

Family of generalized random matrix ensembles

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Using the Generalized Maximum Entropy Principle based on the nonextensive q entropy a new family of random matrix ensembles is generated. This family unifies previous extensions of Random Matrix Theory (RMT) and gives rise to an orthogonal invariant stable Lévy ensemble with new statistical properties. Some of them are analytically derived.

Random matrix theory (RMT) started in physics with the introduction by E. Wigner, in the 50s, of Gaussian matrix ensembles, the Orthogonal (GOE), the Unitary (GUE) and the Symplectic (GSE). Their properties were fully developed by Dyson, Gaudin, Mehta and others [1]. These ensembles have a wide application as models to describe statistical properties of quantum fluctuations of systems of few or many-body particles. They have been useful in discussing nuclear and atomic properties, mesoscopic physics, quantum chaos, theory of amorphous solids, etc (see, for instance [2]). The link between RMT and Information Theory was set by Balian [3] who, by using the Boltzmann-Gibbs-Shannon entropy associated to the ensemble probability distribution, obtained the Wigner ensembles by maximizing it subjected to the normalization

condition and a constraint given by the average norm of the matrices. Ensembles to describe symmetry breaking have been constructed by adding an extra constraint to this scheme [4].

In this letter, we use this framework and consider ensembles within the Generalized Maximum Entropy Principle (GMEP) based on the nonextensive Tsallis entropy [5]. This entropy has been applied to a great variety of phenomena, specially those in which long range correlations are present (see however [6] concerning its physical interpretation). It is dependent on the non-additivity parameter q defined in such a way that when $q \rightarrow 1$ the Boltzman-Gibbs-Shannon entropy is recovered. We show that a new family of ensembles is generated that unifies some important extensions of RMT. In the range $-\infty < q < 1$, it is found to be a restricted trace ensemble that interpolates between the bounded trace ensemble [7] when $q \rightarrow -\infty$ and the Wigner-Gaussian ensembles at $q = 1$. In the domain $1 < q < q_{\max}$, with q_{\max} being a cutoff imposed by the normalization condition, it interpolates between RMT at $q = 1$ and an ensemble of Lévy matrices [8] that appears at the neighborhood of the extremum q_{\max} where the ensemble distribution has divergent moments.

As extensions of RMT that preserve the stability of the universal ensembles, Lévy matrices have attracted recently much attention due to its potential application to many areas ranging from physics to finances [8–10]. Stability means that if H_1 and H_2 are matrices of the ensemble, their sum $H = H_1 + H_2$ also is [11]. This will be the case if the individual matrix elements are distributed according to a Gaussian or a Lévy function. We prove that this indeed happens, in the case of the q -generalized

ensembles, for all allowed values of q , i.e. $-\infty < q < q_{\max}$ when N goes to infinity.

Although the individual matrix element distribution of q -ensembles have the same asymptotic behavior as the Lévy matrices of Ref. [8], there is here a basic difference as they are orthogonal invariant with matrix elements, in principle, correlated. Orthogonal invariance is also satisfied by the ensembles of Refs. [9,10] which are directly defined in terms of the joint distribution of eigenvalues. However no explicit reference to the matrix elements distribution is made there and the spectral statistical measures are obtained expressing them in terms of appropriately defined orthogonal polynomials. Here we do not apply this technique and show that the special relation that q -ensembles have with the Gaussian ensembles allows their spectral properties to be analytically derived.

Applied to matrices whose entries are random variables, the nonextensive entropy can be written as

$$S_q = \frac{1 - \int dH P^q(H)}{q - 1}, \quad (1)$$

where H is a $N \times N$ matrix distributed according to $P(H)$ and dH is the product of differentials of the independent variables of the matrices. For definiteness we consider real symmetric matrices in which case we have $f = N(N+1)/2$ independent matrix elements and the differential in (1) is conveniently defined as $dH = 2^{N(N-1)/4} \prod_{1 \leq i < j \leq N} dH_{ij}$.

