac{1}{4\omega_c}\Delta)^{-1}\Big((f+\frac{1}{4\omega_c}\Delta f)\star_0(g+\frac{1}{4\omega_c}\Delta g)\Big) \tag{49}$$

Thus

$$f\star_1 g = (1+\frac{1}{4\omega_c}\Delta)^{-1}\Big(e^{-\frac{1}{4\omega_c}(\partial_x - i\partial_y)(\partial'_x + i\partial'_y)}(1+\frac{1}{4\omega_c}\Delta)(1+\frac{1}{4\omega_c}\Delta')f(x,y)g(x',y')|_{x=x',y=y'}\Big) \tag{50}$$

Note that it follows from (50) that the canonical commutation relation $x\star_1 y - y\star_1 x = \frac{1}{2i\omega_c}$ is identical to (38), i.e. it is the same as in the LLL.

In terms of the non commutative coordinates X, Y the eigenvalue equation (27) becomes

$$\hat{V}^{(1)}(X,Y)g(X+iY) = (E-3\omega_c)g(X+iY) \tag{51}$$

or equivalently

$$(\widehat{V \star_1 g})^{(1)}(X,Y) = (E - 3\omega_c) g(X + iY) \tag{52}$$

Note that since $g(z)$ is analytic $V \star_1 g = (1 + \frac{1}{4\omega_c}\Delta)^{-1}((V + \frac{1}{4\omega_c}\Delta V) g)$.

Higher Landau level: Let us apply the same procedure than in the LLL and the 1LL to the the n-th Landau level (nLL). An eigenstate is of the form

$$< z, \bar{z}|\psi >= (f_0(z) + \bar{z} f_1(z) + ... + \bar{z}^n f_n(z)) e^{-\frac{\omega_c}{2} z\bar{z}} \tag{53}$$

with $f_n(z)$ analytic and

$$f_i(z) = \omega_c^{(i-n)} (-1)^{(n-i)} \frac{n!}{i!(n-i)!} \frac{\partial^{n-i}}{\partial z^{n-i}} f_n(z) \tag{54}$$

Projecting the Hamiltonian (1) on the nLL implies that $f_n(z)$ satisfies to the eigenvalue equation

$$: \sum_{i=0}^{n} \frac{1}{\omega_c^i} \frac{n!}{i!^2(n-i)!} \frac{\partial^{2i}}{\partial z^i \partial \bar{z}^i} V : f_n(z) = (E - (2n+1)\omega_c) f_n(z) \tag{55}$$

which can be viewed as the generalisation of the LLL Peierls substitution equation (3) to the nLL (details of the derivation of (53,55) can be found in [6]).

In a non commutative plane formulation, the potential $V(x,y)$ is replaced by the nLL operator

$$\hat{V}^{(n)}(X,Y) = \int dkdl : e^{ikX+ilY} : \tilde{V}(k,l) \sum_{i=0}^{n} (-\frac{1}{4\omega_c})^i \frac{n!}{i!^2(n-i)!} (k^2 + l^2)^i \tag{56}$$

with also

$$\hat{V}^{(n)}(X,Y) = \sum_{i=0}^{n} (\frac{1}{4\omega_c})^i \frac{n!}{i!^2(n-i)!} (\hat{\Delta})^i \hat{V}^{(0)}(X,Y) \tag{57}$$

This induces a $\star_n$ product between classical functions such that $\hat{f}^{(n)} \hat{g}^{(n)} = (\widehat{f \star_n g})^{(n)}$: namely one has

$$f \star_n g = (D^{(n)}_{x,y})^{-1} \left(e^{-\frac{1}{4\omega_c}(\partial_x - i\partial_y)(\partial'_x + i\partial'_y)} D^{(n)}_{x,y} D^{(n)}_{x',y'} f(x,y) g(x',y')|_{x=x',y=y'} \right) \tag{58}$$

where $D^{(n)}_{x,y} = \sum_{i=0}^{n} (\frac{1}{4\omega_c})^i \frac{n!}{i!^2(n-i)!} (\Delta)^i$. Note that the canonical commutation relation associated to $\star_n$ again narrows down to (38), i.e. to the LLL situation.

Hilbert space: Equation (56) gives a natural specification of nLL operators $\hat{f}^{(n)}(X,Y)$ associated to classical functions $f(x,y)$. One can also construct a Hilbert space on which these operators act and which turns out to be equivalent to the Bargmann space of analytic functions with the scalar product $< f|g >= (\omega_c/\pi) \int dzd\bar{z}e^{-\omega_c z\bar{z}}\bar{f}g$.

