Deformations of the Tracy-Widom distribution

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In random matrix theory (RMT), the Tracy-Widom (TW) distribution describes the behavior of the largest eigenvalue. We consider here two models in which TW undergoes transformations. In the first one disorder is introduced in the Gaussian ensembles by superimposing an external source of randomness. A competition between TW and a normal (Gaussian) distribution results, depending on the spreading of the disorder. The second model consists in removing at random a fraction of (correlated) eigenvalues of a random matrix. The usual formalism of Fredholm determinants extends naturally. A continuous transition from TW to the Weibull distribution, characteristic of extreme values of an uncorrelated sequence, is obtained.

I. INTRODUCTION

In the beginning of the 90’s, Tracy and Widom (TW) derived the probability distribution of the largest eigenvalue of random matrices belonging to the three Gaussian ensembles, the orthogonal (GOE), the unitary (GUE) and the symplectic (GSE)[1]. Few years later, Baik, Deift and Johansson proved that the longest increasing subsequence of a random permutation fluctuates as the largest GUE eigenvalue[2] and triggered TW applications in combinatorics and other areas such as growing processes[3, 4, 5, 6] (see [7] for a review). The main ingredient in their derivation was the discovery that the formalism of random matrix theory based on Fredholm determinants and Painlevé equations[8] which at the bulk of the spectrum is associated to integral equations with a sine-kernel, at the border of the spectrum it is associated to integral equations with an Airy-kernel. It is by now accepted that these distributions belong to universal class of extreme values of correlated sequences. Deviations from the TW have also been observed and studied. It has been found, for instance, that growing processes in random media may show regimes in which TW compete with other distributions[9].

In this general context it is important to establish links between TW and known universal distributions of extreme values of uncorrelated sequences. In this last case, namely for a sequence of i.i.d. random variables, the probability that the extreme value is less than a given value $t$ is $\exp \left[ -\int_t^\infty \rho(x) dx \right]$, where $\rho(x)$ is their density which extends till $x = T[10]$. Depending on the asymptotic behavior of the function $\rho(x)$, this probability distribution takes three forms. With $T = \infty$, it becomes the Gumbel distribution $\exp \left[ -\exp(-y) \right]$ if $\rho$ has a fast exponential decay and, if it decays with a power $\mu + 1$, it is the Fréchet distribution $\exp(-1/y^\mu)$ (in both cases $y$ is a properly scaled variable). With $T$ finite, that is if $\rho$ has a bounded support, it is the Weibull distribution $\exp(-y)$
where \( y = \int_T^T \rho(x)dx \) has density one.

Largest eigenvalues of non-Gaussian ensembles (Wishart matrices) have been the subject of recent investigations \cite{11}. It has been found that if the matrix elements are taken from a distribution with finite moments then the TW holds \cite{11}. Considering instead the case in which the distribution of the matrix elements have long tails, it has been proven that when the second moment diverges, the largest eigenvalue and the largest matrix element follow a Fréchet distribution with the same power \( \mu \leq 2 \) \cite{12}. This result has more recently been extended til \( \mu = 4 \) \cite{13}.

Models to describe deviations from TW have been discussed by K. Johansson \cite{14}. In one model, he studies the behavior of the largest eigenvalue of a matrix model \cite{15} and finds that it is described by a kernel that goes from a Poisson kernel (see Eq. (31) below) with an exponential density to the RMT Airy kernel. Accordingly, the distribution of the largest eigenvalue goes from Gumbel to TW. In another model, a deformed GUE ensemble is considered in which the eigenvalue density fluctuates in such a way that the largest eigenvalue distribution goes from TW to Gaussian.

Following similar lines to those of \cite{14} the purpose of this note is to investigate other models that describe deformations of the TW. The first model results from superimposing to the Gaussian fluctuations an external source of randomness \cite{16}. This causes the eigenvalue semi-circle density to fluctuate and results in features common to growing processes in random media. The second model is based on the recent recognition that the mathematical structure of random matrix theory (RMT) also describes the statistical properties of the eigenvalues of spectra when a fraction of eigenvalues is randomly removed \cite{17}. As this operation reduces correlations, it describes intermediate statistics between RMT and Poisson statistics. Focusing on eigenvalues at the edge of the spectrum we show here that this leads to a transition from TW to a Weibull distribution.