The GMEP consists in maximizing (1) subjected to normalization

$$\int dH P(H) = 1, \quad (2)$$

and to the constraint [12]

$$\int dHP^q(H) \operatorname{tr} H^2 - \mu \int dHP^q(H) = 0 \quad (3)$$

that fixes the q -average of the norm defined as the trace of the square of the matrices. Following the usual steps of the variational method, we arrive at the probability distribution

$$P(H; \lambda, \alpha) = Z_N^{-1} \left(1 + \frac{\alpha}{\lambda} \operatorname{tr} H^2 \right)^{\frac{1}{1-q}} \quad (4)$$

with λ given by

$$\lambda = \frac{1}{q-1} - \alpha\mu = \frac{1}{q-1} - \frac{f}{2}. \quad (5)$$

Z_N in (4) is the partition function and (3) has been used to determine the relation $\alpha = \frac{f}{2\mu}$. Let us remark that had we used Renyi's entropy [13] instead of Eq. (1) we would also have been led to Eq. (4).

Changing from matrix elements to eigenvalue and eigenvector variables the ensemble distribution factorizes and, after integrating over the eigenvector parameters, we find for the eigenvalues the joint probability distribution

$$P(E_1, \dots, E_N; \lambda, \alpha) = K_N \left(1 + \frac{\alpha}{\lambda} \sum E_k^2 \right)^{\frac{1}{1-q}} \prod |E_j - E_i|, \quad (6)$$

where K_N is the normalization constant. Taking $q \rightarrow 1$ in the above, $\lambda \rightarrow \infty$ and the RMT distributions

$$P_{GOE}(H; \alpha) = Z_{GOE,N}^{-1} \exp(-\alpha \operatorname{tr} H^2) \quad (7)$$

and

$$P_{GOE}(E_1, \dots, E_N; \alpha) = K_{GOE, N} \exp\left(-\alpha \sum E_k^2\right) \prod |E_j - E_i| \quad (8)$$

are recovered.

Considering $-\infty < q < 1$, i.e. $-\frac{f}{2} > \lambda > -\infty$ the condition $\text{tr}(H^2) = \sum E_k^2 < -\frac{\lambda}{\alpha}$ has to be imposed in order to warrant a real positive probability distribution for any q . These two inequalities define hyperspheres in which the matrix elements and the eigenvalues are confined in their respective spaces. Taking in Eq. (4) the limit $q \rightarrow -\infty$ with the partition function given by

$$Z_N(q) = \left(-\frac{\pi\lambda}{\alpha}\right)^{\frac{f}{2}} \frac{\Gamma\left(\frac{2-q}{1-q}\right)}{\Gamma(1-\lambda)} \quad (9)$$

we find that the ensemble goes to the bounded trace ensemble

$$P\left(H; -\frac{f}{2}, \alpha\right) = \left(-\frac{\alpha}{\pi\lambda}\right)^{\frac{f}{2}} \Gamma\left(\frac{f}{2}\right) \Theta\left(\frac{f}{2\alpha} - \text{tr} H^2\right), \quad (10)$$

where $\Theta(x)$ is the step function. The bounded trace ensemble is known to follow the Wigner-Dyson statistics of the Gaussian ensemble when $N \rightarrow \infty$. To show that this is also the case for $-\infty < q < 1$ we consider the probability distribution of a generic matrix element

$$p(x; \lambda, \alpha) = \sqrt{-\frac{\alpha}{\pi\lambda} \frac{\Gamma(1-\lambda)}{\Gamma\left(\frac{1}{2}-\lambda\right)}} \left(1 + \frac{\alpha}{\lambda} x^2\right)^{-\frac{1}{2}-\lambda} \quad (11)$$

and the correlation between two matrix elements h_1 and h_2

$$C(h_1, h_2) = \langle h^2 \rangle^2 - \langle (h_1 h_2)^2 \rangle = \frac{1}{4\alpha^2} \frac{\lambda^2}{(2-\lambda)(1-\lambda)^2}. \quad (12)$$

By taking the limit of large matrices, (11) goes to the Gaussian distribution

$$p(x; \lambda, \alpha) \sim \sqrt{\frac{\alpha}{\pi}} \exp(-\alpha x^2) \quad (13)$$

while $C(h_1, h_2) \rightarrow 0$ indicating that the matrix elements behave as those of the Gaussian ensembles as $N \rightarrow \infty$. Numerical simulations [14] confirm that the level density is given by the Wigner semi-circle law

$$\rho_{GOE}(E; \alpha) = \begin{cases} \frac{2\alpha}{\pi} \sqrt{\frac{N}{\alpha} - E^2}, & |E| < \sqrt{\frac{N}{\alpha}} \\ 0, & |E| > \sqrt{\frac{N}{\alpha}} \end{cases}. \quad (14)$$

and spectral fluctuations follow GOE statistics.

