Parafermionic theory with the symmetry Z_N, for N odd.

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Abstract.

We construct a parafermionic conformal theory with the symmetry Z_N, for N odd, based on the second solution of Fateev-Zamolodchikov for the corresponding parafermionic chiral algebra. Primary operators are classified according to their transformation properties under the dihedral group D_N, as singlet, doublet $1, 2, \ldots, \frac{N-1}{2}$, and disorder operators. In an assumed Coulomb gas scenario, the corresponding vertex operators are accommodated by the weight lattice of the Lie algebra $B_{(N-1)/2}$. The unitary theories are representations of the coset $SO_n(N) \times SO_2(N)/SO_{n+2}(N)$, with $n = 1, 2, \ldots$. Physically, they realise the series of multicritical points in statistical theories having a D_N symmetry.

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1 Introduction

Parafermionic algebras are associated in a natural way with extra discrete group symmetries in two-dimensional statistical systems.

The most used parafermionic conformal theory, which was constructed by Fateev and Zamolodchikov in Ref. [1], describes the self-dual critical points of the Z_N-invariant generalisations of the Ising model. The values of the spins of the parafermions take the minimal possible values admitted by the associativity constraints for the corresponding chiral algebra. In this case, the central charge of the Virasoro algebra is fixed as a function of N only; there is only one conformal field theory for each N.

In the Appendix A of Ref. [1], the same authors showed that there exists another associative chiral algebra, with the next allowed values of the spins of the parafermions. In this second solution, the central charge and the structure constants of the algebra are given, for each N, as functions of a free parameter.

The conformal field theories which correspond to this second solution are known only for particular values of N.

The case $N = 2$ corresponds to the superconformal theory ($\Delta = 3/2$). For $N = 4$, the parafermionic algebra turns out to factorise into two independent superconformal chiral algebras, with fields of dimensions 2 and 3/2. The theory with $N = 3$ has been fully developed by Fateev and Zamolodchikov in Ref. [2]. By imposing the degeneracy condition on the representations fields, they constructed an infinite series of conformal theories. These theories are supposed to describe the multicritical fixed points of physical statistical systems with the Z_3 symmetry. Indeed, the first theory of this series corresponds to the tricritical Potts model.

In a recent paper [3], we presented the theory based on the second solution of the Z_5 parafermionic algebra. In particular, by studying the degenerate representations of this algebra we showed that the representation fields are accommodated by the weight lattice of the Lie algebra B_2.

To obtain this result it was useful observing that the central charge of the second solution of the Z_N parafermionic algebra agrees with that of the coset [4]

$$\frac{SO_n(N) \times SO_2(N)}{SO_{n+2}(N)}, \tag{1.1}$$

which is

$$c = (N-1)\left(1 - \frac{N(N-2)}{p(p+2)}\right), \tag{1.2}$$

$$p = N - 2 + n. \tag{1.3}$$

It is then natural to look for a Kac formula for the dimensions of the primary operators based on the weight lattice of the Lie algebra $B_{\frac{N-1}{2}}$ for $N = 2r + 1$ (with $r \geq 2$), and of $D_{\frac{N}{2}}$ for $N = 2r$ (with $r \geq 3$).

In the present work we will present in detail the whole class of the conformal theories based on the second solution of the Z_{2r+1} parafermionic algebra.

The paper is organised as follows. In Section 2, we introduce the parafermionic algebra presented in Ref. [1]. We then discuss the operator content of the theory together with the modules induced by the primary operators. The commutation relations of the parafermionic modes in each sector are given and some results of the degeneracy calculations are shown. These results, together with the properties of the Coulomb gas realisation of the theory, will be used in Section 3 to fix the values of the boundary terms characterising each sector. Finally, in Section 4 we determine to which sector each operator of the theory belongs. The theory we have built is verified by considering the characteristic equations for three-point functions. The conclusions are given in Section 5.

2 Parafermionic algebra and representation space

As discussed in the Introduction, we will consider the second solution of the associative algebra presented in Appendix A of Ref. [1]. The operator product expansions, defining the algebra of the parafermionic currents Ψ^k (with $k = 1, 2, \ldots, N-1$), have the form:

$$\begin{aligned}\Psi^k(z)\Psi^{k'}(z') &= \frac{\lambda^{k,k'}_{k+k'}}{(z-z')^{\Delta_k+\Delta_{k'}-\Delta_{k+k'}}} \\ &\times \left\{\Psi^{k+k'}(z') + (z-z')\frac{\Delta_{k+k'}+\Delta_k-\Delta_{k'}}{2\Delta_{k+k'}}\partial_{z'}\Psi^{k+k'}(z') + \ldots\right\}, \quad k+k' \neq 0\end{aligned} \tag{2.1}$$

$$\Psi^k(z)\Psi^{-k}(z') = \frac{1}{(z-z')^{2\Delta_k}}\left\{1 + (z-z')^2\frac{2\Delta_k}{c}T(z') + \ldots\right\}. \tag{2.2}$$

In Eqs. (2.1)–(2.2), the conformal dimension Δ_k of the parafermionic current Ψ^k is given by:

$$\Delta_k = \Delta_{N-k} = \frac{2k(N-k)}{N}, \qquad k = \pm 1, \pm 2, \ldots, \pm\frac{N-1}{2}. \tag{2.3}$$

Note that in the above equations and in the rest of the present manuscript, the Z_N-charges k and their sums $k+k'$ are defined modulo N. In particular we write:

$$\Psi^{N-k} \equiv \Psi^{-k} \equiv (\Psi^k)^+, \qquad \Delta_{N-k} \equiv \Delta_{-k}. \tag{2.4}$$

Within the second solution, the structure constants $\lambda^{k,k'}_{k+k'}$ and the central charge c of the Virasoro algebra are given as functions of a single free parameter v:

$$\begin{aligned}(\lambda^{k,k'}_{k+k'})^2 &= \frac{\Gamma(k+k'+1)\Gamma(N-k+1)\Gamma(N-k'+1)}{\Gamma(k+1)\Gamma(k'+1)\Gamma(N-k-k'+1)\Gamma(N+1)} \\ &\times \frac{\Gamma(k+k'+v)\Gamma(N+v-k)\Gamma(N+v-k')\Gamma(v)}{\Gamma(N+v-k-k')\Gamma(k+v)\Gamma(k'+v)\Gamma(N+v)},\end{aligned} \tag{2.5}$$

$$c = \frac{4(N-1)(N+v-1)v}{(N+2v)(N+2v-2)}. \tag{2.6}$$

Changing the parametrisation $v \to n/2$, the connection between Eq. (2.6) and the coset formula (1.2) becomes explicit.

Among the representation fields, primaries with respect to the algebra (2.1)–(2.2), we expect to find singlet operators

$$\Phi^0(z,\bar{z}) \tag{2.7}$$

with Z_N charge equal to 0, and doublet q operators

$$\Phi^{\pm q}(z,\bar{z}), \qquad q = 1, 2, \ldots, \frac{N-1}{2}, \tag{2.8}$$

with Z_N charge equal to $\pm q$. The singlet and doublet operators (2.7)–(2.8) form the usual representations of the group Z_N.

In fact, the theory we are considering is invariant under the dihedral group D_N, which includes Z_N as a subgroup. This can be seen from the symmetry of Eqs. (2.1)–(2.5) under conjugation of the Z_N charge, $q \to N - q$, cf. Eq. (2.4). The representation space should thus include also the N-plet of Z_2 disorder operators, which we denote as:

$$\{R_a(z,\bar{z}), \quad a = 1, \ldots, N\}. \tag{2.9}$$

The presence of disorder operators in the spectrum has been shown explicitly in the case of the first solution for general N [5], and in the case of the second solution for the theories with $N = 3$ [2] and $N = 5$ [3].

The disorder fields thus play a symmetry generating role, since they complete the cyclic group Z_N, generated by the parafermionic currents $\Psi^{\pm k}(z)$, to the dihedral group D_N. However, the disorder fields are non-chiral representation fields, and in particular they do not participate in the chiral algebra [3]. The chiral algebra is therefore based on the *abelian* group Z_N.

Until now we have specified and discussed the operator content of the theory. The next step is to study the degenerate representations of the algebra (2.1)–(2.2). It is indeed well established that for a given chiral algebra with a free parameter, the associated conformal field theory is described by the degenerate representations of that algebra.

In order to study the degenerate representations, we first have to analyse the structure of the modules induced by various primary fields. In Ref. [3] these modules were constructed for $N = 5$. We shall show below how to generalise these results to arbitrary odd N.

2.1 Singlet and doublet sectors

2.1.1 Modules and mode commutation relations

We start by considering the module of the identity. First of all we place the chiral fields $\Psi^{\pm k}(z)$, with $k = 1, 2, \ldots, (N-1)/2$, and the stress-energy tensor $T(z)$ in the module of the identity I. The levels of these operators correspond to their conformal dimensions $\Delta_{\pm k}$. To complete the module with the remaining levels it is sufficient to take into account that, owing to the abelian monodromy of the fields $\Psi^{\pm k}$, the level spacing in each sector is equal to 1.

In the module of the identity there are no states below I and above $\Psi^{\pm 1}$. This is not the case for a more general singlet operator Φ^0; the first descendent levels of the

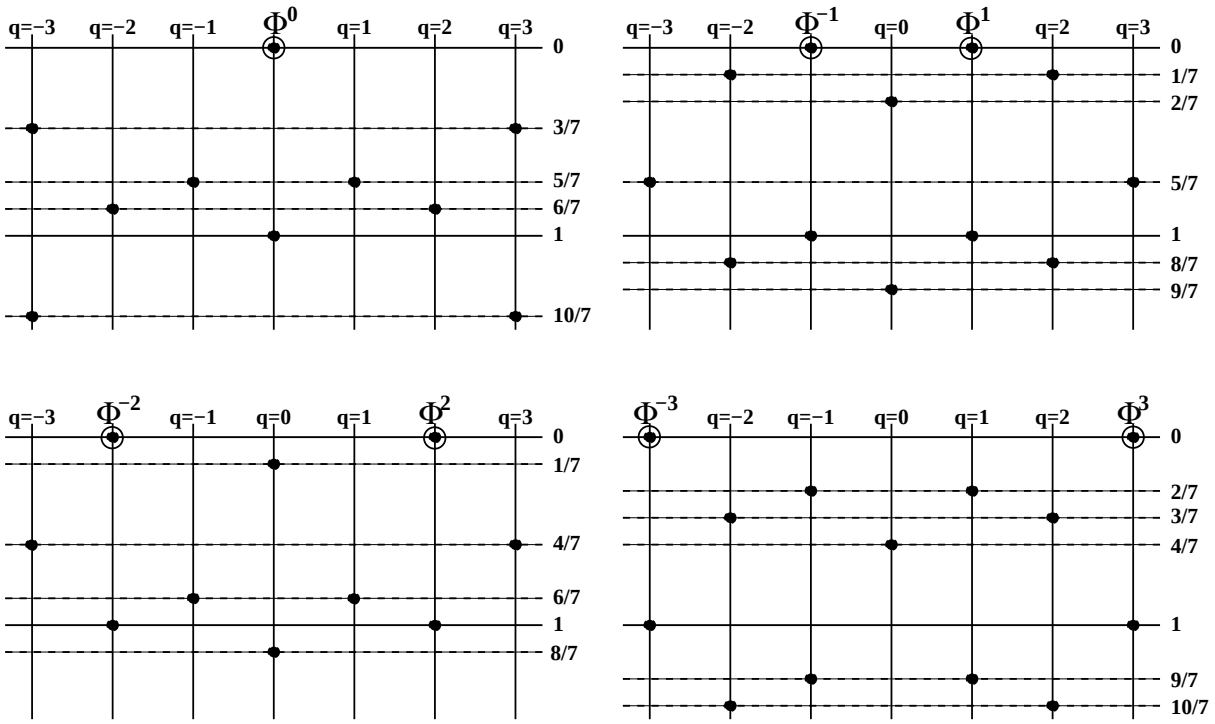

Figure 1: Representation modules for $N = 7$. We show the level structure of the modules of a singlet (top left), a doublet 1 (top right), a doublet 2 (bottom left), and a doublet 3 operator (bottom right).

correspondent module are, at least partially, occupied. Let us denote by δ_k^0 the first descendent level in the Z_N charge sector k. From Eq. (2.3) we readily obtain that

$$\delta_k^0 = -\frac{2k^2}{N} \text{ mod } 1. \tag{2.10}$$

The knowledge of the gaps δ_k^0 completely fixes the level structure of a generic singlet module. In the upper left part of Fig. 1 we show, as an example, the module of a singlet operator for $N = 7$.

The structure of the doublet q modules is extracted from the module of the identity by examining its corresponding submodules. The generalisation to the doublet q module of Eq. (2.10) is:

$$\delta_k^q = 2\frac{(q^2 - k^2)}{N} \text{ mod } 1, \tag{2.11}$$

where δ_k^q is the first level in the module of the doublet q corresponding to the Z_N charge sector k. From now on we will make extensive use of the above notation. In Fig. 1, the three possible doublets for $N = 7$ are shown.

In accordance with the structure of the modules, the developments of the chiral fields in each sector take, for general N, the form:

$$\Psi^k(z)\Phi^q(0) = \sum_n \frac{1}{(z)^{\Delta_k - \delta^q_{k+q}+n}} A^k_{-\delta^q_{k+q}+n}\Phi^q(0), \tag{2.12}$$

$$A^k_{-\delta^q_{k+q}+n}\Phi^q(0) = 0, \qquad n > 0. \tag{2.13}$$

We remind that the sums $k+q$ of the Z_N charges are always defined modulo N. Eq. (2.13) is the usual highest weight condition which expresses that representation fields are primary with respect to the parafermionic algebra. As usual, from Eq. (2.12), the action of the modes in each sector can be given in the form of a contour integral:

$$A^k_{-\delta^q_{k+q}+n}\Phi^q(0) = \frac{1}{2\pi i}\oint_{C_0} \mathrm{d}z\,(z)^{\Delta_k-\delta^q_{k+q}+n-1}\Psi^k(z)\Phi^q(0). \tag{2.14}$$

Note that, by Eq. (2.11), the level δ^q_{k+q} is equal to zero for $k=-2q$. This results in the presence, for each doublet $\Phi^{\pm q}$, of a zero mode $A_0^{\mp 2q}$ which acts between the two states at the summit of the corresponding module:

$$A_0^{\mp 2q}\Phi^{\pm q}(0) = h_q\Phi^{\mp q}(0). \tag{2.15}$$

The eigenvalues h_q defined by the above equation characterise the representations, together with the conformal dimension of the fields $\Phi^{\pm q}$.

