HIGHEST WEIGHT MACDONALD AND JACK POLYNOMIALS

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Fractional quantum Hall states of particles in the lowest Landau levels are described by multivariate polynomials. The incompressible liquid states when described on a sphere are fully invariant under the rotation group. Excited quasiparticle/quasihole states are member of multiplets under the rotation group and generically there is a nontrivial highest weight member of the multiplet from which all states can be constructed. Some of the trial states proposed in the literature belong to classical families of symmetric polynomials. In this paper we study Macdonald and Jack polynomials that are highest weight states. For Macdonald polynomials it is a (q,t)-deformation of the raising angular momentum operator that defines the highest weight condition. By specialization of the parameters we obtain a classification of the highest weight Jack polynomials. Our results are valid in the case of staircase and rectangular partition indexing the polynomials.

I. INTRODUCTION

The fractional quantum Hall effect is a state of electronic matter with elusive physical properties. Its theoretical description pioneered by Laughlin[1] is based on explicit wavefunctions describing the full many-body state of the interacting electrons. These wavefunctions are generically given by a polynomial function of the coordinates of the particles in the plane. They have been studied in several geometries. In this paper we are concerned by the case of the sphere and the unbounded plane. In the plane a generic quantum state is given by an antisymmetric polynomial in the complex coordinates $z_1, \dots, z_N$ (where N is the number of particles) times a universal Gaussian factor which is independent of the state under consideration and hence can be omitted completely. On the sphere a quantum state is an antisymmetric polynomial in the spinor coordinates (u_i, v_i) with $u_i = \cos(\theta_i/2)\exp(-i\phi_i/2)$ and $v_i = \sin(\theta_i/2)\exp(i\phi_i/2)$ where θ_i and ϕ_i are usual spherical coordinates. The polynomials appearing in these two cases can be related through stereographic projection[2]. Since one always factor out $\prod_{i<j}(z_i - z_j)$ from an antisymmetric polynomial one can consider only symmetric polynomials which are acceptable wavefunctions for bosons.

In the physics of the fractional quantum Hall effect, the special polynomials that are relevant[3–5] are not in general solutions of the true eigenvalue problem involving the Coulomb interaction between electrons. Instead they are thought[6] to be adiabatically related to the true eigenstates. In some cases they are exact eigenstates of some auxiliary operator. The most prominent example is the celebrated Laughlin wavefunction whose explicit formula is :

$$\Psi_L = \prod_{i<j}(z_i - z_j)^3. \tag{1}$$

This quantum state is known to be an excellent approximation of the true state of electrons when the lowest Landau level has filling factor 1/3. This polynomial is the eigenvector of smallest degree with zero eigenvalue for the interaction energy $V(\mathbf{r}-\mathbf{r}') = \Delta_{\mathbf{r}}\delta^2(\mathbf{r}-\mathbf{r}')$. Another prominent polynomial is the Moore-Read Pfaffian[7] state :

$$\Psi_{\mathrm{MR}} = \mathrm{Pf}(\frac{1}{z_k - z_l})\prod_{i<j}(z_i - z_j), \tag{2}$$

where Pf stands for the Pfaffian of the matrix $1/(z_k - z_l)$. Here we have written it for bosons i.e. as a symmetric polynomial. This state is one of the candidates to describe the elusive physics at filling factor 5/2. It is also the polynomial of smallest degree that is an eigenvector with zero eigenvalue of the following operator :

$$\mathcal{H}^{(3)} = \sum_{i<j<k}\delta^2(\mathbf{r}_i - \mathbf{r}_j)\delta^2(\mathbf{r}_j - \mathbf{r}_k). \tag{3}$$

This operator forbids three bosonic particles to be in the place. There is a natural generalization to p particles and the corresponding wavefunctions of smallest degree are called the Read-Rezayi (RR) states[8,9]. These wavefunctions are thus multivariate symmetric polynomials with special vanishing properties. Their study was pioneered by Feigin, Jimbo, Miwa and Mukhin[10,11] and it was realized that they belong to the family of Jack polynomials[12–16]. Such polynomials depend upon a parameter and a partition of an integer[17]. A given polynomial can be expanded in powers of the z_i coordinates and a general term in the expansion is characterized by the set of occupation numbers of the one-body orbitals $\{n_m, m = 0, 1, 2, \dots\}$. Note that we consider bosonic quantum Hall states for which one can have $n_m > 1$. A given configuration of occupation numbers $(n_0, n_1, n_2 \dots)$ characterizes each term of the expansion. The set of occupation numbers defines a partition of the integer N since $N = \sum_m n_m$ (for convenience and according to the standard notations used in literature related to symmetric functions, we consider *decreasing* partitions $\lambda = (\lambda_1, \dots, \lambda_k)$ with $\lambda_1 \geq \cdots \geq \lambda_k \geq 0$). Alternatively one can also specify the same configuration by giving all the m values that appear with nonzero occupation numbers $(m_1..m_1 m_2..m_2 \dots)$ where each m is repeated n_m times. This set of numbers then defines equivalently a partition of the total angular momentum $L_z = \sum_m m n_m$. In the physics literature it is common to specify the set of occupation numbers while the mathematical literature[17] on symmetric polynomials uses instead the partitioning of L_z. A partition λ defines also a unique symmetric monomial m_λ given by :

$$m_\lambda = z_1^{k_1} \dots z_N^{k_N} + \text{permutations}. \tag{4}$$

This can be considered as a (unnormalized) wavefunction for N bosons in the lowest Landau level where the quantum numbers k_i of occupied orbitals can be associated in a one to one correspondence to a set of occupation numbers $\{n_m\}$. For example the monomial for N=3 $m = z_1^2 z_2 z_3 + perm$ is defined by the partition (0210. . .) since there are two bosons in the m=1 orbital and one boson in the m=2 orbital. An arbitrary bosonic WF in the LLL can be expanded in terms of such monomials, each of them being indexed by a partition :

$$f = \sum_\lambda c_\lambda m_\lambda, \tag{5}$$

where c_λ are some coefficients. For a given polynomial it may happen that not all partitions appear in the expansion above. Indeed there is a partial ordering on partitions called the dominance ordering : let λ and μ two (decreasing) partitions then $\lambda \geq \mu$ if $\lambda_1 + \cdots + \lambda_i \geq \mu_1 + \cdots + \mu_i$ for all i. This is only a partial order : it may happen that the relation above does not allow comparison of two partitions. Some of the trial wavefunctions proposed in the FQHE literature have the property that there is a dominant partition with respect to this special order and all partitions appearing in the expansion Eq.(5) are dominated by a leading one :

