Spectral statistics of a quantum interval-exchange map

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Curious spectral properties of an ensemble of random unitary matrices appearing in the quantization of a map $p \to p + \alpha$, $q \to q + f(p + \alpha)$ in [1] are investigated. When $\alpha = m/n$ with integer co-prime m, n and matrix dimension $N \to \infty$ is such that $mN \equiv \pm 1 \bmod n$, local spectral statistics of this ensemble tends to the semi-Poisson distribution [2] with arbitrary integer or half-integer level repulsion at small distances: $R_2(s) \to s^\beta$ when $s \to 0$ and $\beta = n - 1$ or $n/2 - 1$ depending on time-reversal symmetry of the map.

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Random matrices appear naturally in many different physical and mathematical problems ranging from nuclear physics to number theory (see e.g. [3]). Though many different ensembles of random matrices were considered, it is well accepted [3] that the level repulsion (i.e. the small-s behaviour of the two-point correlation function $R_2(s) \stackrel{s\to 0}{\longrightarrow} s^\beta$) is determined only from symmetry arguments: $\beta = 1, 2, 4$ for, respectively, real symmetric, complex hermitian, and self-dual quaternion matrices (plus, of course, $\beta = 0$ for diagonal matrices with independent entries). Dyson's arguments leading to these values are so robust that only rarely one considers seriously the possibility that certain models may have different values of β (see e.g. [4]).

The purpose of this Letter is to demonstrate that a quantization of an interval-exchange map leads to a random unitary matrix ensemble which, under certain conditions, reveals level repulsion with arbitrary integer and half-integer exponent β. To the authors knowledge it is the first example which shows naturally level repulsion different from standard values (but see [5]).

According to [1] a suitable quantization of the map

$$\Phi_\alpha : \begin{pmatrix} p \\ q \end{pmatrix} \longmapsto \begin{pmatrix} p + \alpha \\ q + f(p+\alpha) \end{pmatrix} \mod 1 \qquad (1)$$

(where α is a constant and $f(q)$ is a certain function with period 1) leads in the momentum representation to the following $N \times N$ unitary matrix ($k, p = 0, 1, \ldots, N-1$)

$$M_{kp} = e^{i\Phi_k} \frac{1 - e^{2\pi i \alpha N}}{N(1 - e^{2\pi i (k - p + \alpha N)/N})} , \qquad (2)$$

where $\Phi_k = -2\pi N F(k/N)$ and $F'(p) = f(p)$. Its eigenvalues have the form $\Lambda(j) = e^{i\varphi(j)}$ with real $\varphi(j)$ called eigenphases and we are interested in their distribution.

To define an ensemble of random matrices it is convenient to consider Φ_k in (2) not as derived from a function $F(k)$ but as independent random variables [6]. More precisely, the following two cases are considered. In the first one (called non-symmetric matrices) all Φ_k with $k = 0, \ldots N-1$ are independent random variables distributed uniformly between 0 and 2π. In the second case (called symmetric matrices) only a half of Φ_k is independent. The others are obtained from symmetry relations $\Phi_{N-k} = \Phi_k$. These two cases correspond to classical maps without and with time-reversal invariance.

When α is a 'good' irrational number (e.g. $\alpha = \sqrt{5}/2$) numerical calculations show that spectral statistics of symmetric (resp. non-symmetric) matrices is well described for large N by the standard Gaussian orthogonal (resp. unitary) ensemble of random matrices as it follows from usual symmetry considerations.

For rational values of $\alpha = m/n$ with integer co-prime m and n only a finite number of different momenta appear under the iterations and classical map (1) can be identified with an interval-exchange map.

Our main result in this case is the following. When $\alpha = m/n$ and matrix dimension $N \to \infty$ is such that

$$mN \equiv \pm 1 \mod n \qquad (3)$$

local spectral statistics of matrix ensemble (2) is described by the semi-Poisson statistics characterized by a parameter β related with α as follows

$$\beta = \begin{cases} n - 1 & \text{for non-symmetric matrices} \\ n/2 - 1 & \text{for symmetric matrices} \end{cases} . \qquad (4)$$

The general semi-Poisson statistics was investigated in [8] and [2] as the simplest model of intermediate statistics. It is defined in such a way that the probability of having N ordered levels $E_1 \le E_2 \le \ldots \le E_N$ is

$$p_\beta(E_1, \ldots, E_N) \sim \prod_{j=1}^{N-1} |E_{j+1} - E_j|^\beta . \qquad (5)$$