The commutation relations of the mode operators can be deduced from Eq. (2.14) in a way similar to that used in Refs. [2, 3]. Below we give the general form of these relations.

In what follows we define the coefficients D^l_α from the development

$$(1-x)^\alpha = \sum_{l=0}^{\infty} D^l_\alpha x^l. \tag{2.16}$$

By Eq. (2.1), the modes of the parafermions Ψ^k and $\Psi^{k'}$, where $k,k' = \pm 1, 2, \ldots, \pm\frac{N-1}{2}$ with $k+k'\neq 0$, satisfy the relations:

$$\begin{aligned}
&\sum_{l=0}^{\infty} D^l_\alpha \left(A^k_{-\delta^{k'+q}_{k+k'+q}+s(k,k',q)+n-l} A^{k'}_{-\delta^q_{k'+q}+m+l} - A^{k'}_{-\delta^{k+q}_{k'+k+q}+s(k',k,q)+m-l} A^k_{-\delta^q_{k+q}+n+l} \right) \Phi^q \\
&= \lambda^{k,k'}_{k+k'}\eta^{(k,k',q)}(n,m) A^{k+k'}_{-\delta^q_{k+k'+q}+t(k,k',q)+n+m}\Phi^q \qquad (k+k'\neq 0)
\end{aligned} \tag{2.17}$$

with

$$\begin{aligned}
\alpha &= \Delta_k + \Delta_{k'} - \Delta_{k+k'} - 2 \qquad (2.18)\\
\eta^{(k,k',q)}(n,m) &= \Delta_k - \delta^q_{k+q} + n - 1 - \frac{\Delta_{k+k'}+\Delta_k-\Delta_{k'}}{2\Delta_{k+k'}}\left(\Delta_{k+k'} - \delta^q_{k+k'+q} + t(k,k',q) + n + m\right).
\end{aligned}$$

In Eq. (2.17), the integers $s(k,k',q)$ and $t(k,k',q)$ shift the indices of the parafermionic modes. They are given by:

$$\begin{aligned}
s(k,k',q) &= \delta^{k'+q}_{k+k'+q} - \delta^q_{k+q} + \alpha \\
t(k,k',q) &= \delta^q_{k+k'+q} - \delta^q_{k+q} - \delta^q_{k'+q} + \alpha
\end{aligned} \tag{2.19}$$

The remaining commutation relations between the modes of the parafermions Ψ^k and Ψ^{-k} (with $k=1,2,\ldots,\frac{N-1}{2}$) realise the connection with the Virasoro algebra. In fact, it is seen from Eq. (2.2) that the stress-energy operator $T(z)$ is produced in the expansion $\Psi^k\Psi^{-k}$. Defining the integers $u(k,q)$ as:

$$u(k,q) = \delta^{-k+q}_q - \delta^q_{k+q} + \beta \tag{2.20}$$

we can write the following relations:

$$\sum_{l=0}^{\infty} D_\beta^l \left(A^k_{-\delta_q^{-k+q}+u(k,q)+n-l} A^{-k}_{-\delta^q_{-k+q}+m+l} + A^{-k}_{-\delta_q^{k+q}+u(-k,q)+m-l} A^k_{-\delta^q_{k+q}+n+l} \right) \Phi^q$$

$$= \left(\kappa(n)\delta_{n+m+u'(k,q),0} + \frac{2\Delta_k}{c} L_{n+m+u'(k,q)} \right) \Phi^q \qquad (2.21)$$

with

$$\beta = 2\Delta_k - 3 \qquad (2.22)$$

$$\kappa(n) = \frac{1}{2}\left(\Delta_k - \delta^q_{k+q} + n - 1\right)\left(\Delta_k - \delta^q_{k+q} + n - 2\right).$$

On the right-hand side of Eq. (2.21) we have defined $u'(k,q) = -1 + u(k,q)$ for $k \neq -2q$, and $u'(k,q) = u(k,q)$ for $k = -2q$.

In Eq. (2.21), the L_n are the generators of the conformal transformations which form the Virasoro algebra:

$$(L_n L_m - L_m L_n)\,\Phi^q = \left[(n-m)L_{n+m} + \frac{c}{12} n(n^2-1)\delta_{n+m,0}\right] \Phi^q. \qquad (2.23)$$

Finally, the list of commutation relations in the singlet and doublet sectors are completed by the commutators between the parafermionic modes A^k and the L_n:

$$\left(A^k_{-\delta^q_{k+q}+m} L_n - L_n A^k_{-\delta^q_{k+q}+m} \right) \Phi^q = \left[(1-\Delta_{\Psi^k})n + m - \delta^q_{k+q}\right] A^k_{-\delta^q_{k+q}+m+n} \Phi^q. \qquad (2.24)$$

We shall often refer to Eq. (2.17) using the shorthand notation $\{\Psi^k, \Psi^{k'}\}\Phi^q$; likewise, Eq. (2.21) is referred to as $\{\Psi^k, \Psi^{-k}\}\Phi^q$. The commutation relations (2.17)-(2.24) given above allow to compute various matrix elements which enter the analysis of the degeneracies in the modules, to which we now turn our attention.

In Ref. [3], where the case $N = 5$ was considered, the dimensions of all the fundamental operators, i.e., the fundamental singlet, doublet 1, doublet 2 and disorder operators, were calculated by imposing the degeneracy at the first levels of the corresponding modules.

Below we present the analysis for the degeneracies at the first levels of the singlet and of the doublet 1 operators for $N = 7$. In the case of the doublet $(N-1)/2$ and of the disorder operator R we have been able to treat the case of arbitrary odd N. In the next Section we shall show that the results of this analysis, together with the properties of the Coulomb gas, will allow us to construct the theory completely.

2.1.2 Identity operator

In the following we shall call a degenerate operator *fundamental* if its levels of degeneracy are the lowest possible. It is natural to expect that the fundamental singlet operator is the identity operator of the theory, with dimension $\Delta_{\Phi^0} = 0$. Despite of this trivial value of the scaling dimension, it is important to analyse the module of the identity operator properly. Namely, the levels and Z_N charges of its degenerate submodules, and their

multiplicities, provide important information that we shall use in Section 3 to dress the general theory. This analysis was done for $N = 5$ in Ref. [3]; we here extend it to $N = 7$.

In the module of a singlet operator, shown in the top left part of Fig. 1, the first descendent level is the level 3/7 in the $q = 3$ sector. By using the commutation relations $\{\Psi^2, \Psi^1\}\Phi^0$ and $\{\Psi^{-1}, \Psi^{-3}\}\Phi^0$ (cf. Eq. (2.17)) it is easy to verify that all the possible $q = 3$ states at level 3/7 are proportional to one another

$$A^1_{3/7}A^2_{-6/7}\Phi^0 \propto A^{-2}_{3/7}A^{-2}_{-6/7}\Phi^0 \propto A^2_{2/7}A^1_{-5/7}\Phi^0 \propto A^{-3}_{2/7}A^{-1}_{-5/7}\Phi^0 \propto A^{-1}_0 A^{-3}_{-3/7}\Phi^0 \propto A^3_{-3/7}\Phi^0. \tag{2.25}$$

The above relations imply that there is only one $q = 3$ state at level 3/7. Thus, in order to impose degeneracy at level 3/7, we must require that the state

$$\chi^3_{-3/7} = A^3_{-3/7}\Phi^0 \tag{2.26}$$

be primary. This amounts to the condition:

$$A^{-3}_{+3/7}A^3_{-3/7}\Phi^0 = 0. \tag{2.27}$$

Using $\{\Psi^3, \Psi^{-3}\}\Phi^0$ (cf. Eq. (2.21)), it turns out that the constraint (2.27) is equivalent to the following relation between certain matrix elements at lower levels[2]:

$$\left(A^3_{+24/7}A^{-3}_{-24/7} - \frac{27}{7}A^3_{+17/7}A^{-3}_{-17/7} + \frac{270}{49}A^3_{+10/7}A^{-3}_{-10/7}\right)\Phi^0 = \left(1 + \frac{48}{7c}\Delta_{\Phi^0}\right)\Phi^0. \tag{2.28}$$

Having imposed the constraint (2.27)—or equivalently the relation (2.28)—we can eliminate the only state at level 3/7. We thus set

$$A^3_{-3/7}\Phi^0 = 0. \tag{2.29}$$

We then consider the next available level 5/7 in the $q = 1$ charge sector. Once the level 3/7 is empty, there are three main ways of descending to this level:

$$A^1_{-5/7}\Phi^0, \qquad A^2_0 A^{-1}_{-5/7}\Phi^0, \qquad A^{-1}_{+1/7}A^2_{-6/7}\Phi^0. \tag{2.30}$$

However, among the above states only two are independent. This is seen from the commutation relation $\{\Psi^2, \Psi^{-1}\}\Phi^0$ from which it follows that

$$A^2_0 A^{-1}_{-5/7}\Phi^0 - A^{-1}_{+1/7}A^2_{-6/7}\Phi^0 = \frac{\lambda^{2,-1}_1}{6}A^1_{-5/7}\Phi^0. \tag{2.31}$$

In addition to the states (2.30), we have to consider also the states which are obtained by repeated application of the parafermionic zero modes, like for example the state $A^2_0 A^{-2}_0 A^2_0 A^{-1}_{-5/7}\Phi^0$. By using the commutation relations (2.17) and the condition (2.29), we have checked that all these states actually decompose into linear combinations of the states (2.30). To see this, it is sufficent to verify that

$$A^2_0 A^{-2}_0 A^1_{-5/7}\Phi^0 = aA^1_{-5/7}\Phi^0 + bA^2_0 A^{-1}_{-5/7}\Phi^0, \tag{2.32}$$

[2]Eq. (2.28) expresses just one of many equivalent ways of transcribing the constraint (2.27) in terms of relations between lower-lying matrix elements. Using the decompositions (2.25) it may be seen that among the various choices of rewriting the constraint (2.27), the one that digs least deeply into the module involves matrix elements up to level 6/7. In any case, the key point is that the matrix elements occuring in any such relation touch lower levels than those that we degenerate in the present calculation.

where a and b are some numerical coefficients.

Each state at level 5/7 in the $q = 1$ sector can thus be written in terms of the states $A^1_{-5/7}\Phi^0$ and $A^2_0 A^{-1}_{-5/7}\Phi^0$. Then, if we demand the complete degeneracy at level 5/7, we have to impose the following conditions:

$$A^{-1}_{+5/7}\left(A^1_{-5/7}\Phi^0\right) = 0, \tag{2.33}$$

$$A^1_{+5/7}A^{-2}_0\left(A^1_{-5/7}\Phi^0\right) = 0, \tag{2.34}$$

$$A^{-1}_{+5/7}\left(A^2_0 A^{-1}_{-5/7}\Phi^0\right) = 0, \tag{2.35}$$

$$A^1_{+5/7}A^{-2}_0\left(A^2_0 A^{-1}_{-5/7}\Phi^0\right) = 0. \tag{2.36}$$

Using the charge conjugation symmetry, and Eq. (2.32), it is easy to see that if the conditions (2.33)–(2.34) are verified, the remaining equations (2.35)–(2.36) are automatically satisfied. Once the conditions (2.33)-(2.34) have been imposed, all the states at level 5/7 can then be eliminated.

From the commutation relation $\{\Psi^1, \Psi^{-1}\}\Phi^0$ we have that

$$A^{-1}_{+5/7}A^1_{-5/7}\Phi^0 = \frac{48}{7c}\Delta_{\Phi^0}\Phi^0. \tag{2.37}$$

The condition (2.33) then implies that

$$\Delta_{\Phi^0} = 0. \tag{2.38}$$

We have thus found the trivial scaling dimension of the identity.

The second equation (2.34) simply determines the values of some matrix elements, starting from the value of the matrix element $A^1_{+5/7}A^{-2}_0 A^1_{-5/7}\Phi^0$ which is set equal to zero. By using the algebra of the parafermionic modes, other matrix elements can then be fixed. For example, from the condition (2.33) and the commutation relations $\{\Psi^{-2}, \Psi^1\}\Phi^0$ and $\{\Psi^1, \Psi^1\}\Phi^2$, it turns out that

$$A^1_{+5/7}A^{-2}_0 A^1_{-5/7}\Phi^0 = \lambda^{1,1}_2 A^{-2}_{+6/7}A^2_{-6/7}\Phi^0 = 0. \tag{2.39}$$

Once all the states at level 3/7 and at level 5/7 have been eliminated, it can easily be shown that the level 6/7 is automatically empty. In fact, according to the commutation relation $\{\Psi^1\Psi^1\}\Phi^0$, the only possible state $A^2_{-6/7}\Phi^0$ at level 6/7 vanishes, since

$$\lambda^{1,1}_2 A^2_{-6/7}\Phi^0 = A^1_{-1/7}A^1_{-5/7}\Phi^0 = 0. \tag{2.40}$$

Summarising, we see that the fundamental singlet has one degenerate doublet 3 submodule at level 3/7 and two degenerate doublet 1 submodules at level 5/7.

2.1.3 Fundamental doublet 1 operator

We here show how to extend the explicit degeneracy analysis of the fundamental doublet 1 operator from the case $N = 5$ (see Ref. [3]) to $N = 7$.

The first descendent states in this doublet (shown in the upper right part of Fig. 1) are the two conjugate doublet 2 states at level 1/7. We first impose complete degeneracy of these states; by charge conjugation symmetry it suffices to study the one with $q = 2$. The degeneracy condition reads

$$A^{-1}_{1/7}A^{1}_{-1/7}\Phi^1 = 0. \tag{2.41}$$

By the commutation relations $\{\Psi^1, \Psi^{-1}\}\Phi^1$, this determines a matrix element at level 2/7:

$$\mu_{1,-1}\Phi^1 \equiv A^{1}_{2/7}A^{-1}_{-2/7}\Phi^1 = \left(-\frac{6}{49} + \frac{24}{7c}\Delta_{\Phi^1}\right)\Phi^1, \tag{2.42}$$

where Δ_{Φ^1} is the conformal weight of the doublet 1 operator that were are trying to determine.