$$\Psi = \sum_{\mu \leq \lambda} c_\mu m_\mu. \tag{6}$$

This was first noted by Haldane and Rezayi[19] in the case of the Laughlin wavefunction. The dominance plays a key role in the expansion of powers of the VanderMonde determinant[20,21]. This property of dominance is also shared by many special orthogonal polynomials in several variables[17]. The Jack polynomials noted J^α_λ are a family indexed by a partition λ which dominates the expansion in monomials and depend upon one parameter α. In fact we have :

$$\Psi^{(k)}_{RR} = \mathcal{S} \prod_{i_1<j_1} (z_{i_1} - z_{j_1})^2 \cdots \prod_{i_k<j_k} (z_{i_k} - z_{j_k})^2 \propto J^{-(k+1)}_{\lambda_k}(\{z_i\}), \tag{7}$$

where the first equality defines the Read-Rezayi states, one divides the particles into k packets and $\mathcal{S}$ means symmetrization of the product of partial Jastrow factors. In the case of the RR states we have $\alpha = -(k+1)$ and $\lambda_k = (k0k0k0\dots)$. The usual bosonic Laughlin wavefunction is the special subcase when there is only one packet $k = 1$ and the Moore-Read Pfaffian corresponds to the case $k = 2$. In general the filling factor of the order-k RR state is $\nu = k/2$.

In the spherical geometry[2], there is a natural action of the SU(2) rotation on the quantum states. Through the stereographic correspondence it can be translated in an action also on the symmetric polynomials of the planar geometry. The rotation operators are then differential operators acting upon the particle coordinates :

$$L_+ = E_0, \quad L_- = N_\phi \sum_{i=1}^N z_i - E_2, \quad L_z = \frac{1}{2} N N_\phi - E_1, \quad \text{where} \quad E_n = \sum_{i=1}^N z_i^n \frac{\partial}{\partial z_i}. \tag{8}$$

Here we have introduced N_ϕ which is the number of flux quanta through the sphere. The incompressible fractional quantum Hall states are realized for a special fine tuning of N and N_ϕ. Deviations from the ideal relation creates quasiparticle excitations on top of the quantum Hall fluid state. The parent incompressible fluid is invariant by rotation and hence satisfy $L_+\Psi = L_-\Psi = 0$: it is a singlet under rotations. Quasiparticle states are less symmetrical : a one-quasiparticle state can satisfy $L_+\Psi = 0$ if it is located at one the poles of the sphere. The same condition apply to states with quasiparticles or quasiholes if they are piled up on the poles of the sphere. Mathematically these states are highest weight in terms of representations of the rotation group, belonging to a nontrivial multiplet. It is thus important to find among the classical symmetric polynomials what are those that are highest weight ones to characterize possible candidate quantum Hall states.

In this paper, we study the generalization of this problem to a (q,t)-deformation of the raising operator L_+. We find two families of Macdonald polynomials that are highest weight states. These families differ in the type of partition that define them : staircase and rectangular. By specialization of the parameters we find also the Jack polynomials indexed by such partitions that are highest weight. While we cannot fully characterize the highest weight Jacks and Macdonald polynomials, we propose a set of conjectures.

In section II, we define several mathematical tools that are used in this paper and we give general properties of Macdonald and Jack polynomials. The action of the raising angular momentum operator L_+ on the symmetric functions is explained in section III. In section IV we introduce a two-parameter (q,t)-deformation of the angular momentum operator L_+ In section V we characterize the Macdonald polynomials that satisfy the highest weight condition for rectangular and staircase partitions. We specialize this result to Jack polynomials in section VI. Finally section VII contains our conclusions. Two appendices are devoted to the study of some eigenproblems using the method of section III.

II. MACDONALD AND JACK POLYNOMIALS

Let us start this paper with a brief account of Macdonald polynomial theory. The introduction of this two parameter family of symmetric polynomials in this context is motivated by the fact that Jack polynomials can be considered as a degenerate case of the Macdonald polynomials. Hence, properties on Jack polynomials can be studied in a more general way when stated in terms of Macdonald polynomials.

A. Symmetric functions

The symmetric functions in n independent variables $X = \{x_1, \dots, x_n\}$ form the subalgebra Λ_n of the free algebras $K[x_1, \dots, x_n]$ (K being a fixed ring) composed by polynomials which are invariant under the action of the symmetric group $\mathfrak{S}_n$ consisting in permuting the variables

$$\Lambda_n = K[x_1, \dots, x_n]^{\mathfrak{S}_n}. \tag{9}$$

The ring Λ of symmetric functions in countably many independent $x_1, x_2, \dots$ is an algebra obtained by applying the projective limit (see Ref.(17) I.2 for more details)

$$\Lambda = \lim_{\longleftarrow} \Lambda_n. \tag{10}$$

When K is a field, the space Λ has its bases indexed by partitions. For convenience a partition will denoted by a decreasing vectors in $\mathbb{N}$ (as in Refs.(17,18)).

When the number of variables is finite, there is an other system of notations (see *e.g.* Refs.(12–14)) that takes account of the interpretation of partitions as vectors of occupation levels of a system of particles. We will use Latin letters $u,\ v,\ w, \dots$ for these notation instead of Greek letters $\lambda,\ \mu,\ \nu, \dots$ which will be reserved for the decreasing vectors. The "translation" between the two systems of notations is very easy to understand while the v_i component of the vectors v equals the number of parts equal to i in λ. For instance, for 6 variables (particles) one has

$$\lambda = [4421] \sim v = [2, 1, 1, 0, 2, 0, \dots] \tag{11}$$

Note also that the classical results on symmetric functions are clearer when stated in the decreasing vector notation.

