Interaction-induced crossover versus finite-size condensation in a weakly interacting trapped 1D Bose gas

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We discuss the transition from a fully decoherent to a (quasi-)condensate regime in a harmonically trapped weakly interacting 1D Bose gas. By using analytic approaches and verifying them against exact numerical solutions, we find a characteristic crossover temperature and crossover atom number that depend on the interaction strength and the trap frequency. We then identify the conditions for observing either an *interaction-induced* crossover scenario or else a *finite-size* Bose-Einstein condensation phenomenon characteristic of an *ideal* trapped 1D gas.

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One-dimensional (1D) Bose gases are remarkably rich physical systems exhibiting properties not encountered in 2D or 3D [1, 2, 3]. Here we study the 1D model of bosons interacting via a repulsive delta-function potential, which plays a fundamentally important role in quantum many-body physics. The reason is that the model is exactly solvable [2, 3] and it is now experimentally realizable with ultracold alkali atoms in highly anisotropic trapping potentials (see Ref. [4] for a review). This means there are unique opportunities for accurate tests of theory that were previously unavailable, in turn leading to the development of fundamental knowledge of interacting many-body systems in low dimensions.

In this paper, we analyze the properties of the 1D Bose gas in the weakly interacting regime, where the dimensionless interaction parameter $\gamma = mg/(n\hbar^2)$ is small, n being the linear density, m the atom mass, and g the 1D coupling constant. This is opposite to Girardeau's regime of "fermionization" [1] achieved in the limit of strong interactions and subject of many recent studies [5]. Our motivation for the study of the weakly interacting regime is to reveal the nature of the transition to a Bose-condensed state in a harmonically trapped system.

For a uniform weakly interacting 1D Bose gas, one has a smooth interaction-induced crossover to a quasi-condensate which is a Bose-condensed state with a fluctuating phase. This occurs when the temperature T becomes smaller than $T_d\sqrt{\gamma}$ [6, 7, 8, 9], where $T_d = \hbar^2 n^2/2m$ is the temperature of quantum degeneracy (in energy units, $k_B = 1$). For a harmonically trapped 1D gas with weak interactions a similar crossover scenario is expected [7]. However, due to the presence of the trapping potential the interaction-induced crossover enters into a competition with Bose-Einstein condensation (BEC) predicted to occur in the ideal gas limit [10] as a macroscopic occupation of the ground state. For a given atom number N, this condensation phenomenon occurs at temperature $T_C \simeq N\hbar\omega/ln(2N)$. It is a purely finite-size effect and disappears in the thermodynamic limit [11] where N tends to infinity while the peak density n_0 is kept constant (this implies that the trap oscillation frequency ω tends to zero in such a way that $N\omega = \text{const}$). The interaction-induced crossover to a quasi-condensate, on the other hand, persists in the thermodynamic limit.

Thus, for sufficiently weak confinement one expects to observe an interaction-induced crossover to a quasi-condensate, rather than a finite-size BEC. The situation is reversed for strong confinement. Here, we identify the parameters of the interaction-induced crossover and find the conditions that enable the realization of either of these two competing scenarios.

We start by briefly reviewing the physics of a uniform 1D Bose gas in the thermodynamic limit, in the case of very weak interactions $\gamma \ll 1$. For $T \ll T_d\sqrt{\gamma}$, the gas is in the quasi-condensate (Gross-Pitaevskii) regime where the density fluctuations are suppressed and the gas is coherent on a distance scale smaller than the phase coherence length: Glauber's local pair correlation function is reduced below the ideal gas level of 2 and is close to 1 [6, 7, 8, 9]. In this regime the chemical potential is positive and well approximated by $\mu \simeq gn$. For $T \gg T_d\sqrt{\gamma}$, the gas is in the fully decoherent regime: interactions between the atoms have a small effect on the equation of state and the local pair correlation is close to that of an ideal Bose gas [6]. This regime contains the quantum decoherent domain $T_d\sqrt{\gamma} \ll T \ll T_d$. In the decoherent regime, the chemical potential μ is negative and the equation of state is well approximated by that of the ideal Bose gas:

$$n = \int_{-\infty}^{\infty} \frac{dk/(2\pi)}{e^{(\hbar^2 k^2/2m-\mu)/T} - 1} = \sqrt{\frac{mT}{2\pi\hbar^2}} \sum_{j=1}^{\infty} \frac{e^{j\mu/T}}{j^{1/2}}. \quad (1)$$