We then focus on the next available level, which is the singlet ($q = 0$) state at level 2/7. Since there are no states left on level 1/7, our candidate for a singular state at level 2/7 reads

$$\chi^0_{-2/7} = aA^{1}_{-2/7}\Phi^{-1} + bA^{-1}_{-2/7}\Phi^1. \tag{2.43}$$

The degeneracy conditions are $A^{\pm 1}_{2/7}\chi^0_{-2/7} = 0$. Using the charge conjugation symmetry, they can be summarised by

$$(\mu_{1,1})^2 = (\mu_{1,-1})^2, \tag{2.44}$$

where the matrix element $\mu_{1,1}$ is defined by

$$\mu_{1,1}\Phi^1 \equiv A^{1}_{2/7}A^{1}_{-2/7}\Phi^{-1} \tag{2.45}$$

and $\mu_{1,-1}$ has been given above. Using $\{\Psi^1, \Psi^1\}\Phi^{-1}$ we can evaluate

$$\mu_{1,1} = \lambda^{1,1}_2 h_2, \tag{2.46}$$

where the structure constant $\lambda^{1,1}_2$ is given by Eq. (2.1), and the zero mode eigenvalue h_2 is defined in Eq. (2.15).

It would now appear natural to impose the third (and last) degeneracy condition on the doublet 3 states at level 5/7. But we shall now show that this level is actually void, as a result of the degeneracies at levels 1/7 and 2/7. Let us recall that there is by now zero states at level 1/7, and one state at level 2/7 since we have set

$$\chi^0_{-2/7} = A^{1}_{-2/7}\Phi^{-1} - A^{-1}_{-2/7}\Phi^1 \equiv 0. \tag{2.47}$$

The potential states at level 5/7 in, say, the charge $q = 3$ sector therefore read:

$$A^{2}_{-5/7}\Phi^1, \qquad A^{-3}_{-5/7}\Phi^{-1}, \qquad A^{3}_{-3/7}A^{1}_{-2/7}\Phi^{-1}. \tag{2.48}$$

The first of these states is zero as a consequence of the degeneracy at level 1/7, and of the commutation relations $\{\Psi^1, \Psi^1\}\Phi^1$ which imply

$$A^{1}_{-4/7}A^{1}_{-1/7}\Phi^1 = \frac{1}{2}\lambda^{1,1}_2 A^{2}_{-5/7}\Phi^1. \tag{2.49}$$

By charge conjugation we also have $A^{-2}_{-5/7}\Phi^{-1} = 0$. But the commutation relations $\{\Psi^{-1}, \Psi^{-2}\}\Phi^{-1}$ imply that

$$A^{-1}_{0}A^{-2}_{-5/7}\Phi^{-1} = \frac{2}{3}\lambda^{1,2}_3 A^{-3}_{-5/7}\Phi^{-1}, \tag{2.50}$$

and we must therefore have $A^{-3}_{-5/7}\Phi^{-1} = 0$ as well. Finally, it is a consequence of $\{\Psi^1, \Psi^3\}\Phi^{-1}$ that

$$\left(A^1_{-4/7}A^3_{-1/7} - A^3_{-3/7}A^1_{-2/7}\right)\Phi^{-1} = -\frac{1}{4}\lambda^{1,3}_{-3}A^{-3}_{-5/7}\Phi^{-1}. \tag{2.51}$$

We have shown that the first term on the left-hand side vanishes, as does the term on the right-hand side. Therefore $A^3_{-3/7}A^1_{-2/7}\Phi^{-1}$ must also vanish. In conclusion, we have shown that all of the three states (2.48) are zero, and thus level 5/7 is empty.

We therefore try imposing degeneracy on the doublet 1 states at level 1. For definiteness, we consider the charge sector $q = 1$. There are potentially three states at this position in the module:

$$A^2_{-1}\Phi^{-1}, \qquad A^1_{-5/7}A^1_{-2/7}\Phi^{-1}, \qquad L_{-1}\Phi^1. \tag{2.52}$$

However, the first of these two states are dependent, as is seen by using the commutation relations $\{\Psi^1, \Psi^1\}\Phi^{-1}$:

$$A^1_{-5/7}A^1_{-2/7}\Phi^{-1} = \frac{1}{2}\lambda^{1,1}_2 A^2_{-1}\Phi^{-1}. \tag{2.53}$$

It turns out to be most convenient to use this dependency to eliminate the first of the states (2.52). We therefore demand that

$$\chi^1_{-1} = \tilde{a}L_{-1}\Phi^1 + \tilde{b}A^1_{-5/7}A^1_{-2/7}\Phi^{-1} \tag{2.54}$$

be a singular state. Defining four matrix elements

$$\mu_0\Phi^1 \equiv L_1L_{-1}\Phi^1, \tag{2.55}$$

$$\mu_1\Phi^1 \equiv L_1A^1_{-5/7}A^1_{-2/7}\Phi^{-1}, \tag{2.56}$$

$$\mu_2\left(A^1_{-2/7}\Phi^{-1}\right) \equiv A^{-1}_{5/7}L_{-1}\Phi^1, \tag{2.57}$$

$$\mu_3\left(A^1_{-2/7}\Phi^{-1}\right) \equiv A^{-1}_{5/7}A^1_{-5/7}A^1_{-2/7}\Phi^{-1}, \tag{2.58}$$

the degeneracy criterion can be cast in the form

$$\mu_0\mu_3 - \mu_1\mu_2 = 0. \tag{2.59}$$

The first three of the matrix elements (2.96)–(2.57) are easily found from the commutation relations (2.23)–(2.24). The results are:

$$\mu_0 = 2\Delta_{\Phi^1}, \qquad \mu_1 = \frac{10}{7}\mu_{1,1}, \qquad \mu_2 = \frac{10}{7}, \tag{2.60}$$

where $\mu_{1,1}$ is defined by Eq. (2.45). The evaluation of the fourth matrix element is slightly more involved. Letting the commutation relation $\{\Psi^1, \Psi^{-1}\}\Phi^0$ act on the state $A^1_{-2/7}\Phi^{-1}$ we obtain

$$\left(A^{-1}_{5/7}A^1_{-5/7} - \frac{3}{7}A^{-1}_{-2/7}A^1_{2/7} + A^1_{-2/7}A^{-1}_{2/7}\right)A^1_{-2/7}\Phi^{-1} = \frac{24}{7c}L_0\left(A^1_{-2/7}\Phi^{-1}\right). \tag{2.61}$$

This implies that

$$\mu_3 = \frac{3}{7}\mu_{1,1} - \mu_{1,-1} + \frac{24}{7c}\left(\Delta_{\Phi^1} + \frac{2}{7}\right). \tag{2.62}$$

The solution of Eq. (2.59), with the ingredients (2.60) and (2.62), depends on the sign chosen in resolving Eq. (2.44). Choosing $\mu_{1,1} = +\mu_{1,-1}$, we find the solutions

$$\Delta_{\Phi^1}^{(\pm)} = \frac{1}{42}\left(36 - c \pm \sqrt{(c-6)(c-216)}\right). \tag{2.63}$$

On the other hand, the choice $\mu_{1,1} = -\mu_{1,-1}$ leads to

$$\Delta_{\Phi^1} = \frac{1}{84}\left(128 + 5c \pm \sqrt{25c^2 + 680c + 16384}\right). \tag{2.64}$$

We shall see below that the solutions (2.63) are physically acceptable, from the point of view of the general structure of the theory that we are constructing, while (2.64) must be discarded as non-physical. It is useful to rewrite Eq. (2.63) in terms of the parametrisation (1.3):

$$\Delta_{\Phi^1}^{(+)} = \frac{5}{7}\frac{p+7}{p}, \tag{2.65}$$

$$\Delta_{\Phi^1}^{(-)} = \frac{5}{7}\frac{p-5}{p+2}. \tag{2.66}$$

In summary, the fundamental doublet 1 operator has one degenerate doublet 2 submodule at level 1/7, one degenerate singlet submodule at level 2/7, and one degenerate doublet 1 submodule at level 1.

2.1.4 Fundamental doublet $(N-1)/2$ operator

Writing $N = 2r + 1$, the doublet with the largest possible Z_N charge is the doublet r. The fact that the module of the doublet r has its two summits in adjacent charge sectors is a help in the degeneracy computations. For this reason, we have been able to treat the case of arbitrary odd N.

We begin by considering the case of $N = 7$ in some detail. In this case, the structure of the doublet $r = 3$ is shown in the lower right part of Fig. 1. Using the commutation relations, it is easy to see that all the ways of descending to level 2/7 in the charge sector $q = -1$ are in fact linearly dependent:

$$A^3_{-2/7}\Phi^3 \propto A^1_{1/7}A^2_{-3/7}\Phi^3 \propto A^{-3}_{1/7}A^{-1}_{-3/7}\Phi^3 \propto A^{-2}_{0}A^{-2}_{-2/7}\Phi^3 \propto A^{1}_{1/7}A^{1}_{-3/7}\Phi^{-3} \propto A^{2}_{-2/7}\Phi^{-3}. \tag{2.67}$$

In particular, there is only one $q = -1$ state at level 2/7. Imposing degeneracy of this state amounts to the condition

$$A^{-2}_{2/7}A^{2}_{-2/7}\Phi^{-3} = 0. \tag{2.68}$$

This condition does not immediately lead to a fixation of the zero mode eigenvalue h_3. Rather, it gives relations between certain matrix elements deeper down in the module. The reason is essentially that the commutation relations $\{\Psi^2, \Psi^{-2}\}\Phi^{-3}$ have a high value of the parameter β, cf. Eq. (2.22), and thus several terms will contribute on the left-hand side. (We have already encountered a similar phenomenon in Eq. (2.28).)

The next available level consists of the doublet 2 states at level 3/7. Due to the degeneracy (2.68), level 2/7 is now empty, and there are a priori two states in the sector $q = -2$ at level 3/7:

$$A^2_{-3/7}\Phi^3 \quad \text{and} \quad A^1_{-3/7}\Phi^{-3}. \tag{2.69}$$

These states are however dependent, as is seen from $\{\Psi^1, \Psi^1\}\Phi^3$. (Further candidate states obtained by acting by the zero mode at level 3/7 can be similarly eliminated.) The degeneracy criterion reads

$$A^{-1}_{3/7}A^1_{-3/7}\Phi^{-3} = 0. \tag{2.70}$$

By $\{\Psi^1, \Psi^{-1}\}\Phi^{-3}$, this immediately fixes the zero mode eigenvalue:

$$(h_3)^2 = -\frac{5}{49} + \frac{24}{7c}\Delta_{\Phi^3}. \tag{2.71}$$

As a consequence of $\{\Psi^1, \Psi^2\}\Phi^{-3}$ we have that

$$\lambda_3^{1,2}A^3_{-4/7}\Phi^{-3} = A^1_{-2/7}A^2_{-2/7}\Phi^{-3} = -\frac{3}{2}A^2_{-1/7}A^1_{-3/7}\Phi^{-3}. \tag{2.72}$$

Thus, since levels 2/7 and 3/7 are now empty, there are no states left at level 4/7.

With these simplifications, the next available states are in the $q = \pm 3$ sectors at level 1. Focusing on $q = 3$, the case of $q = -3$ being equivalent, we form the candidate singular state

$$\chi^3_{-1} = aL_{-1}\Phi^3 + bA^{-1}_{-1}\Phi^{-3}, \tag{2.73}$$

and we define the matrix elements

$$\mu_{0,0}\Phi^3 = L_1L_{-1}\Phi^3, \tag{2.74}$$

$$\mu_{0,-1}\Phi^3 = L_1A^{-1}_{-1}\Phi^{-3}, \tag{2.75}$$

$$\mu_{1,0}\Phi^{-3} = A^1_1L_{-1}\Phi^3, \tag{2.76}$$

$$\mu_{1,-1}\Phi^{-3} = A^1_1A^{-1}_{-1}\Phi^{-3}. \tag{2.77}$$

The degeneracy criterion then reads, as usual,

$$\mu_{0,0}\mu_{1,-1} - \mu_{0,-1}\mu_{1,0} = 0. \tag{2.78}$$

The first three matrix elements are readily computed

$$\mu_{0,0} = 2\Delta_{\Phi^3}, \qquad \mu_{0,-1} = \mu_{1,0} = \frac{12}{7}h_3, \tag{2.79}$$

and the fourth one is obtained from $\{\Psi^1, \Psi^{-1}\}\Phi^{-3}$, yielding

$$\mu_{1,-1} = \frac{3}{7}(h_3)^2 + \frac{9}{49} + \frac{24}{7c}\Delta_{\Phi^3}. \tag{2.80}$$

Inserting this into Eq. (2.78), and using also Eq. (2.71) for the zero mode eigenvalue, we arrive at the solution

$$\Delta^{(\pm)}_{\Phi^3} = \frac{1}{70}\left(36 - c \pm \sqrt{(c-6)(c-216)}\right). \tag{2.81}$$

This can also be written, using the parametrisation (1.3), as:

$$\Delta^{(+)}_{\Phi^3} = \frac{3}{7}\frac{p+7}{p}, \tag{2.82}$$

$$\Delta^{(-)}_{\Phi^3} = \frac{3}{7}\frac{p-5}{p+2}. \tag{2.83}$$

In conclusion, the doublet 3 module that we have constructed has one degenerate doublet 1 submodule at level 2/7, one degenerate doublet 2 submodule at level 3/7, and one degenerate doublet 3 submodule at level 1.