There are three main multiplicative bases for Λ : The power sums $p^\lambda(\mathbb{X}) = p_{\lambda_1}(\mathbb{X}) \dots p_{\lambda_n}(\mathbb{X})$ where $p_k(\mathbb{X}) = \sum_i x_i^k$, the elementary functions $e^\lambda(\mathbb{X}) = e_{\lambda_1}(\mathbb{X}) \dots e_{\lambda_n}(\mathbb{X})$ where

$$e_k(\mathbb{X}) = \sum_{x_{i_1}, \dots, x_{i_n} \ distincts} x_{i_1} \dots x_{i_n}$$

and the complete functions $h^\lambda(\mathbb{X}) = h_{\lambda_1}(\mathbb{X}) \dots h_{\lambda_n}(\mathbb{X})$ where

$$h_k(\mathbb{X}) = \sum_{x_{i_1}, \dots, x_{i_n}} x_{i_1} \dots x_{i_n}.$$

There is also non multiplicative bases such as monomial functions $m_\lambda(\mathbb{X})$ and Schur functions $s_\lambda(\mathbb{X})$ (see *e.g.* Refs.(17, 18)). When there is no relation between the variables (this implies that the number of variables is infinite), the algebra of symmetric functions is a polynomial algebra over the elementary functions and also over the homogeneous functions.

B. The λ-ring notations

The generating function of the complete symmetric functions for an alphabet $\mathbb{X}$ has a well-known factorized expression

$$\sigma_z(\mathbb{X}) := \sum_i h_i(\mathbb{X}) z^i = \prod_{x\in\mathbb{X}} \frac{1}{1-xz}. \tag{12}$$

If $\mathbb{Y}$ is an alphabet disjoint of $\mathbb{X}$, straightforwardly

$$\sigma_z(\mathbb{X}\cup\mathbb{Y}) = \sigma_z(\mathbb{X})\sigma_z(\mathbb{Y}). \tag{13}$$

But more generally, if $\mathbb{Y}$ contains some letters of $\mathbb{X}$,

$$\sigma_z(\mathbb{X})\sigma_z(\mathbb{Y}) = \sigma_z(\mathbb{X}\cup\mathbb{Y})\sigma_z(\mathbb{X}\cap\mathbb{Y}). \tag{14}$$

In fact it is more convenient to consider an alphabet, not as a set of variables, but as the formal sum of its variables, $\mathbb{X} = x_1 + x_2 + \dots$. In this case, Eq. (14) reads :

$$\sigma_z(\mathbb{X})\sigma_z(\mathbb{Y}) = \sigma_z(\mathbb{X}+\mathbb{Y}). \tag{15}$$

Hence, the binary operator $+$ acting on the alphabets will encode the transformation sending $h_i(\mathbb{X})$ to $\sum_{j+k=i} h_j(\mathbb{X})h_k(\mathbb{Y})$. More generally, the product of an alphabet $\mathbb{X}$ by an element α of the ground field allows to define a new alphabet denoted by $\alpha\mathbb{X}$ and whose complete functions are given by :

$$\sigma_z(\alpha\mathbb{X}) = \sigma_z(\mathbb{X})^\alpha. \tag{16}$$

For example, if $\alpha = -1$, one has :

$$\sigma_z(-\mathbb{X}) = \sigma_z(\mathbb{X})^{-1} = \prod_z (1-xz) = \sum_i (-1)^i e_i(\mathbb{X}) z^i = \lambda_{-z}(\mathbb{X}), \tag{17}$$

where $\lambda_z(\mathbb{X}) = \sum_i e_i(\mathbb{X}) z^i$ is the generating function of the elementary functions. The operation sending $\mathbb{X}$ on $-\mathbb{X}$ can be formally interpreted as the transformation sending $h_i(\mathbb{X})$ on $\pm e_i(\mathbb{X})$.
The last operation we introduce is the multiplication of two alphabets $\mathbb{X}\mathbb{Y}$. The transformation is clearer when stated in terms of power sums :

$$p_\lambda(\mathbb{X}\mathbb{Y}) = p_\lambda(\mathbb{X})p_\lambda(\mathbb{Y}). \tag{18}$$

For interpreting this operation on the complete function, we need to introduce the generating function of the power sums

$$\Psi_z(\mathbb{X}) := \sum_i \frac{p_i(\mathbb{X})}{i} z^i = \log \sigma_z(\mathbb{X}). \tag{19}$$

After a short computation, one finds :

$$\sigma_z(\mathbb{X}\mathbb{Y}) = \sum_i \sum_{\lambda\vdash i} z_\lambda^{-1} p^\lambda(\mathbb{X}) p^\lambda(\mathbb{Y}) z^i \tag{20}$$

where $\lambda \vdash i$ means that λ is a partition of weight i and $z_\lambda = \prod_{i\geq 1} i^{m_i} m_i!$ if m_i denotes the number of parts of λ equal to i. The function $K(\mathbb{X},\mathbb{Y}) = \sigma_1(\mathbb{X}\mathbb{Y})$ has another important role in the theory of symmetric functions : it is the reproducing kernel of the usual scalar product whose evaluation on power sums is :

$$\langle p_\lambda, p_\mu \rangle = z_\lambda \delta_{\mu,\lambda}. \tag{21}$$

By reproducing kernel, we mean that if $(B_\lambda)_\lambda$ and $(C_\lambda)_\lambda$ are two bases in duality for the scalar product $\langle\, , \rangle$ ($\langle B_\lambda, C_\mu\rangle = \delta_{\mu,\lambda}$) then we have :

$$\langle K(\mathbb{X},\mathbb{Y}), B_\lambda(X)\rangle_X = B_\lambda(\mathbb{Y}). \tag{22}$$

The Schur basis is the unique basis (s_λ) orthonormal for $\langle\, , \rangle$ verifying that the dominant monomial in s_λ is $x_1^{\lambda_1}\dots x_n^{\lambda_n}$, hence :

$$K(\mathbb{X},\mathbb{Y}) = \sum_\lambda s_\lambda(\mathbb{X}) s_\lambda(\mathbb{Y}). \tag{23}$$