Consider now $q > 1$. The partition function is given by

$$Z_N(q) = \left(\frac{\pi\lambda}{\alpha}\right)^{\frac{f}{2}} \frac{\Gamma(\lambda)}{\Gamma\left(\frac{1}{q-1}\right)} \quad (15)$$

that requires the restriction $\lambda > 0$ or $q < q_{\max} = 1 + \frac{2}{f}$. We see that the introduction of the parameter λ is crucial to be able to study the limit $N \rightarrow \infty$. It maps the interval $1 < q < q_{\max}$ onto the interval $\infty > \lambda > 0$. The Fourier transform of the distribution of a generic matrix element, Eq. (11), with $\lambda > 0$ is

$$F(k; \lambda, \alpha) = \sqrt{\frac{2}{\pi}} \frac{1}{\Gamma(\lambda)} \left(k\sqrt{\frac{\lambda}{\alpha}}\right)^\lambda K_\lambda\left(k\sqrt{\frac{\lambda}{\alpha}}\right) \quad (16)$$

where $K_\lambda(z)$ is the modified Bessel function [15]. In order to ensure that spectra scale independently of the size of the matrices, α has to go to infinity when N does. This can be seen from the analytic expression of the level density, Eq. (27) below. The requirement is that a characteristic value, say $E_c = \sqrt{\frac{N\lambda}{\alpha}}$, remains finite when N diverges. In this limit, $K_\lambda(z)$ can be replaced by its small z expansion and keeping only the first terms we can write $F(k; \lambda, \alpha) \sim \exp\left(-\Lambda \left|\frac{k}{2}\sqrt{\frac{\lambda}{\alpha}}\right|^\sigma\right)$ with

$$\sigma = 2 \text{ and } \Lambda = \frac{1}{4(\lambda - 1)} \quad \text{if } \infty > \lambda > 1. \quad (17)$$

and

$$\sigma = 2\lambda \text{ and } \Lambda = \frac{\Gamma(1 - \lambda)}{\Gamma(1 + \lambda)} \quad \text{if } 1 > \lambda > 0 \quad (18)$$

Therefore for $\infty > \lambda > 1$ the distribution of a generic matrix element approaches the Gaussian distribution

$$p(x; \lambda, \alpha) \simeq \sqrt{\frac{(\lambda - 1)\alpha}{\pi\lambda}} \exp\left[-(\lambda - 1)\frac{\alpha}{\lambda}x^2\right]. \quad (19)$$

For $1 > \lambda > 0$ the Lévy-Gnedenko generalized central limit holds [16] and $p(x; \lambda, \alpha)$ goes to the Lévy function, $L_\sigma(x, \sigma, \Lambda) = \pi^{-1} \int_0^\infty dt \exp(-\Lambda t^\sigma) \cos(xt)$, with the same asymptotic behavior, i.e.

$$p(x; \lambda, \alpha) \simeq 2\sqrt{\frac{\alpha}{\lambda}} L_{2\lambda}\left[2\sqrt{\frac{\alpha}{\lambda}}x, 2\lambda, \frac{\Gamma(1 - \lambda)}{\Gamma(1 + \lambda)}\right]. \quad (20)$$

Concerning correlations between matrix elements, Eq. (12) shows that only for large values of λ or α the matrix elements behave independently, whereas for small values, $\lambda < 2$, they are strongly correlated. Therefore for large λ or α (19) predicts that the level density goes to the semi-circle $\rho_{GOE}\left[E; (\lambda - 1)\frac{\alpha}{\lambda}\right]$ i.e. Eq. (14) with α replaced by $(\lambda - 1)\frac{\alpha}{\lambda}$.