General case: Based on the similarities of the fundamental doublet $r = \frac{N-1}{2}$ modules for $N = 5$ (see Ref. [3]) and $N = 7$ (see above), we can conjecture the structure of this doublet in the general case. One should demand the existence of r distinct degenerate submodules; however, only two degeneracy conditions are required to fix the zero mode eigenvalue h_r and the dimension Δ_{Φ^r}. We conjecture that these two conditions are obtained by requiring, respectively:

1. The complete degeneracy of the doublet $r-1$ state at level $\delta^r_{r-1} = 1 - 4/N$ (see Eq. (2.11)). Note that we have here supposed that those of the remaining $r-2$ degeneracy conditions that act at levels strictly between 0 and δ^r_{r-1} are such that these levels become empty after factoring out the corresponding degenerate submodules (or at least that no indirect ways of descending to level δ^r_{r-1} exist). Under this assumption, there remains only the two direct ways of descending to level δ^r_{r-1}:
$$A^{-1}_{-\delta^r_{r-1}}\Phi^r \quad \text{and} \quad A^{-2}_{-\delta^r_{r-1}}\Phi^{-r}. \tag{2.84}$$
However, the commutation relation $\{\Psi^{-1}, \Psi^{-1}\}\Phi^{-r}$ of Eq. (2.17), with $n = 1$ and $m = 0$, shows that these two states are in fact proportional to one another. In order to have degeneracy at level δ^r_{r-1}, it thus suffices to require that the state
$$\chi^{r-1}_{-\delta^r_{r-1}} \equiv A^{-1}_{-\delta^r_{r-1}}\Phi^r \tag{2.85}$$
be primary. This condition then fixes the eigenvalue h_r (see below).

2. A degeneracy of the doublet r state at level 1. We here suppose that those of the remaining $r-2$ degeneracy conditions that act at levels strictly between δ^r_{r-1} and 1 are such that these levels become empty after factoring out the corresponding degenerate submodules. If this is so, we can produce a degenerate state of a form analogous to Eq. (2.73)
$$\chi^r_{-1} = aL_{-1}\Phi^r + bA^{-1}_{-1}\Phi^{-r}. \tag{2.86}$$
Demanding the primarity of this state then fixes Δ_{Φ^r} (see below).

Note that apart from eliminating unwanted states between levels 0 and 1, the remaining $r-2$ degeneracy conditions may fix the values of certain lower-lying matrix elements (as was the case for $N = 7$, at level 2/7).

We have verified that the scenario outlined above indeed holds true also for $N = 9$. Now, let us examine its algebraic consequences in the general case.

The primarity of the state (2.85) can be obtained by requiring that

$$A^1_{+\delta^r_{r-1}} A^{-1}_{-\delta^r_{r-1}} \Phi^r = 0. \tag{2.87}$$

This can be rewritten, using the commutation relations $\{\Psi^1, \Psi^{-1}\}\Phi^r$, through

$$\left(A_0^{-1}A_0^1 + A^{-1}_{\delta^r_{r-1}} A^1_{-\delta^r_{r-1}}\right)\Phi^r = \left(-\frac{N-2}{N^2} + \frac{2\Delta_1}{c}\Delta_{\Phi^r}\right)\Phi^r, \tag{2.88}$$

which fixes the eigenvalue (2.15) to be:

$$(h_r)^2 = -\frac{N-2}{N^2} + \frac{2\Delta_1}{c}\Delta_{\Phi^r}, \tag{2.89}$$

thus generalising Eq. (2.71). If we impose the condition (2.89), then the state (2.85) can be set equal to zero, thus reducing the module. With the above set of assumptions, there are no states left in the module strictly between levels 0 and 1.

Turning now to level 1, we demand that the state (2.86) be primary:

$$L_{+1}\chi^r_{-1} = 0, \quad A^{+1}_{+1}\chi^r_{-1} = 0. \tag{2.90}$$

In terms of the matrix elements μ_{ij} defined by

$$L_{+1}L_{-1}\Phi^r = \mu_{11}\Phi^r, \tag{2.91}$$

$$L_{+1}A^{-1}_{-1}\Phi^{-r} = \mu_{12}\Phi^r, \tag{2.92}$$

$$A^{-1}_{-1}L_{-1}\Phi^{-r} = \mu_{21}\Phi^r, \tag{2.93}$$

$$A^{-1}_{+1}A^1_{-1}\Phi^r = \mu_{22}\Phi^r, \tag{2.94}$$

the degeneracy criterion reads

$$\mu_{11}\mu_{22} - \mu_{12}\mu_{21} = 0. \tag{2.95}$$

Using the commutation relations, the required matrix elements are readily computed:

$$\mu_{11} = 2\Delta_{\Phi^r}, \tag{2.96}$$

$$\mu_{12} = \mu_{21} = \Delta_1 h_r, \tag{2.97}$$

$$\mu_{22} = \frac{N-4}{N}(h_r)^2 + \frac{N+2}{N^2} + \frac{2\Delta_1}{c}\Delta_{\Phi^r}, \tag{2.98}$$

Inserting the eigenvalue (2.89) and the matrix elements (2.96)–(2.98) into Eq. (2.95) we get the following solutions:

$$\Delta^{(\pm)}_{\Phi^r} = \frac{1}{2} + \frac{1-c}{2N(N-2)} \pm \frac{1}{2N(N-2)}\sqrt{(c-N+1)(c-(N-1)^3)}, \tag{2.99}$$

which generalise Eq. (2.81).

Using the parametrisation (1.3) in Eq. (2.6), these solutions take the form

$$\Delta^{(+)}_{\Phi^r} = \frac{1}{2}\frac{(N-1)(N+p)}{Np}, \tag{2.100}$$

$$\Delta^{(-)}_{\Phi^r} = \frac{1}{2}\frac{(N-1)(p+2-N)}{(p+2)N}. \tag{2.101}$$

It should be remarked that the assumptions that we have made in order to determine the dimension Δ_{Φ^r} for general r will ultimately be validated by the agreement of Eqs. (2.100)–(2.101) with the Kac formula which we shall discuss in the next Section.

2.2 Sector of the disorder operators R_a

2.2.1 Modes of the chiral operators and commutation relations

The theory of disorder operators R_a ($a = 1, 2, \ldots, N$) has been fully developed in Ref. [5] within the first parafermionic conformal field theory with symmetry Z_N, and in Refs. [2, 3] within the context of the second parafermionic theory with symmetries Z_3 and Z_5 respectively. The general properties (operator product expansions, analytic continuations, etc.) of the disorder sector operators, and the approach used for studying this particular sector, can be adapted to the present theory as well.

The non-abelian monodromy of the disorder operator $R_a(z, \bar{z})$ with respect to the chiral fields $\Psi^{\pm k}(z)$ amounts to the decomposition of the local products $\Psi^k(z)R_a(0)$ into half-integer powers of z:

$$\Psi^k(z)R_a(0) = \sum_n \frac{1}{(z)^{\Delta_k+\frac{n}{2}}} A^k_{\frac{n}{2}} R_a(0), \qquad k = 1, 2, \ldots, \frac{N-1}{2}. \tag{2.102}$$

The expansion of the product $\Psi^{-k}(z)R_a(0)$ (with $k = 1, 2, \ldots, \frac{N-1}{2}$) can be obtained by an analytic continuation of z around 0 on both sides of Eq. (2.102). The result is:

$$\Psi^{-k}(z)R_a(0) = \sum_n \frac{(-1)^n}{(z)^{\Delta_k+\frac{n}{2}}} A^1_{\frac{n}{2}} \mathsf{U}^k R_a(0), \qquad k = 1, 2, \ldots, \frac{N-1}{2}. \tag{2.103}$$

In the above equation, U is a $N \times N$ matrix which rotates the index of the disorder field: $\mathsf{U}R_a(0) = R_{a-1}(0)$.

In accordance with these expansions, the mode operators $A^k_{\frac{n}{2}}$ can be defined by the contour integrals

$$A^k_{\frac{n}{2}} R_a(0) = \frac{1}{4\pi i} \oint_{C_0} \mathrm{d}z\, (z)^{\Delta_k+\frac{n}{2}-1} \Psi^k(z)R_a(0), \tag{2.104}$$

where the integrations are defined by letting z turn twice around the operator $R_a(0)$ at the origin, exactly as described in Ref. [2].

In the calculations of degeneracy we have used two types of commutation relations: The first one is between the modes of two Ψ^1 chiral fields,

$$\begin{aligned} &\sum_{l=0}^{\infty} D^l_{\alpha\beta} \left(A^1_{\frac{n-l}{2}} A^1_{\frac{m+l}{2}} + A^1_{\frac{m-l}{2}} A^1_{\frac{n+l}{2}} \right) R_a \\ &= \lambda_2^{1,1} 2^{\Delta_2-3} A^2_{\frac{n+m}{2}} R_a + (-1)^n 2^{-\Delta_2-2} \left[\kappa(n)\delta_{n+m,0} + \frac{16\Delta_1}{c} L_{\frac{n+m}{2}} \right] \mathsf{U}^{-1} R_a, \end{aligned} \tag{2.105}$$

and the second one is between the Ψ^1 and Ψ^k chiral fields, with $k = 2, 3, \ldots, \frac{N-1}{2}$,

$$\begin{aligned} &\sum_{l=0}^{\infty} D^l_{\nu\mu} \left(A^k_{\frac{n-l}{2}} A^1_{\frac{m+l}{2}} - A^1_{\frac{m-l}{2}} A^k_{\frac{n+l}{2}} \right) R_a \qquad (2.106) \\ = \;& (-1)^{n+m} 2^{\mu-\nu-2} \lambda^{k,1}_{k+1} \left[(2\Delta_k + n - 1) - \frac{\mu-\nu-2}{2} - a_1(2\Delta_{k+1} + n + m) \right] A^{k+1}_{\frac{n+m}{2}} R_a \\ + \;& (-1)^m 2^{\nu-\mu-2} \lambda^{-k,1}_{1-k} \left[(2\Delta_k + n - 1) - \frac{\nu-\mu-2}{2} - a_2(2\Delta_{1-k} + n + m) \right] A^{1-k}_{\frac{n+m}{2}} \mathsf{U}^k R_a, \end{aligned}$$

where the coefficients $D^l_{\nu\mu}$ are defined by

$$(1-x)^\nu(1+x)^\mu = \sum_{l=0}^{\infty} D^l_{\nu\mu} x^l. \tag{2.107}$$

In Eq. (2.105) we have used the following abbreviations:

$$\begin{aligned}
\alpha &= 2\Delta_1 - \Delta_2 - 1, \\
\beta &= 2\Delta_1 - 3, \\
\kappa(n) &= (2\Delta_1 + n - 1)(2\Delta_1 + n - 2) - (2\Delta_1 + n - 1)(\Delta_2 + 1) + \frac{(\Delta_2+1)(\Delta_2+2)}{4}.
\end{aligned}$$

And in Eq. (2.106) we have abbreviated the following quantities:

$$\begin{aligned}
\mu &= \Delta_k + \Delta_1 - \Delta_{k+1} - 2, \\
\nu &= \Delta_{-k} + \Delta_1 - \Delta_{1-k} - 2, \\
a_1 &= \frac{\Delta_{k+1} + \Delta_k - \Delta_1}{2\Delta_{k+1}}, \\
a_2 &= \frac{\Delta_{1-k} + \Delta_k - \Delta_1}{2\Delta_{1-k}}.
\end{aligned}$$

2.2.2 Degeneracy of the disorder modules

The structure of the module of a disorder operator R_a ($a = 1, 2, \ldots, N$) is relatively simple, as witnessed by the expansion (2.102): Each module has N summits, labeled by the components R_a, and has only integer and half-integer levels. Furthermore, it is seen from the expansion (2.102) that there are $(N-1)/2$ zero modes A^k_0 (with $k = 1, 2, \ldots, \frac{N-1}{2}$), associated with the parafermion Ψ^k which acts between the N summits of the module:

$$A^k_0 R_a = h_k \, \mathsf{U}^{2k} R_a. \tag{2.108}$$

This defines the eigenvalues h_k. We recall that $\mathsf{U} R_a = R_{a-1}$. Like in the case of the doublet operators, the eigenvalues h_k characterise, together with the conformal dimension, each representation R_a. Actually, the eigenvalues h_1 and h_2 are linked by a relation which does not depend on the details of the representation R_a. This relation is easily obtained by setting $n = m = 0$ in Eq. (2.105):

$$2h_1^2 = \lambda_2^{1,1} 2^{\Delta_2 - 3} h_2 + 2^{-\Delta_2 - 2} \left[\kappa(0) + \frac{16\Delta_1}{c} \Delta_R \right]. \tag{2.109}$$

We conclude then that each representation R_a is characterised by its dimension Δ_R and $(N-3)/2$ eigenvalues h_k. These values are fixed by studying the degenerate representations of the disorder sector. We shall show an example below.

The first descendants for a given primary operator R_a are found at level $1/2$. For a given value of the index a there are $(N-1)/2$ states:

$$\left(\chi^{(k)}_a\right)_{-\frac{1}{2}} = A^k_{-\frac{1}{2}} \mathsf{U}^{-2k} R_a = \mathsf{U}^{-2k} A^k_{-\frac{1}{2}} R_a, \qquad k = 1, 2, \ldots, \frac{N-1}{2}. \tag{2.110}$$

In the case of $N = 5$, there are two states, $\left(\chi_a^{(1)}\right)_{-\frac{1}{2}}$ and $\left(\chi_a^{(2)}\right)_{-\frac{1}{2}}$, in Eq. (2.110). In Ref. [3], these were both required to be primary, and this choice was shown to be consistent with the properties of the fundamental disorder operator in a Coulomb gas construction based on the algebra B_2. The most natural generalisation for $N > 5$ is to impose that all of the states (2.110) be primary. The corresponding solutions give the dimensions of the fundamental disorder operators, with the $(N-1)/2$ required degeneracies all situated at the first descendant level 1/2.

We therefore impose that

$$A^{k'}_{+\frac{1}{2}}(\chi_a^{(k)})_{-\frac{1}{2}} = 0 \tag{2.111}$$

for each $k' = 0, 1, \ldots, \frac{N-1}{2}$, and for each $k = 0, 1, \ldots, \frac{N-1}{2}$. Using the commutation relations (2.105)–(2.106), the degeneracy condition (2.111) results in a system of $(N-1)/2$ independent equations which allow to determine the $(N-1)/2$ unknown variables, i.e., the $(N-3)/2$ independent eigenvalues h_k and the conformal dimension Δ_R of the disorder operator R_a. (We omit here all the algebraic manipulations since this computation is strictly analogous to the one presented in detail in Ref. [3].)