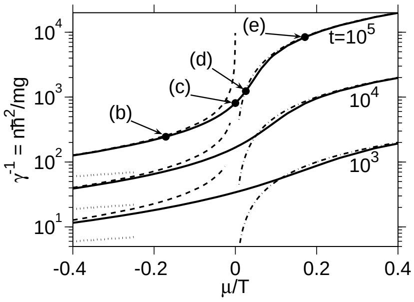

FIG. 1: Equation of state of the uniform weakly-interacting 1D Bose gas for three different values of the temperature parameter $t = 2T\hbar^2/mg^2$. The exact numerical result (solid line) is compared with the behavior in the quasi-condensate regime (dash-dotted lines) and with the ideal Bose gas result of Eq. (1) (dashed lines). The straight dotted lines correspond to the classical (Boltzmann) ideal gas.

The crossover between the decoherent and the quasi-condensate regimes ($T \sim T_d\sqrt{\gamma}$) corresponds to the density of the order of $n_{\rm co} = (mT^2/\hbar^2 g)^{1/3}$. Using the crossover density $n_{\rm co}$ is convenient for analyzing the properties of the gas at a constant temperature and varying n. In this sense, the quantum decoherent regime corresponds to $n_d \ll n \ll t^{1/6}n_d \simeq n_{\rm co}$, where $t = T/T_d\gamma^2 = 2\hbar^2 T/mg^2$ is a dimensionless temperature parameter which is independent of the density and is large, and $n_d = \sqrt{mT}/\hbar$ is the density of quantum degeneracy at a given T. The width of the quantum decoherent region in terms of densities increases with t.

In Fig. 1 we illustrate the properties of the weakly interacting uniform gas by plotting the linear density as a function of the chemical potential for three different values of the temperature parameter t. The exact numerical results [6] based on the finite-temperature solution [3] to the Lieb-Liniger model [2] are compared with both the ideal Bose gas equation of state (1) in the region of $\mu < 0$ and with the quasi-condensate equation of state corresponding to $\mu \simeq gn > 0$. For a given temperature, the crossover from the decoherent regime to the quasi-condensate corresponds to μ going from negative to positive. We obtain $n(\mu = 0, T) \simeq 0.6n_{\rm co}$ within 20% accuracy as long as $t > 10^3$. Note that the values of t as high as 10^3 are required to ensure that the gas is highly degenerate at the crossover.

We now turn to the analysis of a harmonically trapped 1D gas and find the crossover temperature $T_{\rm co}$ and crossover atom number $N_{\rm co}$ around which the gas enters the quasi-condensate regime. For small trap frequencies ω, the density profile of the gas can be described using the local density approximation (LDA) [7]. In this treatment, the 1D density $n(z)$ as a function of the distance z from the trap centre is calculated using the uniform gas equation of state in which the chemical potential μ is replaced by its local value $\mu(z) = \mu_0 - m\omega^2 z^2/2$, where μ_0 is the global chemical potential. Within the LDA, the uniform results remain relevant and imply, in particular, that the gas enters the quasi-condensate regime in the trap centre once μ_0 changes sign. In addition, as long as the peak density $n_0 = n(0)$ fulfills the condition $n_0 \ll n_{\rm co}$ the entire gas is in the decoherent regime and the equation of state is well approximated by Eq. (1) in which n and μ are replaced by $n(z)$ and $\mu(z)$. Integrating $n(z)$ over z and taking the sum over j gives a relation between the total atom number and μ_0:

$$N = -T/(\hbar\omega)\ln(1 - e^{\mu_0/T}), \quad (\mu_0 < 0). \tag{2}$$

As mentioned above, for very large values of t the crossover to the quasi-condensate occurs under conditions where the gas is highly degenerate in the centre, with $n_0 \gg n_d = \sqrt{mT}/\hbar$. Assuming that this is the case and taking into account that the degeneracy condition is equivalent to $|\mu_0|/T \ll 1$, Eq. (2) can be rewritten as

$$N \simeq T/(\hbar\omega)\ln\left(T/|\mu_0|\right). \tag{3}$$

Under these conditions, as Eq. (1) reduces to $n \simeq \sqrt{mT^2/2\hbar^2|\mu|}$ for $|\mu| \ll T$, the density profile develops a sharp central peak which is well approximated by

$$n(z) \simeq \sqrt{\frac{mT^2}{2\hbar^2}}\frac{1}{\sqrt{|\mu_0| + m\omega^2 z^2/2}}, \tag{4}$$

and extends up to distances $|z| \lesssim R_T = \sqrt{2T/m\omega^2}$.