We focus now on the spectral properties of these new ensembles. They are analytically derived by introducing the representation

$$\left[1 + \frac{\alpha}{\lambda} \text{tr}(H^2)\right]^{\frac{1}{1-q}} = \frac{1}{\Gamma\left(\frac{1}{q-1}\right)} \int_0^\infty d\xi \exp(-\xi) \xi^{\frac{1}{q-1}-1} \exp\left(-\frac{\alpha}{\lambda}\xi \text{tr} H^2\right) \quad (21)$$

that allows the joint distribution function of the matrix elements to be written in terms of the joint distribution function of the GOE ensemble as

$$P(H; \lambda, \alpha) = \frac{1}{\Gamma(\lambda)} \int_0^\infty d\xi \exp(-\xi) \xi^{\lambda-1} P_{GOE}\left(H; \frac{\alpha}{\lambda} \xi\right); \quad (22)$$

the joint distribution of eigenvalues becomes

$$P(E_1, \dots, E_N; \lambda, \alpha) = \frac{K_N}{\Gamma\left(\frac{1}{q-1}\right)} \int_0^\infty d\xi \exp(-\xi) \xi^{\frac{1}{q-1}-1} \exp\left(-\frac{\alpha}{\lambda} \xi \sum E_k^2\right) \prod |E_j - E_i| \quad (23)$$

where K_N is the normalization constant. Integrating (23) over all eigenvalues we deduce the relation

$$K_N = \frac{\left(\frac{2\alpha}{\lambda}\right)^{\frac{f}{2}} \Gamma\left(\frac{1}{q-1}\right)}{\Gamma(\lambda)} K_{GOE,N} \quad (24)$$

relating K_N to the corresponding RMT constant in standard units, i.e. $\alpha = \frac{1}{2}$ in Eq. (8) see [1]. Substituting in (23) one finally obtains for the normalized joint eigenvalue density

$$P(E_1, \dots, E_N; \lambda, \alpha) = \frac{1}{\Gamma(\lambda)} \left(\frac{2\alpha}{\lambda}\right)^{\frac{N}{2}} \int_0^\infty d\xi \exp(-\xi) \xi^{\lambda+\frac{N}{2}-1} P_{GOE}\left(\sqrt{\xi}x_1, \dots, \sqrt{\xi}x_N; \frac{1}{2}\right) \quad (25)$$

where we have introduced the rescaled eigenvalues $x_k = \sqrt{\frac{2\alpha}{\lambda}} E_k$. This is one of the central results of this paper and can be taken as the defining equation of the new ensemble. It expresses the eigenvalue distribution of the new ensemble as a sort of Γ function of the GOE eigenvalue distribution. It shows that one may expect that measures of the q -family will be weighted Laplace transforms of the corresponding measures of the Gaussian ensemble.

Integrating (25) over all eigenvalues but one and multiplying by N , the average eigenvalue density is expressed in terms of Wigner's semi-circle law as

$$\rho(E; \lambda, \alpha) = \frac{1}{\Gamma(\lambda)} \sqrt{\frac{2\alpha}{\lambda}} \int_0^{\frac{N\lambda}{\alpha E^2}} d\xi \exp(-\xi) \xi^{\lambda-\frac{1}{2}} \frac{1}{\pi} \sqrt{2N - 2\frac{\alpha}{\lambda} \xi E^2}. \quad (26)$$

The asymptotic power law behavior of this distribution is better seen by rewriting it as

$$\rho(E; \lambda, \alpha) = \frac{N}{|E|^{2\lambda+1} \sqrt{\pi}} \left(\frac{N\lambda}{\alpha} \right)^\lambda \frac{\Gamma\left(\lambda + \frac{1}{2}\right)}{\Gamma(\lambda) \Gamma(\lambda + 2)} M\left(\lambda + \frac{1}{2}, \lambda + 2, -\frac{N\lambda}{\alpha E^2}\right) \quad (27)$$

where $M(a, b, z)$ is the confluent hypergeometric function [15]. In Fig. 1, with $\alpha = \frac{N^{\frac{2}{\sigma}}}{2}$ (see Eqs. (17) and (18)) the density $\rho(E; \lambda, \alpha)$ is plotted for four values of λ , exhibiting the deviation from the semi-circle law as λ moves inside the interval $1 > \lambda > 0$. When $\lambda \rightarrow 0$, the density behaves as $\rho \simeq \frac{N^\lambda}{|E|^{2\lambda+1}}$ approaching the same behavior as for a nonconfining log square potential [17].