C. Macdonald Polynomials

The Macdonald polynomials $(P_\lambda(\mathbb{X};q,t))_\lambda$ form the unique basis of symmetric functions orthogonal for the standard (q,t)-deformation of the usual scalar product on symmetric functions :

$$\langle p^\lambda, p^\mu\rangle = \prod_{i=1}^{l(\lambda)} \frac{1-q^{\lambda_i}}{1-t^{\lambda_i}} z_\lambda \delta_{\lambda,\mu}, \tag{24}$$

see *e.g.* Ref. (17) VI.4 p322, verifying the following equation :

$$P_\lambda(\mathbb{X};q,t) = m_\lambda(\mathbb{X}) + \sum_{\mu\le\lambda} u_{\lambda\mu} m_\mu(\mathbb{X}), \tag{25}$$

where m_λ is a monomial function in the notation of Ref.(17), I.2.1, p8. Their generating function is (see *e.g.* Ref.(17), VI.4.13, p324) :

$$K_{q,t}(\mathbb{X},\mathbb{Y}) := \sum_\lambda \prod_{i=1}^{l(\lambda)} \frac{1-t^{\lambda_i}}{1-q^{\lambda_i}} p^\lambda(\mathbb{X}) p^\lambda(Y) = \sigma_1\left(\frac{1-t}{1-q}\mathbb{X}\mathbb{Y}\right) = \sum_\lambda P_\lambda(\mathbb{X};q,t) Q_\lambda(\mathbb{Y};q,t), \tag{26}$$

where :

$$Q_\lambda(\mathbb{X};q,t) = b_\lambda(q,t) P_\lambda(\mathbb{X};q,t), \tag{27}$$

with :

$$b_\lambda(q,t) = \prod_{(i,j)\in\lambda} \frac{1-q^{\lambda_i-j}t^{\lambda'_j-i+1}}{1-q^{\lambda_i-j+1}t^{\lambda'_j-i}}. \tag{28}$$

Note that the first part of equality (26) is obtained by a straightforward computation from the expression of $K_{qt}(\mathbb{X},\mathbb{Y})$ as the reproducing kernel of the scalar product $\langle\,,\,\rangle$:

$$K_{qt}(\mathbb{X},\mathbb{Y}) = \sum_\lambda \frac{1-t^{\lambda_i}}{1-q^{\lambda_i}} z_\lambda p^\lambda(\mathbb{X}) p^\lambda(\mathbb{Y}). \tag{29}$$

Whilst the second part is a non trivial consequence of the Pieri formula, see *e.g.* Ref.(17), VI.6.19, p339. The notation $\frac{1-t}{1-q}\mathbb{X}\mathbb{Y}$ must be understood in terms of λ-ring as a product of the three alphabets $\frac{1-t}{1-q}$, $\mathbb{X}$ and $\mathbb{Y}$. The resulting alphabet is described by the following formal sum :

$$\frac{1-t}{1-q}\mathbb{X}\mathbb{Y} = \sum_{\substack{x\in\mathbb{X},\,y\in\mathbb{Y}\\ k\in\mathbb{N}}} q^k xy - \sum_{\substack{x\in\mathbb{X},\,y\in\mathbb{Y}\\ k\in\mathbb{N}}} tq^k xy. \tag{30}$$

Macdonald polynomials appears in literature with numerous normalizations, let us recall the main ones. The normalization J is particularly interesting since the coefficients of the monomials are polynomials in q and t. It follows that a polynomial J_λ has no pole (in contrast to the normalization P). These polynomials are defined by :

$$J_\lambda(\mathbb{X};q,t) = c_\lambda(q,t) P_\lambda(\mathbb{X},q,t) = c'_\lambda(q,t) Q_\lambda(\mathbb{X};q,t), \tag{31}$$

where $c_\lambda(q,t) = \prod_{(i,j)\in\lambda}(1-q^{\lambda_i-j}t^{\lambda'_j-i+1})$ and $c'_\lambda(q,t) = \prod_{(i,j)\in\lambda}(1-q^{\lambda_i-j+1}t^{\lambda'_j-i})$ if λ' denotes the partition conjugate with λ. As in Ref.(22), we will denote by $J^\sharp$ the adjoint normalization of J *w.r.t* the scalar product $\langle\,,\,\rangle$:

$$\langle J^\sharp_\lambda, J_\mu\rangle_{q,t} = \delta_{\lambda,\mu}. \tag{32}$$

When the alphabet is finite, Lassalle introduced[22] another normalization denoted by $J^\star$ which is defined by :

$$J^\star_\lambda(x_1+\cdots+x_n;q,t) = \frac{J_\lambda(x_1+\cdots+x_n;q,t)}{J_\lambda(\frac{1-t^n}{1-t};q,t)}. \tag{33}$$

D. Skew functions

We describe first the general process allowing to define skew functions from any bases of symmetric functions. Consider two bases $(B_\lambda)_\lambda$ and $(C_\lambda)_\lambda$ which are adjoint *w.r.t.* a certain scalar product denoted by $\{\,,\,\}$. We will denote by $K^{\{\,\}}(\mathbb{X},\mathbb{Y})$ the reproducing kernel of $\{\,,\,\}$:

$$K^{\{\,\}}(\mathbb{X},\mathbb{Y})=\sum_\lambda B_\lambda(\mathbb{X})C_\lambda(\mathbb{Y}). \tag{34}$$

The aim of the construction of skew functions deals with the problem of the description of the operator which is adjoint of the multiplication by C_λ,

$$i.e.\,\{C_\lambda P,Q\}=\{P,???.Q\},$$

or equivalently to find the polynomials $B_{\lambda/\mu}$ verifying :

$$C_\mu(\mathbb{Y})K^{\{\,\}}(\mathbb{X},\mathbb{Y})=\sum_\lambda B_{\lambda/\mu}(\mathbb{X})C_\lambda(\mathbb{Y}). \tag{35}$$

A straightforward computation gives the equality :

$$B_{\lambda/\mu}=\sum_\mu c^\lambda_{\mu,\nu}B_\nu, \tag{36}$$

where the $c^\lambda_{\mu,\nu}$ are the structure coefficients of the basis $(C_\lambda)_\lambda$:

$$C_\mu C_\nu=\sum_\lambda c^\lambda_{\mu,\nu}C_\lambda. \tag{37}$$

Remark *If $\{\,,\,\}'$ is a second scalar product and $(B'_\lambda)_\lambda$ the basis adjoint to $(C_\lambda)_\lambda$* w.r.t. $\{\,,\,\}'$*, then the decomposition in the basis B' of the polynomials $B'_{\lambda/\mu}$ involves the same coefficients than which appear in the decomposition of $B_{\lambda/\mu}$ in the basis B, that is*

$$B'_{\lambda/\mu}=\sum_\mu c^\lambda_{\mu,\nu}B'_\nu. \tag{38}$$

Now, suppose that the reproducing kernel of $\{\,,\,\}$ is multiplicative for the addition of alphabets :

$$K^{\{\}}(\mathbb{X}+\mathbb{Y},\mathbb{Z})=K^{\{\}}(\mathbb{X},\mathbb{Z})K^{\{\}}(\mathbb{Y},\mathbb{Z}). \tag{39}$$

