The traveling salesman problem, conformal invariance, and dense polymers

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(May 29, 2004)

We propose that the statistics of the optimal tour in the planar random Euclidean traveling salesman problem is conformally invariant on large scales. This is exhibited in power-law behavior of the probabilities for the tour to zigzag repeatedly between two regions, and in subleading corrections to the length of the tour. The universality class should be the same as for dense polymers and minimal spanning trees. The conjectures for the length of the tour on a cylinder are tested numerically.

The traveling salesman problem (TSP) is a classic problem in combinatorial optimization. The basic problem is, given a set of N marked points ("cities") in the plane, to find the closed path (a cycle or "tour") of shortest length that passes through each city once. In the random uniform TSP, the cities are chosen randomly, and are independently and uniformly distributed over some bounded domain $\mathcal{D}$, say a square, with mean density $\bar{n}$. While much effort has been expended on finding algorithms that produce the optimal tour for a given set of cities, for statistical physicists the interest of the problem is in the statistical properties of the optimal tour in the random uniform problem, to which we will refer simply as TSP [1,2]. In the past, much attention has been given to the total length ℓ of the optimal tour, which for N cities in a square behaves as $\ell(N) \sim \beta A/a$ with probability one as $N \to \infty$ [3,4], where A is the area of the square, $a = \sqrt{A/N} \equiv 1/\sqrt{\bar{n}}$ is the typical spacing of the cities, and β is a constant: $\beta \simeq 0.7120$ [5]. Finite N corrections to the mean length in a cube of dimension d with periodic boundary conditions have been studied [5].

In this paper, we consider geometrical properties of the optimal tour, other than the mean length for the square. These properties include the dependence of the mean length on the aspect ratio when the cities are distributed in a rectangle or on the surface of a cylinder. They also include the connectivity of the path, which we quantify by defining the number of times the tour alternately enters (or "zigzags" between) two specified subregions. We formulate conjectures based on statistical conformal invariance [6] of the properties of the optimal tour over length scales much larger than a. For clarity we separate these conjectures and call them TSPI, II, and III. The conjectures are: I) *conformal invariance* of the distribution of tours, and hence power-law behavior of the probability $P_k(r)$ for zigzagging k times between two regions that are a distance r apart (and much further from the boundary of $\mathcal{D}$),

$$P_k(r) \propto r^{-2x_k} \tag{1}$$

for $r \gg a$, for some exponents x_k. This implies that the optimal tour is a random fractal, and the x_k determine the fractal dimensions $D_k = 2 - x_k$ of various sets associated with the tour. These predictions, including the values of x_k, are *universal*; they do not depend on the precise distribution of cities, which might even be correlated, as long as the distribution is translationally and rotationally invariant, and any correlations are short-range; II) the mean length $\overline{\ell}$ of the optimal tour in a domain of area A that has a smooth boundary of length C has the form $\overline{\ell}/a = \beta\bar{n}A + \gamma C/a + \dots$ (a weaker version of this has been proved for the square [7], and implies that $\gamma > 0$). If we define $\Delta\ell/a \equiv \ell/a - \beta\bar{n}A - \gamma C/a$, then we expect

$$\overline{\Delta\ell}/a \sim -\frac{1}{6}\lambda c\chi \ln(|\mathcal{D}|/a) \tag{2}$$

as $N \to \infty$, where $\lambda > 0$ and c are constants, $|\mathcal{D}|$ is the diameter of $\mathcal{D}$, and χ is the Euler number of $\mathcal{D}$ ($\chi = 2 - 2h - b$, where h is the number of handles, b is the number of boundaries). If the tour is also constrained to zigzag k times between two fixed regions far from the boundary, then we expect in addition

$$\overline{\Delta\ell}/a \sim 2\lambda x_k \ln(r/a) \tag{3}$$

for $r/a \to \infty$ [here x_k are the same as in eq. (1)]. The values of β, γ, and λ are not universal; III) the universality class for the TSP is the same as that known as dense polymers, so the exact values are [8–11]

$$x_k = (k^2 - 4)/16, \quad c = -2. \tag{4}$$

After explaining these conjectures further, we study the length as in TSPII numerically by a transfer-matrix-like approach on a cylinder, testing the conformal symmetry proposed in I, and finding reasonable agreement with the quantitative conjectures in II and III.