The analysis made above is valid as long as $n_0 \ll n_{\rm co}$. Using Eq. (4) and the expression for $n_{\rm co}$, the condition $n_0 \ll n_{\rm co}$ can be rewritten as

$$|\mu_0| \gg m^{1/3}(gT/\hbar)^{2/3}. \tag{5}$$

Using Eq. (3) to relate μ_0 to the total atom number, Eq. (5) leads to the condition that the gas is in the decoherent regime as long as $N \ll N_{\rm co}$, where

$$N_{\rm co} \simeq T/(\hbar\omega)\ln\left(\hbar^2 T/mg^2\right)^{1/3} = (T/3\hbar\omega)\ln(t/2) \tag{6}$$

is the characteristic atom number at the crossover. As we mentioned earlier, one should have $t \gg 10^3$ for obtaining a highly degenerate gas at the crossover. Under this condition, Eq. (6) can be approximately inverted to yield, for a given N, a crossover temperature

$$T_{\rm co} \simeq N\hbar\omega/\ln\left(N\hbar^3\omega/mg^2\right)^{1/3}. \tag{7}$$

We emphasize that our results are obtained within the LDA which is valid if the characteristic correlation length l_c of density-density fluctuations is much smaller than the typical length scale L of density variations. The correlation length is $l_c \simeq \hbar/\sqrt{m|\mu_0|}$ in both the quantum decoherent and quasi-condensate regimes [6, 7]. Approaching the crossover from the decoherent regime we replace $|\mu_0|$

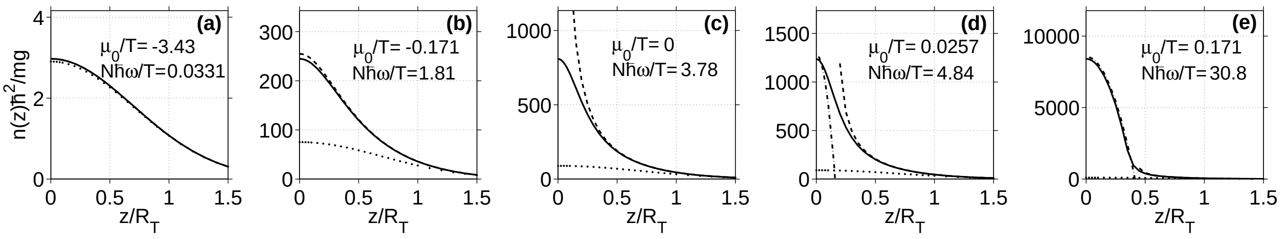

FIG. 2: Density profiles of a 1D Bose gas in a harmonic trapping potential for five different values of the ratio μ_0/T and a fixed value of the temperature parameter $t = 2\hbar^2 T/mg^2 = 10^5$. The exact numerical solution (solid line) is compared with the ideal Bose gas distribution (dashed line), classical Boltzmann distribution (dotted line), and Thomas-Fermi distribution in the Gross-Pitaevskii regime (dashed-dotted line). The resulting values of the dimensionless ratio $N\hbar\omega/T$, following the exact solutions, are also shown. The distance from the trap center z is in units of $R_T = (2T/m\omega^2)^{1/2}$. All calculations are done within the LDA using the equation of state for the homogeneous gas shown in Fig. 1, with μ_0 and $n(0)$ in (b)-(e) being the same as μ and n indicated by the points (b)-(e) in Fig. 1.

by the rhs of Eq. (5), while approaching it from the quasi-condensate regime we use $\mu_0 \simeq gn_{\rm co}$. In both cases, one obtains $l_c \simeq \hbar^{4/3}/(m^2 gT_{\rm co})^{1/3}$. The length scale L can be estimated as the distance from the trap center where the density is halved compared to the peak density n_0. Approaching the crossover from the decoherent side, Eq. (4) gives $L \simeq \sqrt{|\mu_0|/m\omega^2} \simeq (gT_{\rm co}/m\hbar\omega^3)^{1/3}$. On the quasi-condensate side, we use the Thomas-Fermi parabola and obtain $L \simeq \sqrt{2n_{\rm co}g/m\omega^2}$, which gives approximately the same result. One then easily sees that the condition of validity of the LDA, $l_c \ll L$, is reduced to