One has then :

$$\begin{aligned}K^{\{\}}((\mathbb{X}+\mathbb{Y}),\mathbb{Z}) &= K^{\{\}}(\mathbb{X},\mathbb{Z})K^{\{\}}(\mathbb{Y},\mathbb{Z})\\ &=\sum_{\mu,\nu}B_\mu(\mathbb{X})B_\nu(\mathbb{Y})C_\mu(\mathbb{X})C_\nu(\mathbb{Y})\\ &=\sum_\lambda\left(\sum_\mu c^\lambda_{\mu,\nu}B_\mu(\mathbb{X})B_\nu(\mathbb{Y})\right)C_\lambda(\mathbb{Z}).\end{aligned}$$

Hence,

$$\begin{aligned}B_\lambda(\mathbb{X}+\mathbb{Y}) &= \{K^{\{\}}((\mathbb{X}+\mathbb{Y})\mathbb{Z}),B_\lambda(\mathbb{Z})\}_Z\\ &=\sum_\mu B_{\lambda/\mu}(\mathbb{X})B_\mu(\mathbb{Y}).\end{aligned} \tag{40}$$

In particular, this is the case for skew Schur functions $s_{\lambda/\mu}$, skew Jack polynomials and skew Macdonald polynomials.

E. Jack polynomials

If we set $q = t^\alpha$ and tends t to 1 in the previous equalities one recovers the theory of Jack polynomials. Indeed, Jack polynomials are homogeneous symmetric functions orthogonal *w.r.t.* the scalar product defined on power sums by :

$$\langle p^\lambda, p^\mu\rangle_\alpha = \lim_{t\to 1}\langle p^\lambda, p^\mu\rangle_{t^\alpha,t} = \alpha^{l(\lambda)} z_\lambda \delta_{\lambda,\mu}. \tag{41}$$

The reproducing kernel associated to this scalar product is :

$$K_\alpha(\mathbb{X},\mathbb{Y}) = \lim_{t\to 1} K_{t^\alpha,t}(\mathbb{X},\mathbb{Y}) = \sigma_1(\alpha^{-1}\mathbb{X}\mathbb{Y}) = (\sigma_1(\mathbb{X}\mathbb{Y}))^{\frac{1}{\alpha}}. \tag{42}$$

Jack polynomials appear in literature with normalization $P_\lambda^{(\alpha)}$ (resp. $Q_\lambda^{(\alpha)}$, $J_\lambda^{(\alpha)}$, $J^\sharp{}_\lambda^{(\alpha)}$, $J^\star{}_\lambda^{(\alpha)}$) which is deduced from P_λ (resp. Q_λ, J_λ, $J^\sharp{}_\lambda$, $J^\star{}_\lambda$) by putting $q = t^\alpha$ and sending t to 1 in Eq. (25) (resp. Eq. (27), Eq. (31), Eq. (33), Eq (32)). For more details see *e.g.* Ref.(17), VI. 10. The classical case of Schur functions s_λ is recovered when setting $\alpha = 1$ in the previous equalities.

III. HIGHEST WEIGHT SYMMETRIC FUNCTIONS

In this paragraph, we explain briefly the algorithm of A. Lascoux[23] to compute the action of the operator :

$$L_+ = \sum_{i=1}^{n} \frac{\partial}{\partial x_i} \tag{43}$$

on symmetric functions. Its action on power sums is easily described in terms of power sums, since we have :

$$L_+.\Psi_z(\mathbb{X}) = nz + z^2\frac{\partial}{\partial z}\Psi_z(\mathbb{X}), \tag{44}$$

or equivalently $L_+.p_k(\mathbb{X}) = kp_{k-1}(\mathbb{X})$. Since, L_+ is a first order differential operator, this equality is sufficient to describe its action on symmetric functions. Its action on elementary symmetric functions is also very simple to understand :

$$L_+.e_k(\mathbb{X}) = (n-k+1)e_{k-1}(\mathbb{X}), \tag{45}$$

and is obtained from the action of L_+ on the generating function $\lambda_z(\mathbb{X})$:

$$L_+.\lambda_z(\mathbb{X}) = z\sum_i \lambda_z(\mathbb{X} - x_i) \tag{46}$$

Since the set of elementary functions $(e^\lambda(\mathbb{X}))_{\lambda, l(\lambda)\le n}$ is a basis of the space of symmetric functions and that these functions are algebraically independent, the operator L_+ can be rewritten by means of the operators $\partial_{e_k} := \frac{\partial}{\partial e_k}$:

$$L_+ = \sum_{i=1}^{n}(n-i+1)e_{i-1}\partial_{e_i}. \tag{47}$$

This expression is unsatisfactory because it is somewhat difficult to cope with the coefficient $(n-i+1)$. To simplify the problem, one introduces a new alphabet $\tilde{\mathbb{X}}$ of size n which consists of the roots of the polynomial :

$$\tilde{\lambda}_z(\mathbb{X}) = \sum_{i=1}^{n}\frac{(n-i)!}{n!}e_i(\mathbb{X})z^i = \lambda_z(\tilde{\mathbb{X}}). \tag{48}$$

Note that a function is symmetric in $\mathbb{X}$ if and only if it is symmetric in $\tilde{\mathbb{X}}$. To convert an expression in $\mathbb{X}$ to an expression in $\tilde{\mathbb{X}}$, it suffices to apply the formal substitution $e_i(\mathbb{X}) = \frac{n!}{(n-i)!}e_i(\tilde{\mathbb{X}})$.