The behavior of the spectral fluctuations can be illustrated by considering the gap probability function $E(s)$ (usually denoted $E(0, s)$) that gives the probability of finding an eigenvalue-free segment of length s . This function has been investigated in Ref. [10] for Cauchy ensembles and is related to the presence of gaps in the spectrum. For the q -family it is expressed in terms of the corresponding GOE function as

$$E(\theta) = \frac{1}{\Gamma(\lambda)} \int_0^\infty d\xi \exp(-\xi) \xi^{\lambda-1} E_{GOE} \left[y \left(\sqrt{\frac{2\alpha\xi}{\lambda}} \theta \right) \right] \quad (28)$$

obtained integrating the joint eigenvalue density over all eigenvalues outside the interval $(-\theta, \theta)$ around the origin. In (28) $y(x) = 2 \int_0^x dt \rho_{GOE}(t)$. Together with

$$s(\theta) = 2 \int_0^\theta dE \rho(E; \lambda, \alpha) \quad (29)$$

(28) expresses $E(s)$ in a parametric form. Using the Wigner surmise for the nearest neighbor spacing distribution $p(s)$ and the relation connecting $E(s)$ and $p(s)$, E_{GOE} in (28) can be well approximated by $E_{GOE}(y) \simeq 1 - \operatorname{erf}\left(\frac{y\sqrt{\pi}}{2}\right)$. On Fig. 2 results in the Lévy regime are displayed. Notice the large increase of the probability of formation of a gap with respect to the GOE case. The asymptotic behavior in Eq. (28) can be extracted by making the substitution $x = \sqrt{\frac{2\alpha\xi}{\lambda}}\theta$ that leads to

$$E(\theta) = \frac{2}{\Gamma(\lambda)} \left(\frac{\lambda}{2\alpha}\right)^\lambda \frac{1}{\theta^{2\lambda}} \int_0^\infty dx \exp\left[-\frac{\lambda}{2\alpha} \left(\frac{x}{\theta}\right)^2\right] x^{2\lambda-1} E_{GOE}[y(x)]. \quad (30)$$

For large θ , this equation predicts for $\lambda = 1$ a power law decay $E(s) \simeq \frac{1}{2s^2}$, clearly seen in the figure. This very characteristic behavior is exhibited here for the first time.

In summary, we have proved that the q -generalized family of ensembles interpolates between the bounded trace ensemble [7] at the extremum $q \rightarrow -\infty$ and the Wigner-Gaussian ensembles at $q = 1$. In the domain $1 < q < q_{\max}$, it interpolates between RMT at $q = 1$ and an ensemble of Lévy matrices at the neighborhood of the extremum $q_{\max} = 1 + \frac{2}{f}$. These orthogonal invariant stable matrix ensembles have novel spectral properties. Remarkably, several of their distribution functions can be expressed as integral transforms (sort of extended Γ functions) of the corresponding distribution functions of the Gaussian ensembles.

It is premature to exhibit specific applications of these generalized ensembles. However there are worth exploring possibilities, for instance, connections with the so-called critical statistics [18] or the transition from Erdős-Renyi to scale free models in random graph theory [19]. In conclusion, let us remind that stable laws (Lévy laws) were first

introduced and studied. It was correctly anticipated [16] that a large domain of applications would follow [20]. We believe that we are presently facing a similar situation, where the role of a random variable is now being extended to the one of a random matrix. The results presented here should contribute to broaden the applications of random matrix theory.

After completion of this letter, we learned of ref. [21] closely related to the work presented here.

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Figure Captions

Fig. 1 The eigenvalue density for four values of the parameter λ ($= 10, 1, 0.75, 0.5$) in the transition region from the Gaussian to the Lévy regime, with $N = 50$. For the sake of comparison the semi-circle $\rho_{GOE} \left[E; (\lambda - 1) \frac{\alpha}{\lambda} \right]$ with $\lambda = 10$ and $\alpha = \frac{N^{\frac{2}{\sigma}}}{2}$ is also shown (dashed line).

Fig. 2 The eigenvalue-free probability $E(s)$ for $\lambda = 1$. Full line: theory, Eq. (28) and its asymptotics (dotted line); dashed line: $E_{GOE}(s)$; * : numerical simulation with $N = 20$. See text for further explanation.