$$\omega \ll \omega_{\rm co} \equiv (mg^2T^2/\hbar^5)^{1/3}. \tag{8}$$

If this inequality is not satisfied then the LDA breaks down and one has to take into account the discrete structure of the trap energy levels. In this case, analytic approaches incorporating both the finite-size effects and small but finite interaction strength are absent in the vicinity of the transition to a quasi-condensate, and we adopt the ideal gas treatment of Ref. [10]. For a fixed temperature, this treatment predicts a finite-size BEC at a critical atom number $N_C = T/(\hbar\omega)\ln(2T/\hbar\omega)$. It is clear that the finite-size BEC phenomenon will prevail the interaction-induced crossover scenario if $N_C < N_{\rm co}$. In fact, the opposite inequality, $N_C > N_{\rm co}$, is equivalent to that of Eq. (8), which makes our analysis self-consistent and implies that the condition of validity of the LDA, $\omega \ll \omega_{\rm co}$, serves as the simultaneous criterion for observing the interaction-induced crossover, while the opposite condition corresponds to finite-size condensation. At a constant N, the criterion for observing the interaction-induced crossover can be obtained from Eq. (8) by replacing T with $N\hbar\omega/\ln(2N)$. The opposite criterion leading to the finite-size BEC has been previously found in Ref. [12] from the condition $gn_0 \ll \hbar\omega$.

In the following, we analyze the properties of the interaction-induced crossover, subject to inequality (8). Since $t \gg 10^3$ in the regime of interest, Eq. (6) written as $T_{\rm co} = 3N\hbar\omega/\ln(t_{\rm co}/2)$ shows that the crossover temperature is lower than the characteristic temperature of quantum degeneracy of a harmonically trapped gas $N\hbar\omega$. Thus, $T_{\rm co}$ represents a more accurate and lower estimate of the crossover temperature to the quasi-condensate regime compared to the inequality $T \ll N\hbar\omega$ given in Ref. [12]. For extremely large values of t, the present treatment identifies an intermediate temperature interval $T_{\rm co} \ll T \ll N\hbar\omega$ which accommodates the decoherent quantum regime. Here the gas is degenerate and is well described within the ideal Bose gas approach.

Fig. 2 shows density profiles for different values of the chemical potential at a fixed temperature parameter $t = 2\hbar^2T/mg^2 = 10^5$. The graph (e) corresponds to the quasi-condensate regime. The graph (c) shows the density profile at the crossover, and we find that the corresponding atom number, $N \simeq 3.78T/\hbar\omega$, is in good agreement with the value $N_{\rm co} \simeq 3.61T/\hbar\omega$ predicted by Eq. (6). The decoherent regime is clearly seen in graphs (a) and (b). Although the inequality $T_{\rm co} \ll N\hbar\omega$ is barely satisfied, the features of the quantum decoherent regime are seen in (b): the density profile is described to better than 10% by the ideal Bose gas approach and differs strongly from the classical Boltzmann distribution.

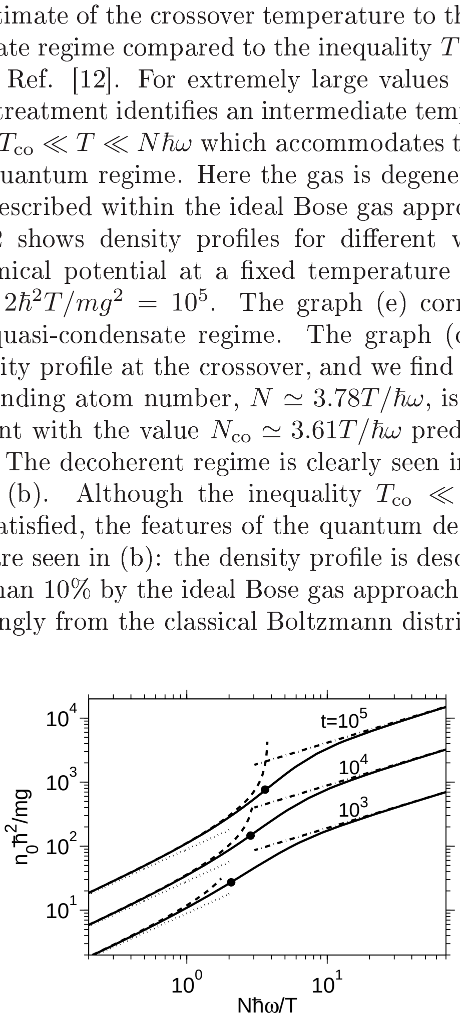

FIG. 3: Peak density n_0 (in units of $mg/\hbar^2$) of a trapped gas versus $N\hbar\omega/T$ for three values of $t = 2\hbar^2T/mg^2$. The three black dots show the respective crossover values of $N_{\rm co}\hbar\omega/T$ from Eq. (6). The different lines are as in Fig. 1.