Setting $\tilde{e}_i(\mathbb{X}) = e_i(\tilde{\mathbb{X}})$, the operator L_+ has the following nice expression :

$$L_+ = \sum_{i=1}^{n}\tilde{e}_{i-1}\partial_{\tilde{e}_i}. \tag{49}$$

We will use the Macdonald notation[17] to denote the basis $(c_\lambda)_\lambda$ which is the adjoint basis to the power sum basis $(p^\lambda)_\lambda$ for the usual scalar product (21). Equivalently, $c_\lambda = z_\lambda^{-1} p^\lambda$. For simplicity, we set $\tilde{c}_\lambda(\mathbb{X}) = c_\lambda(\tilde{\mathbb{X}})$. With this notation, the kernel of L_+ is characterized by a theorem of Mac Mahon[24] corrected by Sylvester[25].

Proposition. (Mac Mahon-Sylvester)
A symmetric polynomial belongs to the kernel of L_+ if and only if its expansion in the basis $(\tilde{c}_\lambda(\mathbb{X}))_\lambda$ does not contains any partition having a part equal to 1.
Equivalently, the kernel of L_+ is the subring generated by $\tilde{c}_2(\mathbb{X})$, $\tilde{c}_3(\mathbb{X}), \dots$.

Note that, in the terminology of Mac Mahon, a polynomial is said *semi-invariant* if and only if it belongs to the kernel of L_+ while Bernevig and Haldane[12–15] use this word for polynomials which are in the kernel of both L_+ and :

$$L_- = N_\phi \sum_i x_i - \sum_i x_i^2 \frac{\partial}{\partial x_i}. \tag{50}$$

To avoid confusion, we will say that a polynomial is *highest weight* (HW) if and only if it is annulled by L_+.
In conclusion, we want to emphasize the relevance of the functions $\tilde{c}_\lambda$ to analyse eigenvalue problems in the context of the fractional quantum Hall effect. In appendix I, we give a closed formula for the expansion of the "yrast" eigenfunctions of the delta function interaction when the angular momentum equals the number of particles in terms of $\tilde{c}_\lambda$. In appendix II, we discuss some spectral properties of the Read-Rezayi operator[8,9] :

$$\mathcal{H}^{(k)} := \sum_{i_1 < \dots < i_{k+1}} \delta^{(2)}(x_{i_1} - x_{i_1}) \dots \delta^{(2)}(x_{i_k} - x_{i_{k+1}}). \tag{51}$$

IV. A (Q,T)-DEFORMATION OF L_+

In Ref.(22), Lassalle introduced generalized binomial coefficients $\binom{\lambda}{\mu}_{q,t}$ in the aim to understand the action of a (q,t)-deformation of L_+ on Macdonald polynomials. These binomial coefficients are the coefficients of $J^\sharp_\lambda$ in the generating series

$$J^\sharp_\mu(\mathbb{X}; q,t) K_{q,t}(1 + t + \dots + t^{n-1}, \mathbb{X}) = \sum_\lambda t^{|\lambda|-|\mu|} \binom{\lambda}{\mu}_{q,t} J^\sharp_\lambda(\mathbb{X}; q,t). \tag{52}$$

These coefficients are equal, up to a multiplicative coefficient which is a power of t, to skew Macdonald polynomials specialized to the alphabet $T_n = 1 + t + \dots + t^{n-1}$. More precisely :

$$\binom{\lambda}{\mu}_{q,t} = t^{|\mu|-|\lambda|} J_{\lambda/\mu}(T_n; q,t) = \sum_\nu {j^\sharp}^\lambda_{\mu,\nu}(q,t) J_\mu(T_n; q,t) \tag{53}$$

where coefficients ${j^\sharp}^\lambda_{\mu,\nu}(q,t)$ denotes the structure coefficients of the basis $(J^\sharp_\lambda)_\lambda$,

$$J^\sharp_\mu J^\sharp_\nu = \sum_\lambda {j^\sharp}^\lambda_{\mu,\nu} J^\sharp_\lambda. \tag{54}$$

Indeed, it suffices to remark that Eq. (52) is a special case of Eq.(35), hence the result is obtained by applying Eq.(36). The (q,t)-deformation of L_+ considered by Lassalle[22] is the same as introduced by Macdonald[17] VI.3. Let us recall it here. First, one has to define the q-deformation of the derivation $\partial/\partial x_i$ by means of the divided difference :

$$\partial/\partial_q x_i f(\mathbb{X}) = \frac{f(\mathbb{X}) - f(\mathbb{X} - (1-q)x_i)}{x_i - q x_i}. \tag{55}$$

Remember that in our notation the alphabet $\mathbb{X} - (1-q)x_i$ is obtained from $\mathbb{X}$ by substituting qx_i to x_i and remark that if q tends to 1, then $\partial/\partial_q x_i$ tends to $\partial/\partial x_i$. The (q,t)-deformation of the operator $L_+^{q,t}$ is defined by

$$L_+^{q,t} := \sum_{i=1}^n \prod_{\substack{j=1 \\ i \neq j}}^n \frac{t x_i - x_j}{x_i - x_j} \frac{\partial}{\partial_q x_i}. \tag{56}$$

The operator L_+ can be recovered from Eq.(56) by taking the limit :

$$L_+ = \lim_{(q,t)\to(1,1)} L_+^{q,t}. \tag{57}$$

In Ref.(22), Lassalle proved the following very interesting identity which describes the action of $L_+^{q,t}$ on the Macdonald polynomials $J_\lambda^\star$ by means of generalized binomial coefficients. For each partition λ of length $l(\lambda) \le n$, one has :

$$L_+^{q,t} J_\lambda^\star(\mathbb{X};q,t) = \sum_i \binom{\lambda}{\lambda_{(i)}}_{q,t} J^\star_{\lambda_{(i)}}(\mathbb{X};q,t), \tag{58}$$

where $\lambda_{(i)}$ denotes the vector obtained by subtracting 1 to the part i of the partition λ and the sum runs over the integers i such that $\lambda_{(i)}$ is a decreasing partition. One way to understand why the coefficients $\binom{\lambda}{\lambda_{(i)}}_{q,t}$ appear is to introduce a new scalar product $\{\,,\,\}$ for which $J_\lambda^\star(\mathbb{X};q,t)$ and $t^{|\lambda|}J_\lambda^\sharp(\mathbb{X};q,t)$ are adjoint. The reproducing kernel of $\{\,,\,\}$ is by definition a generalized hypergeometric function associated to the Macdonald polynomials :

$$K^{\{\,,\,\}}(\mathbb{X},\mathbb{Y}) = {}_0\mathcal{F}_0(\mathbb{X},\mathbb{Y};q,t) := \sum_\lambda t^{|\lambda|} J_\lambda^\sharp(\mathbb{X};q,t) J_\lambda^\star(\mathbb{Y};q,t), \tag{59}$$