For this model all correlation functions are known when $N \to \infty$ [2]. In particular, the nearest-neighbor distribution is

$$p_\beta(s) = A_\beta s^\beta e^{-(\beta+1)s} , \; A_\beta = \frac{(\beta+1)^{\beta+1}}{\Gamma(\beta+1)} , \qquad (6)$$

the two-point correlation form factor takes the form

$$K_\beta(\tau) = 1 + 2 \,\mathrm{Re}\, \frac{1}{(1 + 2\pi i \tau/(\beta+1))^{\beta+1} - 1} , \qquad (7)$$

and the level compressibility which determines the asymptotic behaviour of the number variance is $\chi \equiv K(0) = 1/(\beta+1)$. The two-point correlation function is expressed through the Mittag-Leffler function [9]. For integer $\beta = n-1$ it is a finite sum

$$R_2^{(n)}(s) = e^{-ns}\sum_{k=0}^{n-1}\exp\left[nse^{2\pi ik/n} + 2\pi i\frac{k}{n}\right] \ . \qquad (8)$$

In Figs. 1 and 2 a few examples of nearest-neighbor distributions for quantum maps with different α and symmetries are presented. For each matrix 100 different realizations of random phases were considered and in all figures the spectrum is unfolded to unit mean density. Prediction (6) with (4) agrees very well for all α with numerical calculations.

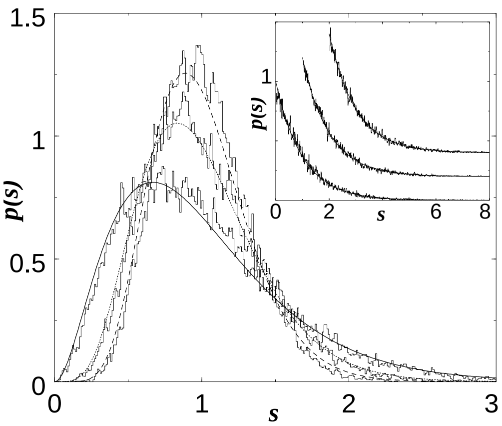

FIG. 1: The nearest-neighbor distribution for non-symmetric matrices with $\alpha = 1/3, 1/6, 1/9$ and resp. $N = 202, 205, 206$. Solid, dotted, and dashed lines are prediction (6) with resp. $\beta = 2, 5, 8$. Insert: the same but for M^n where n is the denominator of α. For clarity pictures are shifted along the diagonal. Solid lines: the Poisson prediction $p(s) = e^{-s}$.

We sketch main steps leading to the above result. Details will be given elsewhere [10]. First, for all α and N eigenphases of matrix (2) are such that for all nearby pairs of eigenphases φ_1 and φ_2 there is one and only one eigenphase φ such that $\varphi + 2\pi\alpha$ falls in-between φ_1 and φ_2 and there is one and only one φ' such that the same is true for $\varphi' - 2\pi\alpha$

$$\varphi_1 \le \varphi + 2\pi\alpha \le \varphi_2 \ , \ \varphi_1 \le \varphi' - 2\pi\alpha \le \varphi_2 \ . \qquad (9)$$

To check it let us consider instead of matrix M_{kp} (2) a new matrix

$$N_{kp} = M_{kp}e^{2\pi i(k-p+\alpha N)/N} \ . \qquad (10)$$

If $u_k(j)$ and $\Lambda(j)$ for $j = 1, \ldots, N$ are eigenfunctions and eigenvalues of matrix M_{kp} ($\Lambda(j)u_k(j) = \sum_p M_{kp}u_p(j)$) then $\Psi_k(j) = e^{2\pi ik/N}u_k(j)$ and $\Lambda'(j) = e^{2\pi i\alpha}\Lambda(j)$ are

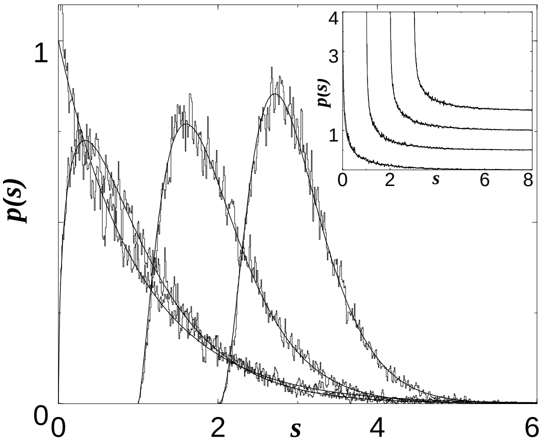

FIG. 2: The nearest-neighbor distribution for symmetric matrices with $\alpha = 1/2, 1/3, 1/5, 1/7$ and resp. $N = 201, 202, 201, 202$. For clarity pictures for $\alpha = 1/5$ and $1/7$ are shifted horizontally by 1 and 2 units. Solid lines are prediction (6) with resp. $\beta = 0, 1/2, 3/2, 5/2$. Insert: the same but for M^n. Solid lines: the super-Poisson distribution (19).