Among the solutions for the conformal dimensions Δ_R admitted by this system of $(N-1)/2$ equations, only two are physical solutions, in the sense that they are consistent with the Kac formula which will be given in the next Section. The two physical solutions for Δ_R read:

$$\Delta_R^{(1)} = \frac{1}{16}\frac{(N-1)(p+N)}{p}, \qquad \Delta_R^{(2)} = \frac{1}{16}\frac{(N-1)(p+2-N)}{p+2}. \tag{2.112}$$

3 Kac formula and boundary terms

3.1 Lie algebra structure

It has already been observed in the Introduction that the central charge of the second parafermionic theory with Z_N symmetry agrees with that of a coset based on the group $SO(N)$. It is therefore natural to suppose some connection with the Lie algebra B_r, when $N = 2r + 1$ is odd. In the case $r = 2$ this connection was made explicit in Ref. [3], by assuming the existence of a Coulomb gas realisation of the theory, based on the algebra B_r. We shall here generalise this construction to arbitrary $r \geq 2$.

Each primary operator Φ of the theory is assumed to be represented by a vertex operator, which can in turn be associated with the weight lattice of B_r as follows:

$$\vec{\beta} \equiv \vec{\beta}_{(n_1,\cdots,n_r)(n'_1,\cdots,n'_r)} = \sum_{i=1}^{r}\left(\frac{1+n_i}{2}\alpha_+ + \frac{1+n'_i}{2}\alpha_-\right)\vec{\omega}_i. \tag{3.1}$$

Here, $\vec{\omega}_i$ are the fundamental weights of the algebra B_r, and $\alpha_\pm$ are the usual Coulomb gas parameters. Their form

$$\alpha_+ = \sqrt{\frac{p+2}{p}}, \qquad \alpha_- = -\sqrt{\frac{p}{p+2}} \tag{3.2}$$

are immediately suggested by Eq. (1.2) for the central charge.

We shall use the notation $\Phi_{(n_1,\ldots,n_r)(n'_1,\ldots,n'_r)}$ to represent the operator associated with $\vec{\beta}_{(n_1,\ldots,n_r)(n'_1,\ldots,n'_r)}$. Note that the integers $(n_1,\ldots,n_r)(n'_1,\ldots,n'_r)$ are essentially the Dynkin labels of the weight $\vec{\beta}$. As usually in the Coulomb gas construction the labels are doubled, with $(n_1,\ldots,n_r)$ representing the α_+ side of the theory, and $(n'_1,\ldots,n'_r)$ the α_- side. The nature of a primary operator $\Phi_{(n_1,\ldots,n_r)(n'_1,\ldots,n'_r)}$, i.e., its transformation properties under Z_N, remains unchanged if the α_+ and α_- sides are interchanged. Actually, such properties only depend on the differences $|n_i - n'_i|$. In most of what follows we shall therefore choose to trivialise the α_+ side, by setting $(n_1,\ldots,n_r) = (1,\ldots,1)$.

The Coulomb gas formula for the conformal dimension of the primary operator $\Phi_{(n_1,\ldots,n_r)(n'_1,\ldots,n'_r)}$ reads

$$\Delta_{(n_1,\cdots,n_r)(n'_1,\cdots,n'_r)} \equiv \Delta^{(0)}_{(n_1,\cdots,n_r)(n'_1,\cdots,n'_r)} + B = \vec{\beta}\cdot\left(\vec{\beta} - 2\vec{\alpha}_0\right) + B, \tag{3.3}$$

where the background charge $\vec{\alpha}_0$ is given by

$$\vec{\alpha}_0 = \left(\frac{\alpha_+ + \alpha_-}{2}\right)\sum_{i=1}^{r}\vec{\omega}_r. \tag{3.4}$$

In Eq. (3.3) we have written the scaling dimension as a sum of two terms, $\Delta^{(0)}_{(n_1,\ldots,n_r)(n'_1,\ldots,n'_r)}$ and B. The *Coulomb term* $\Delta^{(0)}_{(n_1,\ldots,n_r)(n'_1,\ldots,n'_r)}$ depends on the parameters α_+^2 and α_-^2, while the *boundary term* B is a constant that characterises the sector of the representation space to which the operator under consideration belongs.

Note that the identity operator is $I = \Phi_{(1,\ldots,1)(1,\ldots,1)}$. Since $\Delta_I = 0$ and I is a singlet, we must necessarily have $B = 0$ for any singlet operator. However, we still need to find the correct values of B for the doublet q and for the disorder sectors. These values, and their assignment to the different positions of the weight lattice, are arguably the main interest of the theory that we are constructing.

We shall refer to Eq. (3.3) as the Kac formula of the theory. The table of operators $\Phi_{(n_1,\ldots,n_r)(n'_1,\ldots,n'_r)}$, which is associated with the lattice made by the vectors $\vec{\beta}_{(n_1,\cdots,n_r)(n'_1,\cdots,n'_r)}$, shall be called the Kac table of the theory.

To use Eq. (3.3) to calculate actual values of the dimensions, we need the scalar products $\vec{\omega}_i\cdot\vec{\omega}_j$ for the Lie algebra B_r, which are encoded in the quadratic form matrix:

$$\vec{\omega}_i\cdot\vec{\omega}_j = i \text{ for } i \le j < r; \qquad \vec{\omega}_i\cdot\vec{\omega}_r = \frac{i}{2} \text{ for } i < r; \qquad \vec{\omega}_r\cdot\vec{\omega}_r = \frac{r}{4}. \tag{3.5}$$

In what follows, a major role is played by the action of Weyl reflections on the weights. We recall that the Weyl group W is generated by r *simple reflections* $s_{\vec{e}_i}$, with $i = 1, 2, \ldots, r$, acting as follows on the vertex operators (weights):

$$s_{\vec{e}_i}\vec{\beta}_{(1,\ldots,1)(n'_1,\ldots,n'_r)} = \vec{\beta}_{(1,\ldots,1)(n'_1,\ldots,n'_r)} + n'_i\alpha_-\vec{e}_i. \tag{3.6}$$

Here, $\vec{e}_i$ are the Lie algebra's simple roots, whose coordinates in the basis of fundamental weights can be read off from the rows of the Cartan matrix. The total number of Weyl reflections is $|W| = 2^r r!$. We also recall that the Weyl group possesses a unique longest element, which can be written as a word of length r^2 in terms of the generators $s_{\vec{e}_i}$. This longest element simply changes the sign of all the labels $(n'_1,\ldots,n'_r)$.

In the context of the conformal field theory, the physical significance of the Weyl group is linked to the fact that the simple roots represent the screenings operators of the Coulomb gas realisation. As usual, we assume that the screenings commute with the parafermionic algebra.

Given a generic vertex operator, there exists exactly one Weyl reflection that maps it into the fundamental Weyl chamber, the physical domain of the Kac table. Like in Felder's resolution for minimal models [6], the simple Weyl reflections indicate singular states (degeneracies) in the modules of physical operators. More precisely, a simple reflection can be associated with a mapping (realised by integrated screenings) that takes a vertex operator outside the physical domain of the Kac table into the module of a physical vertex operator. The difference between the dimensions of these two operators, computed from Eq. (3.3) with the appropriate boundary terms, gives the level of degeneracy. It is in the sense of this mapping that the operator outside the physical domain is associated with a non-physical (singular) state in the module of the physical operator; for this reason it could be called a *ghost operator*. The nature (singlet, doublet q, disorder) of the corresponding singular state must coincide with the labeling of the ghost operator. In a similar fashion, non-simple reflections can be used to infer the fine structure of the degenerate modules (ghosts within ghosts, etc.). For instance, this aspect has been studied in a certain detail, in the case of the representations of the WA_2 algebra, in the paper [7], and references therein.

3.2 Disorder sector

We shall first consider the problem of the boundary term for the disorder sector, and of the position of the fundamental disorder operators.

The boundary term B_R for this sector is readily accessible, since we have computed explicitly the conformal dimensions of the fundamental disorder operators (2.112) by means of degeneracy conditions. Indeed, once we have identified the position on the weight lattice of these operators by the coefficients of α_+^2 and α_-^2, we may read off the boundary terms from the $\alpha_\pm^2$ independent terms in the formulae for the dimensions.

Having done some algebric manipulation, it is easy to verify—by using the scalar products (3.5) and Eq. (3.1) in the Kac formula (3.3)—that the dimensions (2.112) of the fundamental disorder operators correspond, respectively, to

$$\Delta_{(1,\ldots,1,2)(1,\ldots,1)} \text{ and } \Delta_{(1,\ldots,1)(1,\ldots,1,2)} \tag{3.7}$$

with the boundary term

$$B_R = \frac{N-1}{32}. \tag{3.8}$$

3.3 Doublet sectors, $r = 3$

Identifying the boundary terms B_{D^q} of the doublet q sector is more complicated. Indeed, we have not been able to explicitly compute the dimensions of the fundamental doublet operators for general r. However, in Section 2 we have obtained such explicit results for the case of $r = 3$ ($N = 7$), at least for the doublet 1 and doublet 3 sectors. Before

turning to the general case, let us see how the $r = 3$ case can be completed by exploiting properties of the Coulomb gas representation.

Our explicit results for the dimensions of the fundamental doublet 1 operators, see Eqs. (2.65)–(2.66), correspond respectively to:

$$\Delta_{(1,2,1)(1,1,1)} \text{ and } \Delta_{(1,1,1)(1,2,1)} \tag{3.9}$$

with the boundary term

$$B_{D^1} = \frac{3}{14}. \tag{3.10}$$

Simarly, the dimensions of the fundamental doublet 3 operators, see Eqs. (2.100)–(2.101), are given by:

$$\Delta_{(2,1,1)(1,1,1)} \text{ and } \Delta_{(1,1,1)(2,1,1)}, \tag{3.11}$$

with the boundary term

$$B_{D^3} = \frac{5}{28}. \tag{3.12}$$

To determine the remaining boundary term B_{D^2}, for whose fundamental operator we have no explicit degeneracy computation, we shall use an argument based on Weyl reflections, as discussed in Section 3.1. To this end we consider the fundamental doublet 1 operator $\Phi_{(1,1,1)(1,2,1)}$, whose position was determined in Eq. (3.9) above. In this case the three principal reflections are:

$$\begin{aligned}
s_{\vec{e}_1}\vec{\beta}_{(1,1,1)(-1,3,1)} = \vec{\beta}_{(1,1,1)(1,2,1)} &= \vec{\beta}_{(1,1,1)(-1,3,1)} + \alpha_-\vec{e}_1, \\
s_{\vec{e}_2}\vec{\beta}_{(1,1,1)(3,-2,5)} = \vec{\beta}_{(1,1,1)(1,2,1)} &= \vec{\beta}_{(1,1,1)(3,-2,5)} + 2\alpha_-\vec{e}_2, \\
s_{\vec{e}_3}\vec{\beta}_{(1,1,1)(1,3,-1)} = \vec{\beta}_{(1,1,1)(1,2,1)} &= \vec{\beta}_{(1,1,1)(1,3,-1)} + \alpha_-\vec{e}_3.
\end{aligned} \tag{3.13}$$

(In this expression, and in the following, the reflections are understood to act on the α_- side of the indices only.) These reflections give the following differences of the Coulomb part of the dimensions:

$$\Delta^{(0)}_{(1,1,1)(-1,3,1)} - \Delta^{(0)}_{(1,1,1)(1,2,1)} = \frac{1}{2}, \tag{3.14}$$

$$\Delta^{(0)}_{(1,1,1)(3,-2,5)} - \Delta^{(0)}_{(1,1,1)(1,2,1)} = 1, \tag{3.15}$$

$$\Delta^{(0)}_{(1,1,1)(1,3,-1)} - \Delta^{(0)}_{(1,1,1)(1,2,1)} = \frac{1}{4}. \tag{3.16}$$

In the above equations we have to add the correct boundary terms in order to obtain the level of degeneracy on the right-hand side. This amounts to determining to which sector the ghost operator belongs. The important point is that the labeling of the ghost operator has to coincide with the nature of the corresponding state in the module of the doublet $\Phi_{(1,1,1)(1,2,1)}$. In Section 2 we have seen that the module of the operator $\Phi_{(1,1,1)(1,2,1)}$ contains three singular states: A doublet 2 state at level 1/7, a singlet state at level 2/7, and a doublet 1 state at level 1. It is then easy to verify, using Eqs. (3.10), (3.12), and (3.14)–(3.16), that there is only one correct way of labeling the ghost operators:

$$\begin{aligned}
\Phi_{(1,1,1)(-1,3,1)} &\sim S, \\
\Phi_{(1,1,1)(3,-2,5)} &\sim D^1, \\
\Phi_{(1,1,1)(1,3,-1)} &\sim D^2.
\end{aligned}$$

The boundary term B_{D^2} is then fixed by the condition

$$\Delta^{(0)}_{(1,1,1)(1,3,-1)} + B_{D^2} - \Delta^{(0)}_{(1,1,1)(1,2,1)} - B_{D^1} = \frac{1}{7}, \tag{3.17}$$

which, using Eqs. (3.16) and (3.10), yields:

$$B_{D^2} = \frac{3}{28}. \tag{3.18}$$

Using the same kind of reasoning, we can also determine the nature of the operator $\Phi_{(1,1,1)(1,1,3)}$. The principal reflections are:

$$\begin{aligned} s_{\vec{e}_1}\vec{\beta}_{(1,1,1)(-1,2,3)} = \vec{\beta}_{(1,1,1)(1,1,3)} &= \vec{\beta}_{(1,1,1)(-1,2,3)} + \alpha_-\vec{e}_1 \\ s_{\vec{e}_2}\vec{\beta}_{(1,1,1)(2,-1,5)} = \vec{\beta}_{(1,1,1)(1,1,3)} &= \vec{\beta}_{(1,1,1)(2,-1,5)} + \alpha_-\vec{e}_2 \\ s_{\vec{e}_3}\vec{\beta}_{(1,1,1)(1,4,-3)} = \vec{\beta}_{(1,1,1)(1,1,3)} &= \vec{\beta}_{(1,1,1)(1,4,-3)} + 3\alpha_-\vec{e}_3. \end{aligned} \tag{3.19}$$

Examining the differences of dimensions as before, one easily checks that the only acceptable configuration is:

$$\begin{aligned} \Phi_{(1,1,1)(1,1,3)} &\sim D^2, \\ \Phi_{(1,1,1)(-1,2,3)} \sim \Phi_{(1,1,1)(2,-1,5)} &\sim D^3, \\ \Phi_{(1,1,1)(1,4,-3)} &\sim D^1. \end{aligned} \tag{3.20}$$

We infer from the above analysis that the module of the doublet 2 operator $\Phi_{(1,1,1)(1,1,3)}$ has two singular doublet 3 states at level 4/7 and one singular doublet 1 state at level 6/7. Moverover, the operator $\Phi_{(1,1,1)(1,1,3)}$ must be the *fundamental* doublet 2 operator, since these singular states are situated at the lowest possible levels.