There is a possible relation with the minimal spanning tree (MST) problem (given N cities, find the tree of smallest length with those cities as vertices; the cities are chosen at random as in the TSP). Analogs of parts of our TSPI and III have already been discussed for MSTs [12,13]. Here the analog of TSPIII would involve so-called uniform spanning trees (USTs) in place of dense polymers. The equivalence of USTs and MSTs in $d = 2$ is not excluded by rigorous results [12], and in our view is *supported* by existing numerics [13]. A tree in two dimensions is equivalent to a nonintersecting loop (take the boundary of a "thickened" tree), and the universality classes of USTs and dense polymers are the same in $d = 2$ [14] (and also the same as stochastic Loewner evolution at parameter value $\kappa = 8$ [15]). Hence in two dimensions we expect the universality classes for TSP and MSTs to be the same (the Hausdorff dimension studied in Ref. [13], which is equal to 5/4 for USTs [16], corresponds to our D_4).

We now further explain our predictions. First, we note that the optimal tour cannot cross itself, as that would allow a shorter tour to be found. For a typical set of cities, the tour comes close to every point in the region considered. In the scaling limit $a \to 0$ for a fixed A, the path becomes a random space-filling (Peano) curve with Hausdorff dimension $D = D_2 = 2$ [3]. For a tour within a simply-connected domain, the interior of the curve is well defined, and can be shaded black, leaving the exterior white. The interior and exterior form interlocking trees which appear statistically alike, except near the boundary of the domain. This implies a self-duality to the tour. One strongly suspects that such a random self-dual curve must be scale, and very likely also conformally, invariant, in a sense we must now explain.

Consider a square at arbitrary position well within the interior, with side L much less than the size of the domain, and with edges parallel to the coordinate axes. The tour passes through this square some number of times, entering and leaving on two sides (not necessarily distinct) of the square. We examine the number $n_x \geq 0$ of times the tour *crosses* the square in the x-direction, that is the number of segments of the tour that have one end on each of the two edges parallel to the y-axis. Similarly, we can consider the number of times n_y it crosses the square in the y direction. If $n_x > 0$, then $n_y = 0$, and *vice versa*. We expect that the joint probability distribution for n_x, n_y is concentrated at small values of n_x, n_y. Then the expectation $\overline{n_x}$ will be of order 1. As L/a increases, $\overline{n_x}$ will remain nonzero, as the tour must occasionally cross the square. (The possibility that the probability of $n_x = n_y = 0$ approaches 1 appears to be excluded, because of the requirement that the tour be a single cycle.) Thus we expect that the joint probability distribution for n_x and n_y is scale invariant for large L/a.

Consider two disks $\mathcal{A}$, $\mathcal{B}$ of radius r_0, the centers of which are separated by $r > 2r_0$. Let $P_k(r)$ be the probability for crossing (zigzagging) between the two disks precisely k times (k even); more precisely, it is the probability that k distinct connected segments of the tour lying outside both disks have one end on the boundary of each disk. By standard arguments, scale invariance of the crossing probabilities (applied to annuli concentric with $\mathcal{A}$ or $\mathcal{B}$) leads us to expect eq. (1) to hold as a function of r, for $r \gg a$ and a, $\mathcal{A}$, $\mathcal{B}$, fixed (and with r much less than the distance of $\mathcal{A}$ or $\mathcal{B}$ from the boundary of $\mathcal{D}$), and that $x_k > 0$ for $k > 2$. With $D_2 = 2$, it follows that $x_2 = 0$.