eigenfunctions and eigenvalues of matrix N_{kp}. On the other hand, due to special form of matrix M_{kp}, matrix N_{kp} (10) is a rank-one perturbation of matrix M_{kp}

$$N_{kp} = M_{kp} - \frac{1-e^{2\pi i\alpha N}}{N}e^{i\Phi_k} \ . \qquad (11)$$

It is known (see e.g. [11]) (and can be checked in this case) that for rank-one perturbations new eigenvalues are in-between the unperturbed ones. As both are known, the first inequality in (9) follows. The second can be proved by similar arguments.

These inequalities manifest the existence of long distance correlations between eigenphases of matrix (2). When condition (3) is fulfill, these long correlations induces a very particular short ordering. Namely, consider, instead of matrix M defined by (2) its n^{th} power, M^n, where n is the denominator of α. Of course, eigenvalues of this matrix are just the n^{th} power of eigenvalues of matrix M but, as eigenphases are defined only modulo 2π, new eigenphases are equal to $n\varphi(j)$ mod 2π. Therefore, even if between two eigenphases φ_1 and φ_2 there were no eigenphases of matrix M, between $n\varphi_1$ and $n\varphi_2$ in M^n there will be, in general, a few new eigenphases coming from different numbers of rotations around the circle. Nevertheless, condition (3) is sufficient to insure that in matrix M^n in-between $n\varphi_1$ and $n\varphi_2$ there exist exactly $n-1$ new eigenphases.

To prove it we put all eigenphases of matrix M on the unit circle and divide it in n consecutive sectors of angle $2\pi\alpha = 2\pi m/n$. Let n_k be the number of eigenphases inside the k^{th} sector. Denote by x_k and y_k the (positive) distances between angle $2\pi m(k-1)/n$ (i.e. the beginning of the k^{th} sector) and the two eigenphases closest to this angle in such a way that x_k corresponds to an eigenphase in the k^{th} sector and y_k to one in $(k-1)^{\text{th}}$ sector. Inequalities (9) lead to the following inequality for all k [12]

$$(y_{k+1}-y_k)(x_{k+1}-x_k)<0. \tag{12}$$

As in the $(k+1)^{\text{th}}$ sector it must exist eigenphases which after the shift by $-2\pi\alpha$ come in-between n_k eigenphases of the k^{th} sector, it follows that the number of eigenphases in the $(k+1)^{\text{th}}$ sector is at least n_k-1. The only possibility to get more eigenphases in the $(k+1)^{\text{th}}$ sector is connected with the positions of the last and the first eigenphases in the sector. Straightforward calculations plus (12) give recursive relations

$$n_{k+1}=n_k-1+\Theta(x_{k+2}-x_{k+1})+\Theta(x_k-x_{k+1})\ . \tag{13}$$

Here $\Theta(x)=1$ if $x>0$ and $\Theta(x)=0$ if $x<0$.

Now choose the beginning of the first sector at the position of an eigenphase (i.e. $y_1=0$). From (12) it follows that $x_2<x_1$ and $x_n<x_1$. Direct applications of (13) show that for $j=2,\dots,n$ (with $x_{n+1}=x_1$)

$$n_j=n_1+\Theta(x_{j+1}-x_j)\ . \tag{14}$$

As $\sum_{k=1}^{n} n_k = mN$ (because it corresponds to m full turns around the unit circle) one obtains

$$mN=nn_1+1+\sum_{j=2}^{n-1}\Theta(x_{j+1}-x_j)\ . \tag{15}$$

If $mN\equiv 1 \bmod n$, all the Θ-functions have to be equal to 0 which leads to the following chain of inequalities

$$0<x_n<x_{n-1}<\ldots<x_2<x_1\ . \tag{16}$$

But x_1 is just the distance between a pair of two nearby eigenphases of matrix (2) and the true eigenphase corresponding to x_k is $2\pi m(k-1)/n+x_k$. Distances of all other eigenphases from boundaries of sectors are bigger than x_1. When matrix M^n is considered, multiples of 2π have to be ignored and one concludes that in-between two eigenphases of matrix M^n coming from two nearby eigenphases of matrix M there are exactly $n-1$ other eigenphases ordered as in (16). When $mN\equiv -1 \bmod n$ all the Θ-functions have to be equal to 1 which leads to the same conclusion (but $n-1$ eigenphases appear in the inverted order). In other words, when eigenphases of matrix M^n are known and (3) is satisfied, correct ordering of eigenphases of matrix M corresponds simply to considering each n^{th} eigenphase of M^n (and dividing them by n).