Let us start by recalling known facts about the eigenvalues of these Gaussian ensembles including some recent results regarding the behavior of their largest values. It is well known, for instance, that, to leading order, their eigenvalue density is given by the Wigner’s semi-circle law

\[
\rho(\lambda) = \begin{cases} 
\frac{1}{2\pi\sigma^2} \sqrt{4N\sigma^2 - \lambda^2}, & |\lambda| < 2\sigma \sqrt{N} \\
0, & |\lambda| > 2\sigma \sqrt{N} 
\end{cases}
\]

(2) where \( \sigma = 1/\sqrt{4\alpha} \) is the variance of the off-diagonal matrix elements. To study the behavior of the largest eigenvalues in the limit of large matrix size \( N \), one introduces the scaling

\[
\lambda = (2\sqrt{N} + \frac{s}{N^{1/6}})\sigma
\]

(3) that substituted in (2) leads to the \( N \)-independent density
\[ \rho(s) = \begin{cases} \frac{1}{\pi} \sqrt{-s}, & s \leq 0 \\ 0, & s > 0 \end{cases} \]  

(4)

at the edge of the spectrum. In the scaled variable \( s \), the probabilities \( E(k, s) \) with \( k = 0, 1, 2, \ldots \) that the infinite interval \((s, \infty)\) has, respectively \( k \) eigenvalues, are obtained from the generating function \( G(s, z) \) through the relation \[ G(s, z) = \sum_{n=0}^{\infty} (-1)^n (z - 1)^n E(n, s) \]  

(5)

such that

\[ E(n, t) = \frac{(-1)^n}{n!} \left[ \frac{\partial G(s, z)}{\partial z^n} \right]_{z=1}. \]  

(6)

For the three symmetry classes, the generating functions \( G_\beta(s, z) \) with \( \beta = 1, 2 \) and 4 have been derived. Starting with the unitary case, \( G_2(s, z) \) can be identified with the Fredholm determinant associated to the integral operator acting on the interval \((s, \infty)\) with kernel

\[ K(x, y) = \frac{\text{Ai}(x)\text{Ai}'(y) - \text{Ai}(y)\text{Ai}'(x)}{x - y} \]  

(7)

where \( \text{Ai}(s) \) is the Airy function. \( G_2(s, z) \) is given by

\[ G_2(s, z) = \exp \left[ -\int_s^\infty (x - s)q^2(x, z)dx \right] \]  

(8)

where \( q(s, z) \) satisfies the Painlevé II equation

\[ q'' = sq + 2q^3 \]  

(9)

with boundary condition

\[ q(s, z) \sim \sqrt{z} \text{Ai}(s) \text{ when } s \to \infty. \]  

(10)

For GOE \( (\beta = 1) \) and GSE \( (\beta = 4) \) the generating functions are

\[ [G_1(s, z)]^2 = G_2(s, \tilde{z}) \frac{z - 1 - \cosh \mu(s, \tilde{z}) + \sqrt{\tilde{z}} \sinh \mu(s, \tilde{z})}{z - 2} \]  

(11)

and

\[ [G_4(s, z)]^2 = G_2(s, z) \cosh^2 \frac{\mu(s, z)}{2} \]  

(12)

where \( \tilde{z} = 2z - z^2 \) and

\[ \mu(s, z) = \int_s^\infty q(x, z)dx. \]  

(13)

We remark that the above expression for the GSE case was obtained in Refs. \[19, 20\] using a scaling that assume \( N/2 \) eigenvalues.
The above equations give a complete description of the fluctuations of the eigenvalues at the edge of the spectra of the Gaussian ensembles. In particular, for the largest eigenvalue, the TW distributions are expressed in terms of these generating functions as 
\[ E_{G,\beta} (E_{\text{max}} < \lambda) = G_{\beta}(s, 1). \]

II. DISORDERED ENSEMBLES

To investigate the modifications these probabilities undergo when an external source of randomness is superimposed to the Gaussian fluctuations, we consider disordered ensembles whose matrices, \( H(\xi, \alpha) \), are defined as
\[ H(\xi, \alpha) = \frac{H_G(\alpha)}{\sqrt{\xi/\bar{\xi}}} \]  
(14)
where \( H_G \) is a matrix of (1) and \( \xi \) is a positive random variable with distribution \( w(\xi) \) with average \( \bar{\xi} \) and variance \( \sigma_w \). From (14) and (1) it is deduced that the joint density distribution of the matrix elements is a superposition of the Gaussian ensembles distributions weighted with \( w(\xi) \), namely
\[ P(H; \alpha) = \int d\xi w(\xi) P_G(H; \alpha \xi/\bar{\xi}). \]
(15)