In summary, we have shown how to determine the boundary term B_{D^2} by exploiting the reflections on the weight lattice together with some partial explicit results for the dimension of fundamental operators.

3.4 Unitary theories. Finite Kac table

All the considerations done so far on the Kac table are valid for general values of α_+ and $\alpha_- = -1/\alpha_+$. As usual, the Kac table becomes finite when $\alpha_+^2 = (p+2)/p$ takes rational values. The unitary theories correspond to p taking integer values, according to the coset realisation (1.1)–(1.3).

The finiteness of the Kac table can be shown by considering the reflections with respect to the hyperplane defined by

$$n'_1 + 2\sum_{i=2}^{r-1} n'_i + n'_r = p + 2 \tag{3.21}$$

on the α_- side, and by

$$n_1 + 2\sum_{i=2}^{r-1} n_i + n_r = p \tag{3.22}$$

on the α_+ side. Geometrically, this hyperplane is perpendicular to the following combination of screenings:

$$\vec{e}_1 + 2\sum_{i=2}^{r} \vec{e}_i. \tag{3.23}$$

Note that the differences between the dimensions of the operators linked by such reflections are in agreement with the levels of the corresponding modules. The operators above the hyperplane (3.21) are expected to be ghost operators and decouple from the theory. For p integer, the physical part of the Kac table is therefore delimited by:

$$\begin{aligned} 2(r-1) &\leq n_1' + 2\sum_{i=2}^{r-1} n_i' + n_r' \leq p+1, \\ 2(r-1) &\leq n_1 + 2\sum_{i=2}^{r-1} n_i + n_r \leq p-1. \end{aligned} \tag{3.24}$$

The finite Kac table defined by Eq. (3.24) contains an extra symmetry. It is easy to verify that the Coulomb part of the conformal dimension $\Delta^{(0)}_{(n_1,\ldots,n_r)(n_1',\ldots,n_r')}$ in Eq. (3.3), i.e., with the boundary term B being neglected, is invariant under the operation

$$\begin{aligned} n_1' &\to p+2-n_1'-2\sum_{i=2}^{r-1} n_i' - n_r', \qquad n_i' \to n_i' \quad i=2,3,\ldots,r \\ n_1 &\to p-n_1-2\sum_{i=2}^{r-1} n_i - n_r, \qquad n_i \to n_i \quad i=2,3,\ldots,r. \end{aligned} \tag{3.25}$$

Now, if the label of each operator (singlet, doublet q, or disorder) is invariant under (3.25), the above reflection will stay a symmetry when the boundary term is included. We shall see in Section 4 below that the correct labeling indeed has this property. Note that this amounts to saying that the correct labeling has periodicity 2 in the $-\vec{\omega_1}/2$ direction.

3.5 Values of the doublet boundary terms

3.5.1 Trivial theory: $c = 0$

Having defined the finite Kac tables, we can make an important observation. From the formula (2.6), with the parametrisation (1.3), the central charge is equal to zero for $p = N-2$. (Note also that the coset (1.1) trivialises.) The corresponding theory is therefore expected to be trivial, i.e., with all conformal weights equal to zero.

When $p = N-2$, it follows from Eq. (3.24) that all physical operators have trivial indices on the α_+ side: $(n_1,\ldots,n_r) = (1,\ldots,1)$. Moreover, using the symmetry (3.25) we see that the only allowed excitations on the α_- side are the following:

$$\begin{aligned} \Phi_0 &\equiv \Phi_{(1,\ldots,1)(1,1,\ldots,1,1,1,\ldots,1,1)}, \\ \Phi_k &\equiv \Phi_{(1,\ldots,1)(1,1,\ldots,1,2,1,\ldots,1,1)}, \qquad 0<k<r, \\ \Phi_r &\equiv \Phi_{(1,\ldots,1)(1,1,\ldots,1,1,1,\ldots,1,3)}, \\ \Phi_R &\equiv \Phi_{(1,\ldots,1)(1,1,\ldots,1,1,1,\ldots,1,2)}, \end{aligned} \tag{3.26}$$

where in the second line we have $n'_k = 2$. As already discussed, the operators Φ_0 and Φ_R are, respectively, the identity operator and a fundamental disorder operator R.

We assume that every physical operator in a $c = 0$ theory should have $\Delta = 0$. When applied to the operators (3.26), this allows us to compute the values of all the boundary terms, for each odd N, by using the Kac formula (3.3). One obtains:

$$\begin{aligned} B_R &= \frac{N-1}{32} \\ B_k &= \frac{k(N-2k)}{4N}, \qquad k = 0, 1, \ldots, \frac{N-1}{2}. \end{aligned} \tag{3.27}$$

In the above equation B_R and B_k denote, respectively, the boundary terms of the operators Φ_R and Φ_k, whose positions have been defined in Eq. (3.26). The value of B_R confirms Eq. (3.8)—the boundary term for a disorder operator—as it should, since Φ_R is a fundamental disorder operator, cf. Eq. (3.7). Also, Φ_0 is trivially a singlet and corresponds to the boundary term $B_0 \equiv B_S = 0$.

The task of finding the boundary terms B_{D^q} of the doublets D^q has therefore been reduced to the problem of associating each value B_k to the correct charge sector or, in other words, to determine the nature of the operators Φ_k with $k = 1, 2, \ldots, r$. We shall assume that among the operators listed in Eq. (3.26), each sector of the theory is represented exactly once. Since the singlet and disorder sectors have already been accounted for, it follows that the operators Φ_k with $k = 1, 2, \ldots, r$ are necessarily a permutation of the (fundamental) doublet operators. Below, we shall consider the problem of determining *which* permutation, for general r.

3.5.2 Position of the fundamental operators

In Section 3.3 we have succeeded in determining the positions on the weight lattice of the fundamental operators in each sector, in the special case of $r = 3$. The general structure of these results is most easily seen if one introduces a new notation for the doublet sectors, by doubling the Z_N charges:

$$Q = 2q \bmod N. \tag{3.28}$$

We shall distinguish between the two notations by placing an asterisk after the indices which are to be read in the Q notation. Note that since N is odd, the Q labels are simply a permutation of the q labels. The results of Section 3.3 can now be written in the form:

$$\begin{aligned} \Phi_{(1,1,1)(2,1,1)} &\sim D^3 \equiv D^{1^*}, \\ \Phi_{(1,1,1)(1,2,1)} &\sim D^1 \equiv D^{2^*}, \\ \Phi_{(1,1,1)(1,1,3)} &\sim D^2 \equiv D^{3^*}. \end{aligned} \tag{3.29}$$

Comparing the positioning (3.29) with Eq. (3.27), we obtain the results (3.10), (3.12), and (3.18).

Moreover, we have explicitly computed the dimensions of the fundamental doublet D^{1^*} for each r in Section 2.1.4. By using the Kac formula (3.3), it is easy to verify that the dimensions (2.100)–(2.101) correspond, respectively, to:

$$\Delta_{(2,1,\ldots,1)(1,\ldots,1)} \text{ and } \Delta_{(1,\ldots,1)(2,1,\ldots,1)} \tag{3.30}$$

with the boundary term

$$B_{D^{1^*}} = \frac{N-2}{4N}. \tag{3.31}$$

Thus, $\Phi_1 \sim D^{1^*}$, and the value of the bondary term B_1 confirms (3.31).

It is now not difficult to guess the labeling of the operators Φ_k for general r:

$$\begin{aligned}
\Phi_0 \equiv \Phi_{(1,\ldots,1)(1,1,\ldots,1,1,1,\ldots,1,1)} &\sim S = I, \\
\Phi_k \equiv \Phi_{(1,\ldots,1)(1,1,\ldots,1,2,1,\ldots,1,1)} &\sim D^{k^*}, \qquad 0 < k < r, \\
\Phi_r \equiv \Phi_{(1,\ldots,1)(1,1,\ldots,1,1,1,\ldots,1,3)} &\sim D^{r^*}.
\end{aligned} \tag{3.32}$$

Below we shall give a (partial) demonstration of (3.32).

Consider a general physical operator $\Phi_{(1,\ldots,1)(n'_1,\ldots,n'_r)}$, for which the r simple reflections $s_{\vec{e}_i}$ give:

$$\begin{aligned}
\Delta^{(0)}_{s_{\vec{e}_i}\vec{\beta}} - \Delta^{(0)}_{\vec{\beta}} &= \frac{n'_i}{2}, \qquad i = 1, 2, \ldots, r-1, \\
\Delta^{(0)}_{s_{\vec{e}_r}\vec{\beta}} - \Delta^{(0)}_{\vec{\beta}} &= \frac{n'_r}{4}
\end{aligned}$$

with $\vec{\beta} \equiv \vec{\beta}_{(1,\ldots,1)(n'_1,\ldots,n'_r)}$. As discussed in Section 3.1, the ghost operator $\Phi_{s_{\vec{e}_i}\vec{\beta}}$ (charge sector q_2) indicates a singular state in the module of the physical operator $\Phi_{\vec{\beta}}$ (charge sector q_1). Their respective boundary terms $B_{D^{q_1}}$ and $B_{D^{q_2}}$ must then satisfy

$$B_{D^{q_2}} - B_{D^{q_1}} + \frac{\tilde{n}}{4} = \delta^{q_1}_{q_2} + k_{1,2}. \tag{3.33}$$

Here, the gap $\delta^{q_1}_{q_2}$ is defined by Eq. (2.11), $k_{1,2}$ is a non-negative integer, and $\tilde{n}$ is a positive integer (which is even if $i < r$ above). The right-hand side of Eq. (3.33) is the degeneracy level of $\Phi_{\vec{\beta}}$ with respect to the singular state $\Phi_{s_{\vec{e}_i}\vec{\beta}}$.

The gaps in a generic singlet module are given by Eq. (2.10). In the Q-notation this can be rewritten as

$$\delta^0_Q = \frac{Q(N-Q)}{2N} \bmod 1. \tag{3.34}$$

(Note that this is not obtained naively, by substituting $Q = 2q$ in Eq. (3.28).) Thus, choosing in particular the physical operator to be a singlet, $q_1 = 0$ and $B_0 = 0$, the condition (3.33) fixes the general form of the boundary term B_{D^Q}:

$$B_{D^Q} = \frac{Q(N-Q)}{2N} - \frac{k_Q}{4}, \tag{3.35}$$

where k_Q is an integer which depends on Q. Note that the set of values k_Q can be fixed, for example, from knowledge of the levels of degeneracy of the fundamental operators.

If N is a prime number it is easy to prove, by comparison of Eqs. (3.35) and (3.27), that $k_Q = Q$ and so

$$B_{D^{q*}} = B_{q*}, \qquad q* = 0, 1, \ldots, r. \tag{3.36}$$

This then proves Eq. (3.32).

If N is not prime, some ambiguity remains. Let us consider the quantity

$$\delta^0_{Q'} - \delta^0_{Q''} = \left(\frac{Q' - Q''}{2} - \frac{(Q' + Q'')(Q' - Q'')}{2N} \right) \text{ mod } 1. \tag{3.37}$$

Note that $|Q' \pm Q''| < N$, and so if N is prime the right-hand side can only vanish if $Q' = Q''$. On the other hand, if $N = st$ for some integers $s, t > 1$, the right-hand side can vanish for $Q' \neq Q''$ provided that $|Q' - Q''|$ mod $s = 0$ and $|Q' + Q''|$ mod $t = 0$ (or vice versa). In this case we would have $\delta^0_{Q'} = \delta^0_{Q''}$, and the argument leading to (3.36)—which was based on the level structure of the modules—would break down; we would not be able to distinguish between the charge sectors Q' and Q''. In particular, in addition to (3.36) the solution

$$\begin{aligned} B_{D^Q} &= B_Q, \qquad Q = 0, 1, \ldots, r; \quad Q \neq Q', Q'', \\ B_{D^{Q'}} &= B_{Q''}, \\ B_{D^{Q''}} &= B_{Q'} \end{aligned} \tag{3.38}$$

would a priori be acceptable.[3]

Summarising, we have explicitly proved for $r \leq 6$ that the positioning (3.32) is the only correct one. For $r > 6$ we have no strict arguments to exclude some additional possibilities, such as (3.38). However, we find it natural to assume that the theories that we are constructing should have a similar structure for all r. We shall therefore accept the solution (3.36) for all $r \geq 1$.

Summarising, we write here the final result for the boundary terms:

$$\begin{aligned} B_R &= \frac{N-1}{32}, \\ B_S &= 0, \\ B_{D^Q} &= \frac{Q(N-2Q)}{4N}, \qquad Q = 1, 2, \cdots, \frac{N-1}{2}. \end{aligned} \tag{3.39}$$

3.6 Alternative derivation of doublet boundary terms

We now briefly show that it is possible to derive the boundary terms of the doublet sectors without making reference to the $c = 0$ theories. In this second argument, which is based on the technique of Weyl reflections, we make two important assumptions:

- The positions of the fundamental operators are given by (3.32).
- The levels of degeneracy δ of a fundamental operator belong to the interval $0 < \delta \leq 1$.

[3]For $N = s^2$, one would have $Q' = 0$ and $Q'' = s$. In this case there is no ambiguity as the boundary term of the singlet sector is trivially $B_0 = 0$. The positioning (3.32) of the fundamental operators is then the only one that is acceptable.