$\mathbb{X}$ and $\mathbb{Y}$ being two alphabets with at most n letters. The difficult part of Lassalle's reasoning consists in proving that the operator ${L_+}^{q,t}$ and e_1. are adjoint :

$$i.e.\; {L_+^{q,t}}_{\mathbb{X}} . {}_0\mathcal{F}_0(\mathbb{X},\mathbb{Y};q,t) = e_1(\mathbb{Y})\, {}_0\mathcal{F}_0(\mathbb{X},\mathbb{Y};q,t). \tag{60}$$

This is a rather technical computation that we do not repeat here. But, this fact being established, Remark (II D), combined with $e_1(\mathbb{X}) = J_1^\star(\mathbb{X};q,t) = J_1^\sharp(\mathbb{X};q,t)$, explains completely why generalized binomial coefficients appear in Eq.(58). In the sequel, we will use the normalization P instead of $J^\star$. The action of $L_+^{q,t}$ on $P_\lambda(\mathbb{X};q,t)$ can be easily deduced from Eq.(58). First, one plugs successively the definition of $J^\star$ (Eq.(33)) and J (Eq.(31)) in Eq.(58) and obtains :

$$L_+^{q,t}.P_\lambda(\mathbb{X};q,t) = \sum_i \binom{\lambda}{\lambda_{(i)}}_{q,t} J_\lambda(T_n;q,t)/J_{\lambda_{(i)}}(T_n;q,t)\frac{c_{\lambda_{(i)}}(q,t)}{c_\lambda(q,t)} P_{\lambda_{(k)}}(\mathbb{X};q,t). \tag{61}$$

Knowing the value of $J_\mu(T_n;q,t)$:

$$J_\mu(T_n;q,t) = \prod_{(i,j)\in\lambda} (t^{i-1} - q^{j-1}t^n), \tag{62}$$

see *e.g.* Ref.(17), VI.8, Eq. (8), and using the value of c_λ given in Eq.(31), one finds :

$$L_+^{q,t}.P_\lambda(\mathbb{X};q,t) = \sum_i \frac{1-t^{n-i}q^{\lambda_i}}{1-q} \prod_{j=i+1}^{n} \frac{1-q^{\lambda_i-\lambda_j-1}t^{j-i+1}}{1-q^{\lambda_i-\lambda_j}t^{j-i}} \frac{1-q^{\lambda_i-\lambda_j}t^{j-i-1}}{1-q^{\lambda_i-\lambda_j-1}t^{j-i}} P_{\lambda_{(i)}}(\mathbb{X};q,t). \tag{63}$$

For simplicity, a polynomial belonging in the kernel of $L_+^{q,t}$ will be called *highest weight*.

V. HIGHEST WEIGHT MACDONALD POLYNOMIALS

In this section, one investigates two families of highest weight Macdonald polynomials and we will suppose that $t^{k-1}q^{r+1} = 1$ for some integers $k, r \in \mathbb{N}$.

A. Weakly admissible partitions

Let us recall some results contained in the paper of Feigin *et al.*[11]. The aim of Ref.(11) is to study ideals of polynomials defined by certain vanishing conditions (called wheel conditions). For their purpose, Feigin *et al.* defined the notion of admissible partitions. A partition λ is said (r,k,n)-*admissible* if for each $i = 1 \ldots n-k$, one has $\lambda_i - \lambda_{i+r} \ge k$. In this definition, one considers that the partition λ encodes an element of a basis of the symmetric functions algebra for an alphabet of size n, hence the partition λ is completed with 0 at the right by setting $\lambda_i = 0$ if

$i > l(\lambda)$. They proved the following property : Suppose $1 \le i < j \le n$ and λ is a (r,k,n)-admissible partition. Then one has :

$$q^{\lambda_i-\lambda_j}t^{j-i} \neq 1,\ q^{\lambda_i-\lambda_j-1}t^{j-i+1} \neq 1,\ q^{\lambda_i-\lambda_j-1}t^{j-i} \neq 1. \tag{64}$$

In addition, if $\lambda_j < \lambda_{j+1}$ then $q^{\lambda_i-\lambda_j}t^{j-i-1} \neq 1$. This can be straightforwardly adapted to slightly more general partitions. A partition λ will be called *weakly* (r,k,n)*-admissible* if for each $i = 1, \dots, n-k$, one has $\lambda_i - \lambda_{i+r} \ge k$ or $\lambda_i = 0$. Let λ be weakly (r,k,n)-admissible partition and $1 \le i < j \le l(\lambda) + r$. Then, remarking that λ is a $(r,k,l(\lambda)+r)$-admissible partition and applying Eq.(64), one obtains :

$$q^{\lambda_i-\lambda_j}t^{j-i} \neq 1,\ q^{\lambda_i-\lambda_j-1}t^{j-i+1} \neq 1,\ q^{\lambda_i-\lambda_j-1}t^{j-i} \neq 1. \tag{65}$$

In addition, if $\lambda_j < \lambda_{j+1}$ then $q^{\lambda_i-\lambda_j}t^{j-i-1} \neq 1$. Feigin *et al*[11] proved an interesting condition for the poles of $P_\lambda(\mathbb{X};q,t)$ when λ is (r,k,n)-admissible.

Lemma 1. (*Feigin, Jimbo, Miwa and Mukhin*)
Assume either λ is (r,k,n)-admissible or else λ is obtained from a (r,k,n)-admissible partition by adding or removing one node. Then $P_\lambda(\mathbb{X};q,t)$ has no pole at $(t,q) = \left(u^{\frac{k-1}{m}}, \omega_1 u^{-\frac{r+1}{m}}\right)$ where $\omega_1 = \exp\left\{\frac{2i\pi(1+dm)}{r-1}\right\}$ with $m = \gcd(r+1, k-1)$ and $d \in \mathbb{Z}$.