Changing variables from matrix elements to eigenvalues and eigenvectors, it is also found, after integrating out the eigenvectors, that the joint probability distribution of the eigenvalues is obtained by averaging over the joint distribution of the Gaussian ensembles. As a consequence, measures of this average ensemble are averages over the Gaussian measures. The eigenvalue density, for instance, turns out to be an average over Wigner’s semi-circles with different radii, that is
\[ \rho(\lambda; \alpha) = \int d\xi w(\xi) \sqrt{\frac{4\sigma^2(\xi) N - \lambda^2}{2\pi \sigma^2(\xi)}} \]  
(16)
where the \( \xi \)-dependent variance, \( \sigma(\xi) \) is given by
\[ \sigma(\xi) = \sigma \sqrt{\frac{\bar{\xi}}{\xi}} \]  
(17)

The probability that the largest eigenvalue \( \lambda_{\text{max}} \) is smaller than a given value \( t \) can be calculated by evaluating the probability that the interval \((t, \infty)\) is empty. This is obtained by integrating the joint probability distribution of the eigenvalues in the interval \((-\infty, t)\) over all eigenvalues, we find
\[ E_{\beta} (\lambda_{\text{max}} < t) = \int d\xi w(\xi) E_{G,\beta}[S(\xi, t)] \]  
(18)
with the argument of \( S(\xi, t) \) obtained by plugging in Eq. (3) the above \( \xi \)-variance, Eq. (17), namely
\[ S(\xi, t) = N^{1/6} \left[ \frac{t}{\sigma(\xi)} - 2\sqrt{N} \right]. \]  
(19)
Equations (18) and (19) give a complete analytical description of the behavior of the largest eigenvalue once the function \( w(\xi) \) is chosen. But even without specifying \( w(\xi) \), asymptotic results can be derived by comparing its localization, given by the ratio \( \sigma_w/\bar{\xi} \), with that of \( E_{G,\beta} \) considered as functions of the integrand variable \( \xi \) through Eq. (19). Since the widths of these last ones depend on the matrix size \( N \), let us introduce a positive parameter \( z \) such that \( \sigma_w/\bar{\xi} = N^{1-z} \) which is kept fixed when the limit \( N \to \infty \) is taken.

As \( z > 0 \), when \( N \) increases, the \( w(\xi) \) distribution becomes more and more localized and, if asymptotically it can be approximated by a Gaussian, by changing the integration variable to

\[
\xi = \bar{\xi} - v \sigma_w, \tag{20}
\]

Eq. (18) can be written as

\[
E_\beta(\lambda_{\text{max}} < t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dv \exp \left( -\frac{v^2}{2} \right) E_{G,\beta} \left[ S(v, t) \right], \tag{21}
\]

where the argument \( S(v, t) \), after neglecting higher order terms in \( 1/N \), takes the form

\[
S(v, t) = N^{1/6} \left( \frac{t}{\sigma} - v N^{1/2-z} - 2\sqrt{N} \right) = s - N^{2/3-z} v. \tag{22}
\]

In the last step of (22), Eq. (3) was used and it was assumed that \( z > 2/3 \). Taking now the limit \( N \to \infty \), the vanishing of the second term in the r.h.s. of (22) makes \( S(v, t) \) independent of \( v \) and the distributions \( E_{G,\beta}(s) \), i.e. TW, are recovered. Values of the parameter \( z \) greater than \( 2/3 \) correspond to situations in which the distribution \( w(\xi) \) collapses faster than \( E_{G,\beta} \), for \( z < 2/3 \), on the other hand, it is the opposite that happens. To be able to get \( N \)-independent results in this range of values of \( z \), it is necessary to modify the scaling to

\[
s = N^{z-1/2} \left( \frac{t}{\sigma} - 2\sqrt{N} \right), \tag{23}
\]

in which case, the argument \( S[v, t(s)] \) becomes

\[
S[v, t(s)] = N^{2/3-z} (s - v). \tag{24}
\]