We may define similar crossing or "k-leg" probabilities for k odd by allowing the path to end at any two of the marked cities, such that the total length is minimized. For open paths, we define $P_k(r)$ for k odd as the probability that the path starts in $\mathcal{A}$ and ends in $\mathcal{B}$, crossing between them k times. We expect $x_1 < 0$, meaning that the optimal path will usually have its ends far apart. These definitions of the 2-point correlations can be easily extended to general n-point functions which are probabilities for the path to pass between n disks in some specified sequence. As for $n = 2$, we expect these to possess scaling limits as $a \to 0$, and these define a probability distribution on non-self-intersecting self-dual space-filling curves, which will be universal, and which we wish to characterize. The definition of the k-leg events can be generalized to the case when a disk is close to a boundary of the domain, which is assumed here to be straight. For $n = 2$, if the distances of $\mathcal{A}$ and $\mathcal{B}$ to the boundary are both much less than their separation r, then we expect $P_k(r) \propto (r/a)^{-2\widetilde{x}_k}$, with universal exponents $\widetilde{x}_k$ different from x_k (again, $\widetilde{x}_1 < 0$, and $\widetilde{x}_2 = 0$).

Conformal invariance is expected in scale-invariant statistical problems when they are defined by *local* processes. In the TSP, the length which is to be minimized is local in the sense that it is the sum of small local steps. There may be a concern that the global constraint of visiting each city once violates locality. However, such a condition is also present in dense polymers, so is not necessarily an issue.

For the following arguments, and for numerical purposes, it is convenient to consider the TSP on a cylinder, with circumference W (i.e. in the region $0 < x < W$ in the plane with a periodic boundary condition in the x-direction), and length L, so $A = LW$. The cities are uniformly distributed over this region. In this case, scaling arguments suggest that the probability that the path crosses at least k times between two regions within r_0 of the ends behaves for $N \to \infty$ as $P_k(W, L) \propto e^{-2\pi x_k L/W}$ for $L/W \gg 1$. If conformal invariance holds, then the exponents x_k here are the same as those defined above in the plane, by using a conformal mapping of the plane to the cylinder [17].

Now suppose that, instead of the tour being unconstrained, the tour is *required* to cross at least k times between the end regions, no matter what the positions of the cities. As the tour must minimize its length, for $k > 2$ the mean length $\overline{\ell_k}$ of the constrained tour will be greater than or equal to that of the unconstrained ($k = 2$) one, by an amount $\propto L$. In fact, from scale invariance we expect that, for each k, as $N \to \infty$ with L/W fixed, $\overline{\Delta\ell_k}/a \equiv \overline{\ell_k}/a - \beta\bar{n}A - 2\gamma W/a$ is proportional to a universal function of L/W, and this function is $\propto L/W$ as $L/W \to \infty$. We further expect that, for k even, the change in length of the constrained tour $(\overline{\ell_k} - \overline{\ell_2})/a$ is proportional to the logarithm of the probability for k crossings in the unconstrained case: $(\overline{\ell_k} - \overline{\ell_2})/a \sim -\lambda \ln P_k(W, L)$ as $a \to 0$. Thus for large L/W, we have

$$(\overline{\ell_k} - \overline{\ell_2})/a \sim 2\pi\lambda x_k L/W, \tag{5}$$

where $\lambda \geq 0$ is a non-universal constant, but is the same for all k, and we expect this form to hold for k odd also [conformally mapping this to the plane yields eq. (3)]. We expect similar behavior also for the unconstrained tour, and so we define a universal constant c by

$$\overline{\Delta\ell_2}/a \sim -\frac{\pi\lambda c}{6} L/W. \tag{6}$$

One expects then $\lambda > 0$ and $c \leq 0$.

The appearance here of only a single non-universal constant λ, and the various scaling forms suggested, require further explanation. We are using an analogy with conformal field theories (CFTs) in two-dimensional critical phenomena. There, the variation of the free energy (the logarithm of the partition function) with respect to the metric of the space defines the stress tensor of the theory. This leads to universal scaling forms for the subtracted free energy in various geometries or with constraints or sources imposed, as in a correlation function [17–19]. Our central conjecture is that, up to a non-universal factor λ, the length behaves as the free energy in some CFT (c is then the central charge). The reason is that the length ℓ/a is the integral over space of a local density, and in many CFTs (including the dense polymer theory considered below) the stress tensor is (up to a factor) the only local spin-2 operator of the correct symmetry and lowest scaling dimension that can appear as the variation of the length with metric. As free energy differences also determine probabilities for events, the relation of the length differences with the probability also follows. By such arguments in the plane, or in other smooth domains (including non-planar ones, such as the sphere or torus), one obtains the scaling form (2) [19]. For geometries in which χ is zero, the term in eq. (2) is replaced by λ times a universal scale-invariant functional of the geometry [18], as for the cylinder in eq. (6).