But the n^{th} power of the classical map (1) with rational $\alpha=m/n$ is classically integrable (as the momentum is conserved)

$$\Phi_\alpha^n:\begin{pmatrix}p\\q\end{pmatrix}\longmapsto\begin{pmatrix}p\\q+\sum_{j=1}^{n}f(p+j\alpha)\end{pmatrix}\ \bmod 1\ . \tag{17}$$

Quantization of this map as in [1] leads to a quantum map whose eigenphases are $\varphi_0(k)=-2\pi N\sum_{j=1}^{n}F(k/N+j\alpha)$. If in the semiclassical limit $N\to\infty$ classical behaviour dominates, it is natural that local spectral statistics of M^n (for all N) will be Poissonian as it is conjectured for all generic integrable models [14]. It seems that it is the case for non-symmetric matrices (with an arbitrary function $f(k)$) because all $\varphi_0(k)$ are different and small corrections from omitted terms are negligible. But for symmetric matrices $F(-k)=F(k)$, consequently, $\varphi_0(N-k)=\varphi_0(k)$ and these corrections are important as unperturbed eigenphases form degenerate pairs. Nevertheless, one can argue that each second eigenphase of M^n (i.e. either odd or even in the ordered sequence) still form a Poisson sequence but degenerate levels are split. Let $p(s)$ be the distribution of nearest-neighbor odd-even eigenphases with mean density 1. As odd or even eigenphases of matrix M^n are assumed to obey the Poisson statistics (with density equals one half of the total density) one gets a convolution equation

$$\frac{1}{2}e^{-s/2}=\int_0^s p(l)p(s-l)dl \tag{18}$$

whose solution is

$$p(s)=\frac{1}{\sqrt{2\pi s}}e^{-s/2}\ . \tag{19}$$

Higher correlation functions are obtained in the same manner and one concludes that the odd-even distribution is exactly the semi-Poisson distribution with $\beta=-1/2$ (cf. (6)). For brevity we refer to it as to the super-Poisson distribution because instead of level repulsion it shows level attraction ($p(s)\overset{s\to 0}{\longrightarrow}\infty$) [13].

The above arguments demonstrate that for large N eigenphases of matrix M^n where n is the denominator of α have universal distribution (independent of α), namely, the Poisson distribution for non-symmetric matrices and the super-Poisson distribution (i.e. the semi-Poisson distribution with $\beta=-1/2$) for symmetric ones (see inserts in Figs. 1 and 2).

When N obeys (3) eigenphases of M are obtained simply by jumping on each n^{th} eigenphase of matrix M^n which leads directly to (4). In [15] it was noted that for integer β the semi-Poisson distribution can be obtained from a Poisson sequence by considering each $(\beta+1)$ level. It is interesting that this completely artificial mechanism appears naturally in the quantization of the map (2) (for symmetric matrices one jumps over not a Poisson sequence but a super-Poisson one).

The described result applies only to local statistics in the universal limit of unit mean density and $N\to\infty$. For long-range statistics (on distances of the order of N in this scale) the above-mentioned long-range correlations between eigenphases of matrix M become apparent. In Fig. 3 the two-point correlation function for non-symmetric matrices with $\alpha = 1/20$ and $N = 801$ is presented. In the insert the correlation function till $s = 5$ is given together with the prediction (8) with $n = 20$. The very strong level repulsion ($\sim s^{19}$) and prominent oscillations described by this formula agree very well with the numerical calculations. Nevertheless, for s close to αN one observes a big peak which is a clear signal of long-range correlations. At higher s new peaks (with smaller amplitudes) appear at multiples of this quantity.