Taking now the limit of \( N \to \infty \), \( E_{G,\beta}(S) \) becomes a step function centered at \( v = s \) and the distribution goes to the normal distribution \( N(0, 1) \). Finally, at the critical value \( z = 2/3 \), from both sides, (21) converges to the convolution of the normal distribution and TW. In summary we have the three regimes

\[
E_\beta(\lambda_{\text{max}} < t) = \begin{cases} 
\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dv \exp \left( -\frac{v^2}{2} \right) E_{G,\beta}(s), & z > 2/3 \\
\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{s} dv \exp \left( -\frac{v^2}{2} \right), & z = 2/3 \\
\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{s} dv \exp \left( -\frac{v^2}{2} \right), & z < 2/3
\end{cases} \tag{25}
\]

Consider now the case in which the distribution \( w(\xi) \) is independent of \( N \). In this case, it is appropriate to make the parameter \( \alpha \) equal to the matrix size \( N \). With this scaling, the eigenvalues of the matrices of the average ensemble are located in the interval \((-1, 1)\), and the \( \xi \)-dependent argument \( S(\xi, t) \), Eq. (19), takes the simple form \( S(\xi, t) = \)
This expression makes evident that when the matrix size $N$ increases, for the three invariant ensembles, the function $E_{G, \beta}$ become a step function centered at $\xi = \bar{\xi}/t^2$. Therefore, in this regime, the probability distribution for the largest eigenvalue converges to

$$E_{\beta}(\lambda_{\text{max}} < t) = \int_{\bar{\xi}/t^2}^{\infty} d\xi w(\xi) \quad (26)$$

with density

$$\frac{dE_{\beta}(t)}{dt} = \frac{2\bar{\xi}w(\bar{\xi}/t^2)}{t^3} \quad (27)$$

A special choice of $w(\xi)$ that already has appeared in previous studies of disordered ensembles\[21, 22\], is that in which it is the one parameter family of Gamma distributions

$$w(\xi) = \exp(-\xi)\xi^{\bar{\xi}-1}/\Gamma(\bar{\xi}) \quad (28)$$

In this case, (27) becomes

$$\frac{dE_{\beta}(t)}{dt} = \frac{2\bar{\xi}\exp(-\bar{\xi}/t^2)}{\Gamma(\bar{\xi})t^{2\bar{\xi}+1}} \quad (29)$$

that defines a long-tailed distribution of extreme values of a correlated set of points (we remark that this distribution distribution has recently been considered in random covariant matrices\[23\]).

Notice that in (27) the variable $t$ is the eigenvalue itself without any edge scaling. This means that, in this case, fluctuations become of the order of the size of the average ensemble spectrum. Although (27) resembles a Fréchet distribution, the fact that the power of $t$ in the exponent is fixed at the value two makes it a different distribution. Nevertheless, for $\bar{\xi} = 1$ it is indeed a Fréchet distribution. We remark that $\bar{\xi} = 1$ corresponds to the critical distribution of the family defined by Eq. (28) that separates the ones that converge from those that diverge at the origin. The asymptotical power-law decay of (29), similar to that of Fréchet distribution, suggest that in the asymptotic region the extreme value behaves independently of the other eigenvalues while, in the internal region, the presence of the others are felt.

### III. FROM TRACY-WIDOM TO WEIBULL

Let us now turn to the case of a model to describe largest eigenvalues of spectra in the intermediate regime between RMT and Poisson. This model is brought on by the fact that the generating function, Eq. (5), can be interpreted as a probability. Indeed, assuming with $0 < z < 1$ that the factor $(1-z)^n$ that multiplies $E(n, s)$ in (4) is the probability that the $n$ eigenvalues in the interval $(s, \infty)$ have been removed, then summing all the terms gives the probability that there is no level in the interval.

A realization of this situation was considered in Ref. [24] in which the effect of removing at random a fraction $1 - f$ of eigenvalues of RMT spectra was investigated. In this case, $1 - f$ is the probability that a given eigenvalue has been dropped from the spectrum. Therefore with the identification of $z$ with $f$ the generating function Eq. (5)
becomes the probability distribution of the largest eigenvalue for this kind of randomly incomplete spectra.

In Refs. [17, 24], in studying the effect of incompleteness in the spectral statistics at the bulk, an interval of length $s$ is increased by a factor of $1/f$ to compensate the reduction in the average number of levels inside it. Following the same idea at the edge, we want a scaling of the variable $s$ such that the average number of eigenvalues in the interval $(s, \infty)$ remains the same when a fraction of levels is removed. This average is

$$<n> = \frac{2}{3\pi} (-s)^{3/2},$$

obtained by integrating (4) from $s$ to $\infty$.

Therefore, in order to keep it invariant when the density of eigenvalues is reduced by a factor of $f$, $s$ has to be divided by $f^{2/3}$.