Let us define the creation-annihilation operators $A^\dagger = \omega_c^{1/2}(X+iY)$ and $A = \omega_c^{1/2}(X-iY)$ and the corresponding Hilbert space spanned by the vectors $A^{\dagger n}|0>$ acting on the vacuum $|0>$ with $A|0>= 0$. Since $[A, A^\dagger] = 1$ we have

$$< 0|e^{i(kX+lY)}|0 >= e^{-\frac{1}{2}<0|(kX+lY)^2|0>} = e^{-\frac{1}{8\omega_c}(k^2+l^2)} \tag{59}$$

If we now consider in the Hilbert space two states $\hat{f}^{(n)}|0>$ and $\hat{g}^{(n)}|0>$ their scalar product becomes

$$\begin{aligned} < 0|\hat{f}^{(n)\dagger}\hat{g}^{(n)}|0 > &= < 0|\widehat{(\bar{f} \star_n g)}^{(n)}|0 > \\ &= \int dkdle^{-\frac{1}{4\omega_c}(k^2+l^2)}\widetilde{(\bar{f} \star_n g)}(k,l)\sum_{i=0}^{n}(-\frac{1}{4\omega_c})^i\frac{n!}{i!^2(n-i)!}(k^2+l^2)^i \\ &= \frac{\omega_c}{\pi}\int dxdye^{-\omega_c(x^2+y^2)}D^{(n)}_{x,y}(\bar{f} \star_n g)(x,y) \end{aligned} \tag{60}$$

The scalar product (60) is equivalent to the scalar product of the Bargmann space. Indeed for any classical function $f(x,y)$, one can always define a polynomial (analytic) function of $A^\dagger$ such that $\hat{f}^{(n)}|0>= p_f(A^\dagger)|0>$. The analyticity then implies that $p_f \star_n p_g = (D^{(n)}_{x,y})^{-1}(\bar{p}_f p_g)$ so that

$$< 0|\hat{f}^{(n)\dagger}\hat{g}^{(n)}|0 >= \frac{\omega_c}{\pi}\int dxdye^{-\omega_c(x^2+y^2)}\bar{p}_f(x-iy)p_g(x+iy) \tag{61}$$

Equations (40) and (52) can be viewed as bona fide eigengenvalue equations on the Hilbert space spanned by $(A^\dagger)^n|0>$,

$$\hat{V}^{(n)}(X,Y)f(X+iY)|0 >= (E-(2n+1))f(X+iY)|0 > \tag{62}$$

Note finally that in (60) scalar products have been expressed in the operator language or equivalently in terms of star products, leading to non trivial identities[2].

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[2]The scalar product

$$< 0|\hat{f}^{(n)\dagger}\hat{f}^{(n)}|0 >= \frac{\omega_c}{\pi}\int dxdye^{-\omega_c(x^2+y^2)}D^{(n)}_{x,y}(\bar{f} \star_n f)(x,y) \tag{63}$$

must be non negative for any $f(x,y)$ although

$$D^{(n)}_{x,y}(\bar{f} \star_n f)(x,y) = \sum_{j=0}^{\infty}\frac{(-1)^j}{j!}(4\omega_c)^{-j}|(\partial_x + i\partial_y)^j D^{(n)}_{x,y}f(x,y)|^2 \tag{64}$$

is an alternating sum. As an illustration consider in the LLL $(n=0)$ $f(x,y) = (x+iy)^k(x-iy)^l$.

Conclusion: One has obtained the class of star products $\star_n$ which generalize to the nth Landau level the standard $\star_0$ product in the LLL. A non commutative space can be thus defined each time the two dimensionnal Hilbert space is projected on a given Landau level. Accordingly, a nLL Peierls subtsitution equation is obtained which generalizes the standard LLL Peierls substitution equation. In each nLL subspace the space is non commutative with a canonical commutation relation $x \star_n y - y \star_n x = \frac{1}{2i\omega_c}$. However, by considering altogether all the nLL projections one should recover the full Landau spectrum and therefore the commutative two dimensionnal space.

Then $\hat{f}^{(0)}(X,Y) = (X-iY)^l(X+iY)^k$ so that for $l > k$ we have $\hat{f}^{(0)}|0>= 0$. This is satisfied if, evaluating the right hand side of (63), the combinatorial identity

$$\sum_{j=0}^{l}(-1)^j \frac{(j+k)(j+k-1)...(j+1)}{(l-j)!j!} = 0, \qquad l > k \geq 0 \tag{65}$$

is verified. An independent check can be made using the binomial expansion of $\frac{d^k}{dx^k}(1-x)^l|_{x=1} = 0$ for $l > k \geq 0$.

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