To provide a better connection with experimentally measurable quantities we plot in Fig. 3 the peak density n_0 versus $N\hbar\omega/T$ for three different values of the temperature parameter t. In all cases we give the comparison with the classical Boltzmann gas, the ideal Bose gas, and the quasi-condensate predictions. The ideal Bose gas prediction connects the Boltzmann behavior $n_0 = N\omega\sqrt{m/2\pi T}$ to the degenerate behavior $n_0 = (\sqrt{mT}/\hbar)\exp(N\hbar\omega/2T)$, whereas in the quasi-condensate regime n_0 scales as $\propto N^{2/3}$. The scaling of the peak density n_0 as a function of N and the sequence of changes between power laws and an exponential can serve as a signature of the transitions between different regimes. This includes the quantum decoherent regime, which becomes more pronounced when increasing the parameter t and is already seen for $t = 10^5$.

The sufficient condition for realizing the 1D regime in a harmonically trapped, weakly interacting gas is $T \ll \hbar\omega_\perp$, where $\omega_\perp$ is the transverse oscillation frequency. If the oscillator length $l_\perp = \sqrt{\hbar/m\omega_\perp}$ is much larger than the 3D scattering length a, the 1D coupling is given by $g \simeq 2\hbar^2 a/ml_\perp^2$ [13]. The condition for the interaction-induced crossover, $\omega \ll \omega_{\rm co}$, can then be rewritten as

$$\omega \ll \omega_\perp (T/\hbar\omega_\perp)^{2/3}(a/l_\perp)^{2/3}. \tag{9}$$

Taking $\omega_\perp/2\pi$ in the range from 1 to 30 kHz and $T \simeq 0.2\hbar\omega_\perp$ (T is ranging from 10 to 300 nK), one can see that for most of the alkali atoms with typical scattering lengths in the range of few nanometers, the inequality (9) is well satisfied with ω of a few Hertz commonly used in practice. Thus, the conditions for realizing the interaction-induced crossover are relatively easy to satisfy, unless the scattering length is extremely small ($a < 0.1$ nm). On the other hand, the condition to observe the quantum decoherent regime before the interaction-induced crossover is more demanding as it requires, in addition to Eq. (9), a very large value of the parameter t. Rewriting the 1D inequality $T \ll \hbar\omega_\perp$ as $a \ll l_\perp/\sqrt{2t}$ we immediately see that even at $t = 10^5$, where one only starts to see the features of this regime, one needs to use light atoms (large $l_\perp$) and/or a very small scattering length in order to satisfy $a \ll 2\times10^{-3} l_\perp$.

A favorable system for fulfilling these conditions is a 1D gas of ^{7}Li atoms in the $F = 1, m = -1$ hyperfine state, where the scattering length can be tuned from very large to extremely small values using an open-channel dominated Feshbach resonance [14]. By taking, for example, $\omega/2\pi \simeq 4$ Hz, $\omega_\perp/2\pi \simeq 4$ kHz, $T \simeq 0.2\hbar\omega_\perp$ (40 nK), and varying a from 20 to 0.2 nm one can increase t from 60 to 6×10^5 and see how a direct interaction-induced crossover from a classical gas to a quasi-condensate regime transforms to accommodate the intermediate quantum decoherent regime. The same system can also be used to observe the finite-size BEC scenario, which requires the inequality opposite to Eq. (9) and hence a reduction of the scattering length to $a \simeq 0.01$ nm.

In conclusion, we have identified the conditions for realizing either a finite-size BEC phenomenon or an interaction-induced crossover to a coherent, quasi-condensate state in a harmonically trapped 1D Bose gas. In the later case, we distinguish between a direct crossover from the classical decoherent regime and a crossover through the intermediate quantum decoherent regime. Furthermore, one can expect that the physics of the interaction-induced crossover remains approximately valid for $T \sim \hbar\omega_\perp$, where the gas is no longer in the 1D regime but is near the 3D-1D boundary. This conjecture is supported by the results of recent experiments [15, 16]. In Ref. [15] a gas at $T \simeq 2\hbar\omega_\perp$ was produced with a density profile well described within a degenerate ideal gas approach. This means that the crossover to a quasi-condensate was likely to involve the features of the decoherent quantum regime. Finally, we note that the interaction-induced crossover through a well pronounced decoherent quantum regime would be easier to produce in a quartic or box-like potential [17].

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