There is a probability distribution for non-self-intersecting self-dual space-filling curves that arises from statistical mechanics. This is the Nienhuis dense-polymer phase that originally arose from the low-temperature phase of the O(m) loop model at $m \to 0$ [8]. It is a model of closed loops on the honeycomb graph, on which each edge of the graph is occupied at most once. The partition function of the model is $Z = \sum K^E m^{\mathcal{L}}$, where the sum is over all such loop configurations, K plays the role of inverse temperature, E is the number of edges occupied, and $\mathcal{L}$ is the number of distinct loops. When $m = 0$, $Z = 0$, but the partition function for a single closed loop on the honeycomb graph can be obtained by differentiating, $Z' = \partial Z/\partial m|_{m=0}$, and then probabilities $P_k(r)$ can be defined analogously to those above. The model has a critical point at $K = K_c = (2 + \sqrt{2})^{-1/2}$, and the region $K_c < K < \infty$ is a conformally-invariant low-temperature phase in which the scaling exponents are independent of K. The scaling limit of the probability distribution is highly robust: no local perturbations are relevant, except for that of allowing the loops to cross [20]. Therefore we find it natural to propose TSPIII, that the TSP and dense-polymer universality classes are the same. For dense polymers, $c = -2$, and the k-leg exponents are given by eq. (4) for the bulk, and by [10]

$$\widetilde{x}_k = k(k-2)/8 \tag{7}$$

for the boundary.

We have studied the length of a tour on a cylinder by extensive numerical calculations. The approach is similar to transfer matrix methods in statistical mechanics. The tour on a cylinder is "grown" starting from one end of the cylinder. The distance y along the cylinder plays the role of time and will be denoted t from here on. We set $a = 1$. The transfer process starts with a city p_1 at $x_1 = 0$, $t_1 = 0$ from which k lines emerge (we say it has valence k), and terminates with a similar city p_N at some x_N and t_N. The remaining $N-2$ cities p_i have coordinates x_i, t_i, and valence 2. The variables x_i and differences $t_i - t_{i-1}$ are all independent for $i > 1$. The x_i's are chosen uniformly in $[0, W]$, while the differences $t_i - t_{i-1} > 0$ are exponentially distributed with mean $1/W$ (this reproduces the Poisson process with density 1).

For a given set of cities, and for each time t_i, the information needed to complete a tour and find the optimal one consistent with the constraint is encoded in a set of states, each of which has a weight (length of path) associated with it. The transfer matrix will evolve these states and weights to t_{i+1}. A state consists of a list of the M cities p_{i_a}, $a = 1, \ldots, M$ $(i_1 < i_2 < \cdots i_M \leq i)$, whose valence has not yet been satisfied (by connection to other cities), plus connections among these cities. Of the M cities, there may be some that are not connected to any other one, some that are connected to just one other, and there is always one distinguished set of at most k cities that are connected to each other. To each of these states, we can associate the set of paths (with minimum total length and no closed loops) that form the connections and satisfy the valence of all the cities p_j, $j \leq i$, not in the list; the distinguished set of k cities are connected to the initial city p_1 (which itself will be in the list during the early stages). The length of this set of paths is the weight associated with the state.

When a city p_{i+1} of valence 2 is added, the states and their weights at t_{i+1} are related to those at t_i by one of three moves: (1) p_{i+1} may be unconnected to any other city; (2) p_{i+1} may be directly connected to one other city p_{i_a} in the list; (3) p_{i+1} may be directly connected to two