We start by applying the r simple Weyl reflections $s_{\vec{e}_i}$ to the identity operator $I = \Phi_{(1,\ldots,1)(1,\ldots,1)}$. This yields the following differences of dimensions:

$$\Delta^{(0)}_{(1,\ldots,1)s_{\vec{e}_i}(1,\ldots,1)} - \Delta_{(1,\ldots,1)(1,\ldots,1)} = \frac{1}{2}, \qquad i = 1, 2, \ldots, r-1, \tag{3.40}$$

$$\Delta^{(0)}_{(1,\ldots,1)s_{\vec{e}_r}(1,\ldots,1)} - \Delta_{(1,\ldots,1)(1,\ldots,1)} = \frac{1}{4}, \tag{3.41}$$

with $\Delta_{(1,\ldots,1)(1,\ldots,1)} = 0$. The only labeling consistent with Eqs. (3.40)–(3.41) is

$$\begin{aligned} \Phi_{(1,\ldots,1)s_{\vec{e}_i}(1,\ldots,1)} &\sim D^1, \qquad i = 1, 2, \ldots, r-1, \\ \Phi_{(1,\ldots,1)s_{\vec{e}_r}(1,\ldots,1)} &\sim D^r, \end{aligned} \tag{3.42}$$

and in addition we deduce the boundary term $B_{D^1} = \frac{N-4}{2N}$. Note that this implies that the module of the identity has a singular doublet r state at level $(N-1)/2N$ and $r-1$ singular doublet 1 states at level $(N-2)/N$. In the special case of $N = 7$, our explicit degeneracy computations have indicated the mechanism by which the doublet 1 states may proliferate, as required by this scenario. In particular, we expect that for $N > 7$ the relation analogous to (2.32) no longer holds true.

With the boundary term B_{D^1} in hand, we next consider the simple Weyl reflections of the fundamental doublet 1 operator $\Phi_{(1,\ldots,1)(1,2,1,\ldots,1)}$. In particular, the reflection $s_{\vec{e}_r}$ gives a doublet r singular state which, according to Eq. (2.11), should be found on level $\delta^1_{r-1} = (N-5)/2N$. This leads to the identity

$$\Delta^{(0)}_{(1,\ldots,1)(1,2,1,\ldots,1,2,-1)} + B_{D^{r-1}} - \Delta^{(0)}_{(1,\ldots,1)(1,2,1,\ldots,1)} - B_{D^1} = \frac{N-5}{2N}, \tag{3.43}$$

from which we determine the boundary term $B_{D^{r-1}} = \frac{3(N-6)}{4N}$.

Knowing now the corresponding boundary term, we can consider the reflections of the fundamental doublet $r-1$ operator, etc. Proceeding like this in a systematic way, one can determine all boundary terms (3.39).

4 General theory

Having fixed the positions of the fundamental operators and determined the values of all the boundary terms of the theory, the remaining problem is to fill the rest of the Kac table. In other words, we have to assign correctly the boundary terms (3.27) to the Kac formula (3.3) for every vertex of the weight lattice. This amounts to determining the sector label with respect to the group Z_N of each operator in the theory (the case of the fundamental operators was settled in Section 3.5).

After having addressed this question, we dedicate the rest of the present Section to a study of the characteristic equations of various three-point functions. These will give us independent verifications of a number of aspects of our theory, and also yield some new results.

4.1 Filling the lattice

Although the Kac table of the Z_N theory, with $N = 2r + 1$, is based on a $2r$-dimensional lattice, it suffices to fill one "basic layer", for instance on the α_- side, whose vertices have the positions $\vec{\beta}_{(1,\ldots,1)(n'_1,\ldots,n'_r)}$. The filling of the other layers is obtained by shifting the filling of the basic layer. Namely, the nature of the operator $\Phi_{(n_1,\ldots,n_r)(n'_1,\ldots,n'_r)}$ depends only on the differences $|n_i - n'_i|$.

Until now we have extensively used the fact that the degeneracies in the modules could be read off from the Weyl reflections of the weight lattice. In this section we shall exploit another useful rule of the Coulomb gas representation.

We consider the multiplication (fusion) of two operators in the basic layer,

$$\Phi_{(1,\ldots,1)(n'_1,\ldots,n'_r)} \cdot \Phi_{(1,\ldots,1)(m'_1,\ldots,m'_r)}. \tag{4.1}$$

According to the Coulomb gas rules this produces, in the principal channel, an operator $\Phi_{(1,\cdots,1)(l'_1,\cdots,l'_r)}$ with

$$\begin{aligned} \vec{\beta}_{(1,\ldots,1)(l'_1,\ldots,l'_r)} &= \vec{\beta}_{(1,\ldots,1)(n'_1,\ldots,n'_r)} + \vec{\beta}_{(1,\ldots,1)(m'_1,\ldots,m'_r)} \\ l'_i &= n'_i + m'_i - 1, \qquad i = 1, 2, \ldots, r. \end{aligned} \tag{4.2}$$

The non-principal channels follow the principal one by shifts realised by linear combinations (over the integers) of the simple roots $\vec{e}_i$ (with $i = 1, \ldots, r$).

The above rule is extremely useful in fixing the distribution of operators on the lattice. It should however be used with some care, since the amplitude of the principal channel may vanish (see below).

4.1.1 Disorder operators

Postponing for the moment the potential difficulty of vanishing amplitudes, we begin by examining for example the fusion of the fundamental disorder operator $\Phi_{(1,\ldots,1)(1,\ldots,1,2)}$ and the fundamental doublet r operator $\Phi_{(1,\ldots,1)(2,1,\ldots,1)}$. The result must necessarily be another disorder operator, since in the dihedral group the product of a reflection and a rotation yields another reflection. Thus, $\Phi_{(1,\ldots,1)(2,1,\ldots,1,2)}$ is a disorder operator. Proceeding in this way it is easy to verify that the operator $\Phi_{(1,\ldots,1)(n'_1,\ldots,n'_r)}$ with $|n'_r - 1|$ odd is a disorder operator R. More generally:

$$\Phi_{(n_1,\ldots,n_r)(n'_1,\ldots,n'_r)} \sim R, \text{ when } |n_r - n'_r| \text{ is odd.} \tag{4.3}$$

We have thus filled half of the lattice with disorder operators.

An independent verification of this result can be obtained by the method of reflections. First, it is not difficult to see that all Weyl reflections, applied to an arbitrary operator $\Phi_{(1,\ldots,1)(n'_1,\ldots,n'_r)}$ in the basic layer, conserve the parity of n'_r. This is at least consistent with the above separation between disorder and singlet/doublet operators.

To perform a more detailed check, we consider the r simple Weyl reflections of the fundamental disorder operators $\Phi_{(1,\ldots,1)(1,\ldots,1,2)}$. These map the ghost operator $\Phi_{(1,\ldots,1)(m_1'^{(i)},\ldots,m_r'^{(i)})}$,

with $i = 1, 2, \ldots, r$, into the module of the disorder operator $\Phi_{(1,\ldots,1)(1,\ldots,1,2)}$. The indices of the ghost operator read

$$\begin{aligned} m_j'^{(i)} &= 1, \text{ for } j < i-1 \text{ and } i+1 < j < r; \\ m_{i-1}'^{(i)} &= m_{i+1}^{(i)} = 2; \qquad m_i'^{(i)} = -1; \qquad m_r'^{(i)} = 2 \end{aligned}$$

for $i < r$, while for $i = r$ they read

$$m_j'^{(r)} = 1, \text{ for } j < r-1; \qquad m_{r-1}'^{(r)} = 3; \qquad m_r^{(r)} = -2. \tag{4.4}$$

All the r simple reflections give the result $1/2$ for the difference of dimensions

$$\Delta^{(0)}_{(1,\ldots,1)(m_1'^{(i)},\ldots,m_r'^{(i)})} - \Delta^{(0)}_{(1,\ldots,1)(1,\ldots,1,2)} = \frac{1}{2}, \text{ for } i = 1, 2, \ldots, r. \tag{4.5}$$

This on one hand confirms the fact that the fundamental disorder operator $\Phi_{(1,\ldots,1)(1,\ldots,1,2)}$ has r singular states at level $1/2$, as was supposed in Section 3. On the other hand, it classifies all the concerned ghost operators $\Phi_{(1,\ldots,1)(m_1'^{(i)},\ldots,m_r'^{(i)})}$ as disorder operators.

More generally, it is not difficult to see that the difference between the Coulomb term of the dimension of any operator $\Phi_{(1,\ldots,1)(n_1',\ldots,n_r')}$ with even n_r' and the ghost operators obtained by acting on it with (not necessarily simple) reflections equals a half-integer. This is in strong support of the result (4.3).

4.1.2 Singlet operators. Periodicity of the filling

It remains to assign the doublet D^q operators and the singlet operators S to the lattice sites $\Phi_{(1,\ldots,1)(n_1',\ldots,n_r')}$ with n_r' odd.

The singlet operator S form a subalgebra, as the fusion between two singlets $S \cdot S$ produces another singlet. Furthermore, the fusion $D^q \cdot S$ gives another doublet D^q. The importance of the above consideration is that once the positions of the first non-trivial singlets (different from the identity) have been found, along each of the principal lattice directions, these will set the periodicity of the labeling of all lattice sites. To see this, one first considers the fusion of these non-trivial singlets among themselves, the result being a periodic distribution of singlet operators throughout the lattice. The periodicity in each lattice direction is given by the distance between the first non-trivial singlet in that direction and the identity operator. The remaining (non-singlet) labels can now be translated throughout the lattice by fusing the corresponding operators with all available singlets.

As noticed above, it might happen that the amplitude of the principal channel vanishes. But in view of the role of the singlets in setting the periodicity of the filling, it seems natural to suppose that this does not happen in fusions involving singlets.

Notice that the symmetry (3.25) of the finite Kac tables implies that the periodicity in the n_1' direction must be 2. The remaining part of the task is to find the periodicities in the other lattice directions, and to specify the labeling of the sites in a r-dimensional hypercube of the lattice that corresponds to the given periodicity.

4.1.3 Doublet operators

To determine this periodicity, and to position the required non-fundamental doublet operators, we have first tried using the method of fusions. Consider for instance fusing the fundamental doublet D^{1*} with itself

$$\Phi_{(1,\ldots,1)(2,1,\ldots,1)} \cdot \Phi_{(1,\ldots,1)(2,1,\ldots,1)} \sim \Phi_{(1,\ldots,1)(3,1,\ldots,1)}. \tag{4.6}$$

The result can either be a doublet D^{2*} or a singlet S, since the addition of Q-charges $(\pm 1) + (\pm 1)$ is ambiguous. Further fusions can be used to establish that there is only a finite number of possible periodicities along each lattice direction (in particular, the periodicity cannot exceed r), but no definite answer emerges. Moreover, in the Z_{2r+1} theory with $r = 3$, which we have studied in great detail, the application of the fusion method leads to a number of inconsistencies with the method of Weyl reflections. We conclude that the fusion method should be abandoned, since the fusion of two particular doublets can (and does; see below) lead to vanishing amplitudes in a few cases.

We therefore turn to the method of reflections. The idea is that the number of elements in the complete Weyl group, $|W| = 2^r r!$, is in general very large. Thus, classifying $|W| - 1$ ghost operators for each fundamental operator—whose positions are supposed to be given by Eq. (3.32)—will fix the labels of a large number of lattice sites close to the origin, hopefully enabling us to discern the correct periodicity.

We have pursued this idea for $r = 3$ and $r = 4$, using a computer. As a first step, the Weyl group was generated in a recursive fashion, as follows. Recall that the elements of the Weyl group can be represented as words made out of an r-letter alphabet, $s_{\vec{e}_1}, s_{\vec{e}_2}, \ldots, s_{\vec{e}_r}$, each word having it minimal possible length. In the case of the algebra B_r, there is a unique longest element represented by a word of length $\ell_0 = r^2$; moreover, all lengths ℓ satisfying $1 \le \ell \le \ell_0$ correspond to at least one element. Therefore, all elements of length ℓ can be generated by prefixing the words representing elements of length $\ell - 1$ by each of the r letters in the alphabet. For each word generated, one tests whether it represents a new element of the Weyl group by considering its action on a fixed lattice site in the fundamental chamber, and comparing to the action of elements represented by shorter words. The recursion starts from the simple Weyl reflections, and terminates once an element of length ℓ_0 has been generated.

For $r = 3$, the resulting classification of $(r+1)|W| = 192$ singlet and doublet operators made it evident that the correct filling has periodicity 2 in the n'_i direction, for $i = 1, 2, \ldots, r-1$, and periodicity 4 in the n'_r direction. In particular, the first non-trivial singlets (different from the identity) along each lattice direction are situated at

$$\begin{aligned} \Phi_{(1,\ldots,1)(1,1,\ldots,1,3,1,\ldots,1,1)} &\sim S, \qquad 0 < k < r, \\ \Phi_{(1,\ldots,1)(1,1,\ldots,1,1,1,\ldots,1,5)} &\sim S, \end{aligned} \tag{4.7}$$

where in the first line $n'_k = 3$. Fusions by these singlets subsequently permitted us to label all operators in the hypercubic unit cell

$$\begin{aligned} &1 \le n'_k \le 2, \qquad 0 < k < r, \\ &1 \le n'_r \le 4, \end{aligned} \tag{4.8}$$

whose translation throughout the lattice yields the complete filling. Note that the sites in the cell (4.8) with $n'_r = 2, 4$ have already been classified as disorder operators, by Eq. (4.3).

The results (4.7)–(4.8) on the periodicity was confirmed by the case $r = 4$, where $(r+1)|W| = 1920$ singlet and doublet operators on the four-dimensional weight lattice were classified.[4]

We now give the general result on the labeling of the unit cell (4.8). It is most easily stated in terms of the Q-notation for the doublets, cf. Eq. (3.28). Defining:

$$\begin{aligned} x_i &= |n_i - n'_i| \bmod 2, \qquad i = 1, 2, \ldots, r-1, \\ x_r &= \frac{|n_r - n'_r|}{2} \bmod 2, \end{aligned} \tag{4.9}$$

the doublet charge Q associated with the position $\vec{\beta}_{(n_1,\ldots,n_r)(n'_1,\ldots,n'_r)}$ is given by the recursive formula

$$Q(x_1, x_2, \ldots, x_{k-1}, 1, 0, \ldots, 0) = k - Q(x_1, x_2, \ldots, x_{k-1}, 0, 0, \ldots, 0), \tag{4.10}$$

with the initial condition $Q(0, \ldots, 0) = 0$. For the case of $r = 3$ this reads explicitly:

$$\begin{aligned} &\Phi_{(1,1,1)(1,1,1)} \sim S, \qquad \Phi_{(1,1,1)(1,2,1)} \sim D^{2^*}, \quad \Phi_{(1,1,1)(1,1,3)} \sim D^{3^*}, \quad \Phi_{(1,1,1)(1,2,3)} \sim D^{1^*}, \\ &\Phi_{(1,1,1)(2,1,1)} \sim D^{1^*}, \quad \Phi_{(1,1,1)(2,2,1)} \sim D^{1^*}, \quad \Phi_{(1,1,1)(2,1,3)} \sim D^{2^*}, \quad \Phi_{(1,1,1)(2,2,3)} \sim D^{2^*}. \end{aligned}$$

It is easy to prove from Eq. (4.10)—using an induction argument—that the number of doublet D^Q operators in the unit cell (4.8) equals $\binom{r}{Q}$, for $Q = 0, 1, \ldots, r$.