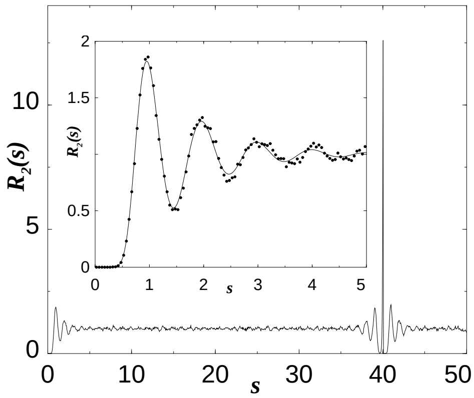

FIG. 3: The two point correlation function for non-symmetric matrices with $\alpha = 1/20$ and $N = 801$. Insert: small-s behaviour of this quantity. Solid line: Eq. (8) with $n = 20$.

We also investigate numerically eigenfunctions $u_k(j)$ of matrix M. As expected for models with intermediate statistics they show fractal properties which manifest e.g. in a non-trivial power increase of participation ratios

$$R_q = \langle \sum_k |u_k(j)|^{2q} \rangle_j \overset{N\to\infty}{\longrightarrow} N^{D_q(q-1)} \qquad (20)$$

where $\sum_k |u_k(j)|^2 = 1$ and $\langle \ldots \rangle_j$ denotes an averaging over the spectrum. For example, for non-symmetric matrices with $\alpha = 1/2$, $D_2 \approx D_3 \approx D_4 \approx .4$.

In conclusion, we investigate unusual spectral statistics of random unitary matrices coming from a quantization of map (2). The most interesting case corresponds to $\alpha = m/n$ (when the map is an interval-exchange map) and $mN \equiv \pm 1 \bmod n$ [16]. When $N \to \infty$ local spectral statistics of these matrices tends to the semi-Poisson statistics with parameter β which can take arbitrary integer and half-integer values depending on the denominator of α and the time-reversal invariance of the map. The level compressibility is non-zero [17]. Eigenphases of M^n for all N have the Poisson distribution for non-symmetric matrices and the super-Poisson distribution for symmetric ones. When $mN \equiv \pm 1 \bmod n$ eigenphases of M are obtained by jumping over each n^{th} eigenphase of M^n thus explicitly realizing the *ad hoc* mechanism proposed in [15]. Eigenfunctions have fractal properties in momentum representation.

The model considered seems to be the first example of a new class of random matrix ensembles and its detailed analysis may lead to a better understanding of spectral statistics of pseudo-integrable plane polygonal billiards whose classical mechanics is described by interval-exchange maps.

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[1] O. Giraud, J. Marklof, S. O'Keefe, nlin.CD/0403033 (2004).
[2] E. Bogomolny, U. Gerland, and C. Schmit, Eur. Phys. J. B **19**, 121 (2001).
[3] M.L. Mehta, *Random matrices*, 2nd ed. (Academic Press, New York, 1991).
[4] T. Prosen and M. Robnik, J. Phys. A: Math. Gen. **27**, 8059 (1999).
[5] I. Dumitriu and A. Edelman, math-ph/0206043 (2002).
[6] In [1] the case $\Phi(k) = k^2$ was considered but the answer depends strongly on arithmetic nature of N due to special properties of the Gauss sums.
[7] M.V. Berry, N.L. Balasz, M.A. Tabor, and A. Voros, Ann. Phys. **122**, 26 (1979).
[8] E. Bogomolny, U. Gerland, and C. Schmit, Phys. Rev. E **59**, R1315 (1999).
[9] A. Erdélyi ed., *Higher Transcendental Functions*, vol. 3, 206 (Mc Graw Hill, New York, London, Toronto, 1955).
[10] E.Bogomolny and C. Schmit, in preparation (2004).
[11] S. Albeverio et al., *Solvable Models in Quantum Mechanics* (Springer, New York, 1988).
[12] For simplicity, degenerate cases like $x_{k+1} = x_k$ are ignored.
[13] Let $x_1 \le x_2 \le \ldots \le x_N$ form a Poisson sequence. Put in-between x_k and x_{k+1} a new point $y_k = x_k + (x_{k+1} - x_k)(1 - \cos\Theta_k)/2$ where Θ_k are independent random variables with uniform distribution between 0 and 2π. Then x_k and y_k together form the super-Poisson sequence [10].
[14] M.V. Berry and M. Tabor, Proc. Roy. Soc. Lond. **356**, 375 (1977).
[15] H. Hernández-Saltana, J. Flores, and T.H. Seligman, Phys. Rev. E **60**, 449 (1999).
[16] Other cases are less drastic and will be considered elsewhere [10].
[17] Semi-Poisson results $\chi = 1/n$ and $2/n$ for, resp. non-symmetric and symmetric matrices agree with the computation of χ from periodic orbits as in [1].