Using this scaling in the Airy-kernel, Eq. (7), we expect that when the limit $f \to 0$ is taken it converges to the Poisson-kernel [3]

$$K_P(x, y) = \begin{cases} 0, & x \neq y \\ \rho(x), & x = y \end{cases}$$

with density given by (4). In fact, we have

$$\lim_{f \to 0} \frac{\text{Ai}(x/f^{2/3})\text{Ai}'(y/f^{2/3}) - \text{Ai}(y/f^{2/3})\text{Ai}'(x/f^{2/3})}{(x - y)/f^{2/3}} = K_P(x, y).$$

Denoting by $\hat{e}(s, f)$ the probability that the semi-infinite interval $(s, \infty)$ is empty, for the incomplete spectra it is given by

$$\hat{e}_\beta(s, f) = \sum_{k=0}^{\infty} (1 - f)^k E_\beta(k, s/f^{2/3}) = G_\beta(s/f^{2/3}, f),$$

that means that in an incomplete spectrum the largest eigenvalue can be anyone other of the $n$th largest eigenvalues. The last equality in (33) follows from Eq. (5) and shows that the generating functions for the three symmetry classes contain a comprehensive description of the largest eigenvalues of complete and incomplete RMT spectra.

To investigate the limit $f \to 0$ we remark that as the scaling factor $f^{2/3}$ appears in the denominator, for small $f$ the function $q(x, z)$ can be replaced by its asymptotic form at $x \to \pm\infty$. For $x > 0$, this is given by the exponential decay of the asymptotic behavior of the Airy function. So, at this positive side, $q(x, z)$ vanishes and $\hat{e}_\beta(s, f) = 1$. For $s < 0$, by the same argument, the integrals can be performed from $s$ to zero with $q(x, z)$ replaced by its asymptotic expression for large negative values. For $0 < z < 1$, this expression has been worked out by Hastings and McLeod [25], that found

$$q(x, z) \sim d(-x)^{-1/4} \sin \left[ \frac{2}{3}(-x)^{3/2} - \frac{3}{4}d^2 \log(-x) - c \right]$$

where $d^2 = -\frac{1}{\pi} \log(1 - z)$.

The integral in (8) contains the square of $q(x, z)$ so, if we use the relation $2\sin^2 x = 1 + \cos 2x$, we find that the generating functions have a smooth and an oscillating part. As the period and the amplitude of the oscillations decrease with $f$, in the limit of small $f$, the oscillating part averages out to zero and can be neglected. For the same reason, in this limit, the function $\mu(s, f)$, Eq. (13), vanishes. Taking then all this into account and
after evaluating integrals, we find that when \( f \to 0 \), the largest eigenvalue distributions, for the three symmetry classes, converge to the Weibull distribution

\[
\hat{\rho}_\beta(s) = \begin{cases} \exp \left[ -\frac{2g_\beta}{3\pi} (-s)^{3/2} \right], & s \leq 0 \\ 1, & s > 0 \end{cases},
\]

where \( g_\beta = 1 \) for \( \beta = 1, 2 \) and \( g_\beta = 1/2 \) for \( \beta = 4 \) (this parameter reflects the fact mentioned above that, in the GSE case, we are using a scaling with \( N/2 \)). This Weibull distribution describes the extreme value of a set of uncorrelated points with a semi-circle density distribution. In Fig. 1, the transition from TW \((f = 1)\) to Weibull \((f = 0)\) is illustrated for the GUE case \((\beta = 2)\). The oscillations at intermediate values are clearly seen.

### IV. CONCLUSION

In conclusion, we have investigated the behavior of the largest eigenvalue of two models that generalize the RMT Gaussian ensembles. In the first one, by introducing disorder in the ensemble, the eigenvalue density is lead to fluctuate generating at the edge a competition between the Tracy-Widom distribution and the normal distribution. This kind of behavior has been observed in growing processes in random media. In the second model, the eigenvalue density is kept fixed but the correlations among the eigenvalues are progressively reduced. It is then observed a continuous transition from the TW to the Weibull distribution, characteristic of uncorrelated variables.

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Figure Captions

Fig. 1 Density distribution of largest eigenvalue for GUE ($\beta = 2$). For complete sequence ($f = 1$, Tracy-Widom), for uncorrelated sequence (Weibull, limit $f \to 0$; Eq. (35)) and partially incomplete sequences ($f = 0.3, 0.6$, from (33)).