Note that the filling (4.10) implies that some of the doublet-doublet fusions must have a vanishing amplitude in the principal channel. For $r = 3$, an example is

$$\Phi_{(1,1,1)(1,2,1)} \cdot \Phi_{(1,1,1)(2,1,3)} \to \Phi_{(1,1,1)(2,2,3)}. \tag{4.11}$$

According to Eq. (4.10), both operators on the left-hand side are D^{2*}. Using the Coulomb gas fusion rules, the resulting operator should thus be either a S or a D^{3*}. However, the operator $\Phi_{(1,1,1)(2,2,3)}$ appearing on the right-hand side has been classified as D^{2*} in Eq. (4.10). Therefore the amplitude of the principal channel must vanish.

4.2 Characteristic equations

We now present some additional verifications of the theory we have constructed. To this end, we consider the characteristic equations for the conformal dimensions of the operators which participate in a given fusion (operator product expansion). These equations

[4]It is worth mentioning a slight problem that appears in the analysis. Namely, for $r = 4$ the reflection method cannot distinguish between S and D^2 operators. This is essentially because the Coulomb part of the dimensions (3.3) of two operators linked by a Weyl reflection always differ by $p/4$, where p is an integer. It is this difference, taken modulo 1, that serves to classify a given ghost operator. With $r+1$ distinct singlet and doublet sectors, this will necessarily lead to ambiguities for $r \geq 4$. (In the case $r = 4$ the ambiguity can be resolved by using the known positions of the fundamental operators.) These ambiguities are related to those discussed near Eq. (3.38).

can be derived from the study of the three-point correlation functions in a way analogous to that of Refs. [2].

The three-point correlation functions considered below have the same analytical properties as the ones studied in Ref. [3] for $N = 5$. Therefore, for the more general case $N = 2r+1$, the derivation of the corresponding characteristic equations is the same. The details of the calculations can be found in the Appendix C of Ref. [3].

We have considered the following correlation functions:

- $\left\langle R^{(2)}(z_2)\Phi^{q*}(z_3)R^{(1)}(z_1)\right\rangle$

  Here, $R^{(1)}(z_1)$ is a disorder operator whose module is supposed to be completely degenerate at level 1/2, i.e., $R^{(1)}(z_1) = \Phi_{(1,\ldots,1,2)(1,\ldots,1)}(z_1)$ or $\Phi_{(1,\ldots,1)(1,\ldots,1,2)}(z_1)$. The operator $R^{(2)}(z_2)$ is a generic disorder operator, and $\Phi^{q*}(z_3)$ with $q^* = 0, \pm r^*$ is a singlet or a doublet r^* operator (we remind that r^* is the Z_N charge in the notation (3.28)).

  The necessary condition for the above function to be non-zero is given by the following equation on the dimensions of the operators $R^{(1)}$, $R^{(2)}$ and Φ^{q*}:

$$\Delta_{R^{(2)}} - \Delta_{\Phi^{q*}} + \frac{1}{(N-1)}\left(1 - \frac{2}{N}\delta_{q^*,r^*}\right)\Delta_{R^{(1)}} = (-1)^{\delta_{q^*,r^*}}\frac{N}{N-1}\frac{h_1^{(2)}}{h_1^{(1)}}\Delta_{R^{(1)}}. \quad (4.12)$$

  The values $h_1^{(1)}$ and $h_1^{(2)}$ are the zero mode eigenvalues of the operators $R^{(1)}$ and $R^{(2)}$, as defined in Eq. (2.108). While $h_1^{(1)}$ has been fixed by the computations presented in Section 2, we do not know the eigenvalue $h_1^{(2)}$ for a generic disorder operator. If we set $R^{(2)} = \Phi_{(1,\ldots,1,2)(1,\ldots,1)}$ or $R^{(2)} = \Phi_{(1,\ldots,1)(1,\ldots,1,2)}$, the eigenvalue $h_1^{(2)}$ in Eq. (4.12) is known and the dimension $\Delta_{\Phi^{q*}}$ can be easily calculated for each of the two channels $q^* = 0$ and $q^* = r^*$.

  Let us discuss, for example, the case $R^{(1)} = R^{(2)} = \Phi_{(1,\ldots,1)(1,\ldots,1,2)}$. In this case we obtain:

$$\Delta_{\Phi^0} = \Delta_{(1,\ldots,1)(1,\ldots,1)} = 0, \qquad \Delta_{\Phi^{r*}} = \Delta_{(1,\ldots,1)(1,\ldots,1,3)}. \quad (4.13)$$

  This result is in agreement both with the fact that $\Phi_{(1,\ldots,1)(1,\ldots,1,3)}$ was deduced to be a doublet r^*, see Eq. (3.32), and with the value of the boundary term $B_{D^{r*}} = (N-1)/8N$. Note also that the Coulomb gas fusin rules are well respected. Indeed, as discussed previously, we expect that the fusion $\Phi_{(1,\ldots,1)(1,\ldots,1,2)} \cdot \Phi_{(1,\ldots,1)(1,\ldots,1,2)}$ produces in the principal channel the operator with $\vec{\beta}_{(1,\ldots,1)(1,\ldots,1,3)} = \vec{\beta}_{(1,\ldots,1)(1,\ldots,1,2)} + \vec{\beta}_{(1,\ldots,1)(1,\ldots,1,2)}$. The $q^* = 0$ channel, corresponding to the identity operator, follows the principal one by a shift realised by the combination $-\sum_{k=1}^{r} k\vec{e}_k$ of screening vectors.

  Actually, even when $h_1^{(2)}$ is unknown, we can still do some amount of verification by using Eq. (4.12). Indeed, one of the two channels of this equation could be used to define $h_1^{(2)}$, by assuming a given value of Δ_{Φ^0} for instance, chosen at a particular position in the Kac table. The other channel, with Φ^{r*}, in which enters the same $h_1^{(2)}$, could then serve to check for the presence of $\Delta_{\Phi^{r*}}$ at the appropriate position in the Kac table, having the value calculated from the characteristic equation (4.12). We have in this way verified the compatibility of the theory with Eq. (4.12).

- $\left\langle \Phi^r_{(2)} \Phi^0 \Phi^{-r}_{(1)} \right\rangle$

  The module of the operator $\Phi^{-r}_{(1)}$ is supposed to be degenerate at levels δ^r_{r-1} and 1, i.e., $\Phi^{-r}_{(1)} = \Phi_{(2,1,\ldots,1)(1,\ldots,1)}$ or $\Phi^{-r}_{(1)} = \Phi_{(1,\ldots,1)(2,1,\ldots,1)}$ (see Section 2). We derive the following equation for the dimensions:

$$\Delta_{\Phi^0} - \Delta_{\Phi^r_{(2)}} - \frac{1}{N-1}\Delta_{\Phi^r_{(1)}} = -\frac{N}{N-1}\frac{h^{(2)}_1}{h^{(1)}_1}\Delta_{\Phi^r_{(1)}}, \tag{4.14}$$

  where $h^{(2)}_1$ and $h^{(1)}_1$ are respectively the eigenvalues of the operators $\Phi^r_{(2)}$ and $\Phi^{-r}_{(1)}$, as defined in Eq. (2.15).

  We consider the operators $\Phi_{(1,\ldots,1)(2n,1,\ldots,1)}$ and $\Phi_{(2n,1,\ldots,1)(1,\ldots,1)}$, with n integer. The above operators have Z_N charge $q = r$, as it can be seen from Eq. (4.10). We know from the reflection-type arguments that these operators are degenerate at level δ^r_{r-1}. If we set $\Phi^r_{(2)} = \Phi_{(1,\ldots,1)(2n,1,\ldots,1)}$ or $\Phi^r_{(2)} = \Phi_{(2n,1,\ldots,1)(1,\ldots,1)}$, then the correspondent eigenvalue $h^{(2)}_1$ is determined by Eq. (2.89). In this case, Eq. (4.14) has respectively the solutions:

$$\Delta_{\Phi^0} = \Delta_{(1,\ldots,1)(2n-1,1,\ldots,1)} \text{ and } \Delta_{\Phi^0} = \Delta_{(2n-1,1,\ldots,1)(1,\ldots,1)}. \tag{4.15}$$

  We conclude that the singlets $\Phi_{(2n-1,1,\ldots,1)(1,\ldots,1)}$ and $\Phi_{(1,\ldots,1)(2n-1,1,\ldots,1)}$ are produced in a non principal channel of the fusion $\Phi^r_{(2)} \cdot \Phi^{-r}_{(1)}$. This is in agreement with the Coulomb gas rules and with the filling of the weight lattice that we have given previously.

- $\left\langle \Phi^r_{(2)} \Phi^1 \Phi^r_{(1)} \right\rangle$.

  In the above function we suppose $\Phi^r_{(1)} = \Phi_{(2,1,\ldots,1)(1,\ldots,1)}$ or $\Phi^r_{(1)} = \Phi_{(1,\ldots,1)(2,1,\ldots,1)}$, and $\Phi^r_{(2)} = \Phi_{(1,\ldots,1)(2n,1,\ldots,1)}$ or $\Phi^r_{(2)} = \Phi_{(2n,1,\ldots,1)(1,\ldots,1)}$. Taking into account that the modules of these operators are degenerate at level δ^r_{r-1}, we obtain the characteristic equation:

$$\Delta_{\Phi^1} = \Delta_{\Phi^r_{(2)}} + \frac{N-3}{N-1}\Delta_{\Phi^r_{(1)}}. \tag{4.16}$$

  Once again, the above equation is consistent with the theory. For example, if we consider the fusion $\Phi_{(1,\ldots,1)(2,1,\ldots,1)} \cdot \Phi_{(1,\ldots,1)(2n,1,\ldots,1)}$ we have from Eq. (4.16) that

$$\Delta_{\Phi^1} = \Delta_{(1,\ldots,1)(2n-1,2,1,\ldots,1)}. \tag{4.17}$$

  The doublet 1 operator $\Phi_{(1,\ldots,1)(2n-1,2,1,\ldots,1)}$ is expected to be produced in a non-principal channel of the fusion considered above, since

$$\vec{\beta}_{(1,\ldots,1)(2n-1,2,1,\ldots,1)} = \vec{\beta}_{(1,\ldots,1)(2,1,\ldots,1)} + \vec{\beta}_{(1,\ldots,1)(2n,1,\ldots,1)} - \vec{e}_1. \tag{4.18}$$

  The placing of the doublet 1 operator at the position $\vec{\beta}_{(1,\ldots,1)(2n-1,2,1,\ldots,1)}$ is confirmed by the above result.

5 Discussion

In this paper, we have constructed and analysed the conformal field theories based on the second solution of the Z_N parafermionic algebra, with $N = 2r + 1$ ($r = 1, 2, \ldots$).

We have achieved the principal goal of giving the Kac formula for these theories (3.3), i.e, the formula for the conformal dimensions of operators which realise the degenerate representations of the parafermionic algebra. To obtain this result, we had to determine the boundary terms for each sector, Eq. (3.39), and to determine to which sector belongs each operator in the Kac table (see Eq. (4.10)).

We want to stress here that the weight lattice of the Lie algebra B_r is also known to accommodate the primary fields of the the WB_r conformal field theories. The Kac formula of these theories, was defined, among those of other W theories, by Fateev and Luk'yanov in Ref. [8]. It should be noticed that the Kac formulae of both the present theory and of WB_r is invariant under the symmetry (3.25).

However, the content of these two conformal theories, WB_r and Z_{2r+1}, second solution, are completely different. In the case of the WB_r theory, the B_r weight lattice is filled with two kinds of operators, corresponding to the Neveu-Schwarz and Ramond sector (alias the Z_2 neutral sector and the spin operator sector). In the case of the Z_{2r+1} theory, second solution, the same weight lattice accommodates singlet, doublet q and disorder operators. The important point is that the distribution of singlets, doublet q and disorder operator on the lattice is consistent with the symmetry (3.25), as well as with the decoupling of the operators outside the physical domain (3.24).

It is natural to assume that, for each $N = 2r + 1$, the infinite set of unitary theories with $p = N-2+n$ and $n = 1, 2, \ldots$ describe multicritical fixed points in statistical systems with Z_{2r+1} symmetry. It would be interesting to verify this assumption by finding lattice realisations of the corresponding models and studying them numerically.

From a theoretical point of view, it is important to remark that it is not in general (except for some special cases; see Section 2) practicable to find the dimensions of the fundamental operators by means of explicit degeneracy computations. We bypassed this problem by exploiting the properties of the assumed Coulomb gas realisation of the theory.

This leads us to conjecture that, assuming a particular lattice on which the primary operators could be accommodated, this kind of approach could greatly simplify the study of the representations of a given chiral algebra with a free parameter. The first application of the above assumption would be the study of the representations of the theory Z_N, with N even, which should be based on the $D_{N/2}$ weight lattice [9].

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Note added in proof.

After the completion of this work, P. Furlan attracted our attention to the paper [10] in which the representation theory of the chiral algebra (2.1)–(2.2) was considered.

Ref. [10] also gives partial results for an explicit free field construction of the coset (1.1), based on earlier work of Gepner [11], and of Christe and Ravanini [12]. Our work differs from that of Ref. [10] in classifying the primary fields representing the coset (1.1) according to their Z_N transformation properties, as singlet, $(N-1)/2$ different doublets, and disorder operators. The Kac table proposed in Eq. (4.71) of Ref. [10] contains three sectors for each odd N, in sharp contrast with the $(N+3)/2$ sectors identified in the present work. Accordingly, we do not agree on the dimensions of primary operators proposed in Ref. [10].

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