	$W=2$	$W=3$	$W=4$	$W=5$	$W=6$
$\overline{\ell_2}/(LW)$	1.07497(2)	0.8330(1)	0.7571(1)	0.7330(2)	0.720(2)
$\overline{\ell_1}/(LW)$	0.7457(4)	0.7175(3)	0.715(1)	0.713(2)	
$\overline{\ell_3}/(LW)$	1.5286(1)	1.075(1)	0.872(1)	0.80(1)	0.76(1)
λc	−2.39	−3.33	−2.98(1)	−2.04(2)	−2.03(15)
λx_1	−0.15(1)	−0.20(2)	−0.15(2)	−0.10(5)	
λx_3		0.30(2)	0.33(2)	0.29(1)	0.26(2)

TABLE I. Length $\overline{\ell_k}$ of tour per unit area, for each width W, extrapolated to $i_{max} \to \infty$, with values of λc, λx_k deduced from ℓ_k for pairs W, $W-1$. Data for $W=1$ are not shown.

other cities p_{i_a}, p_{i_b}, so it will not appear in the list (if p_{i_a}, p_{i_b} were connected to each other at time t_i, this move is forbidden). In moves (2) and (3), cities p_{i_a} or p_{i_b} whose valence becomes satisfied disappear from the list, and the connections of other cities change accordingly. The weight of a state at time t_{i+1} is equal to that of the state it came from at time t_i plus the length of the connections that have been added. If a state at time t_{i+1} can arise from more than one state at time t_i, its weight is taken as the minimum of the various possibilities. At the final time t_N, a k-valent city is added (using similar moves), and we take the length ℓ_k to be the length of the state at t_N in which all valences are satisfied (the condition of not connecting already-connected cities is dropped at this step).

Clearly, with the above moves, the size of the state space grows without limit as $i \to \infty$. In the spirit of heuristic (local optimization) algorithms we deal with this by discarding all states at each time t_i that contain a city p_{i_a} with $i_a < i - i_{\max}$. The transfer matrix is then finite dimensional, with a size that grows exponentially with $i_{\max}$. This truncation means that steps with a long t-component are suppressed. For $i_{\max} \gg W$, they would be rare on the optimal path anyway.

We have produced results for $\ell_k(i_{\max})$ with $1 \leq W \leq 6$ and $4 \leq i_{\max} \leq 8$ or 9. In each case, we find the optimal tour for $N = 10^5 \cdot W$ cities, and then average over 10 independent samples. The systematic error due to the finite value of N is negligible as compared to the statistical error (sample-to-sample fluctuations). To extrapolate to the $i_{\max} \to \infty$ limit we use the Ansatz $\overline{\ell_k}(i_{\max})/(LW) = \overline{\ell_k}(\infty)/(LW) + Ae^{-Bi_{\max}}$, A and B being k-dependent constants, which matches the data very well, especially when $i_{\max} > W$. In Table I we display the extrapolated data for $k = 2, 1, 3$. Estimates for λc (λx_k, $k = 1, 3$) were based on eq. (6) [eq. (5)], using $\overline{\ell_2}$ ($\overline{\ell_k} - \overline{\ell_2}$) for pairs W, $W-1$ (γ was neglected). We obtained a value $\beta = 0.7119(3)$, in good agreement with Ref. [5]. Our final best estimates are $\lambda c = -2.0(2)$, $\lambda x_1 = -0.15(5)$, and $\lambda x_3 = 0.28(4)$. We note that the values $\lambda = 1$, $c = -2$, $x_1 = -3/16$, and $x_3 = 5/16$ lie within the error bars. At present, we have no explanation for why λ should be close to one.

Finally, for TSP in a three-dimensional region of fixed thickness L_3 in the z-direction and a domain $\mathcal{D}$ of area $A \gg L_3^2$ in the x–y plane, projecting the tour into the plane produces a two-dimensional problem, but the projected tour will occasionally cross itself. If our conjectures hold for $L_3 = 0$, then by Ref. [20] any small, positive L_3 causes a crossover to a "Goldstone phase". It then seems very likely that TSP in all $d > 2$ is also described by the Goldstone phases, which would mean that the segments of the tour in a box of side L have Hausdorff dimension 2, and behave as Brownian walks on large scales. Conformal invariance would be lost in these cases.

We thank P.W. Jones, D.L. Stein, and A.A. Middleton for useful discussions. This work was supported by NSF grant no. DMR-02-42949 (NR) and by the DOE (HS).

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