They obtained this result by investigating the coefficients of $P_\lambda(\mathbb{X};q,t)$ in the expansion in the monomial basis. This expansion is known (see *e.g.* Ref.(17), VI 7, (7.13')) :

$$P_\lambda(\mathbb{X};q,t) = \sum_T \psi_T(q,t)x^T, \tag{66}$$

where the sum runs over tableaux T of shape λ, and involves rational fractions $\psi_T(q,t)$ which are an explicit product of quotients of fractions given by :

$$b_\lambda(i,j;q,t) = \frac{1-q^{\lambda_i-j}t^{\lambda'_j-i+1}}{1-q^{\lambda_i-j+1}t^{\lambda'_j-i}}, \tag{67}$$

where (i,j) is a node of λ. Hence, Lemma 1 remains true for weakly admissible partitions.

Lemma 2.
Assume either λ is weakly (r,k,n)-admissible or else λ is obtained from a (r,k,n)-admissible partition by adding or removing one node. Then $P_\lambda(\mathbb{X};q,t)$ has no pole at $(t,q) = \left(u^{\frac{k-1}{m}}, \omega_1 u^{-\frac{r+1}{m}}\right)$.

Proof The expansion of $P_\lambda(\mathbb{X};q,t)$ on monomial functions involves coefficients whose denominators are constituted by products of $1-q^{\lambda_i-j}t^{\lambda'_j-i+1}$ or $1-q^{\lambda_i-j+1}t^{\lambda'_j-i}$ with $(i,j) \in \lambda$. This implies that $i \le l(\lambda) \le l(\lambda)+r$. Since λ is a $(r,k,l(\lambda)+r)$-admissible partition, the result is a direct consequence of Lemma 1.□

Let us give an example. Consider the partition $\lambda = (2,2)$ which is weakly $(2,2,n)$-admissible for any $n > 2$. The polynomial $P_{22}(\mathbb{X};q,t)$ admits the following decomposition over the monomial functions

$$P_{22}(\mathbb{X};q,t) = m_{2,2} + \frac{(q+1)(-1+t)\,m_{2,1,1}}{qt-1} + \frac{(q+1)(-1+t)^2(t+2\,qt+2+q)\,m_{1,1,1,1}}{(qt-1)(qt^2-1)}$$

This equality is independent of the size n of the alphabet. Hence, the only possible poles are such that $qt = 1$ or $qt^2 = 1$. Lemma 2 predicts that $(t,q) = (u,-u)$ is not a pole of $P_{22}(\mathbb{X};q,t)$.

B. Rectangular partitions

In this subsection, one investigates a family of highest weight Macdonald polynomials indexed by rectangular partition. More precisely, one proves the following result :

Theorem I. *If $n \ge 2r$ then the Macdonald polynomial $P_{k^r}(x_1+\cdots+x_n;q,t)$ belongs to the kernel of $L_+^{q,t}$ for the specialization $(t,q) = (u^{\frac{k-1}{g}}, u^{\frac{r-1-n}{g}}\omega_1)$ where $g = \gcd(k-1, n-r+1)$ and $\omega_1 = \exp\left\{\frac{2i\pi(1+dg)}{k-1}\right\}$ with $d \in \mathbb{Z}$.*

Proof We start with the Lassalle identity for the normalization P (Eq.(63)). This identity involves a unique Macdonald polynomial in its right hand side :

$$L_+^{q,t}P_{k^r}(x_1+\cdots+x_N;q,t) = \frac{1-q^k}{1-q}\frac{1-q^{k-1}t^{N+1-r}}{1-q^{k-1}t}P_{k^{r-1}k-1}(x_1+\cdots+x_N;q,t). \tag{68}$$

Suppose now that $(t,q)=(u^{\frac{k-1}{s}},u^{\frac{r-1-n}{g}}\omega_1)$. First remark that the partition (k^r) is weakly $(k,n-r,n)$-admissible when $n\geq 2r$. Hence, from Lemma 1, the polynomial $P_{k^r}(x_1+\cdots+x_N;q,t)$ is well defined. It follows that a necessary condition for $L_+^{q,t}P_{k^r}(x_1+\cdots+x_N;q,t)=0$ is $\frac{1-q^k}{1-q}\frac{1-q^{k-1}t^{N+1-r}}{1-q^{k-1}t}=0$. If $n\geq 2r$, the polynomial $1-q^{k-1}t^{N+1-r}$ is not divisible by $1-q^{k-1}t$ and vanishes for our specialization. It remains to prove that $P_{k^{r-1}k-1}(x_1+\cdots+x_N;q,t)$ has no pole at $(t,q)=(u^{\frac{k-1}{s}},u^{\frac{r-1-n}{g}}\omega_1)$. Since the partition $(k^{r-1},k-1)$ is obtained from the weakly admissible partition (k^r) by subtracting 1 to the last part, this is again a consequence of Lemma 1. This ends the proof.□

Let us give some examples to illustrate this result. The following polynomials are highest weight Macdonald polynomials,

1. $P_4(x_1+x_2+x_3;\exp(2i\pi/3)u^{-1},u)$,
2. $P_5(x_1+x_2+x_3;u^{-3},u^4)$,
3. $P_5(x_1+x_2+x_3+x_4;iu^{-1},u)$,
4. $P_{33}(x_1+x_2+x_3+x_4;u^{-3},u^2)$,
5. $P_{33}(x_1+x_2+x_3+x_4+x_5;-u^{-2},u)$.

Whilst the following polynomials are not highest weight Macdonald polynomials :

1. $P_4(x_1+x_2+x_3;u^{-1},u)$. Indeed,
$$L_+^{q,t}(q,t)P_4(x_1+x_2+x_3;u^{-1},u)=3\frac{(u+1)(1+u^2)(u^2+u+1)}{u^3}x_1x_2x_3.$$
2. $P_5(x_1+x_2+x_3+x_4;u^{-1},u)$. Indeed ,
$$L_+^{q,t}P_5(x_1+x_2+x_3+x_4;u^{-1},u)=-4\frac{(u+1)(u^2+1)(u^4+u^3+u^2+u+1)}{u^4}x_1x_2x_3x_4$$
3. $P_{33}(x_1+x_2+x_3+x_4+x_5;u^{-2},u)$. Indeed,
$$L_+^{q,t}P_{33}(x_1+x_2+x_3+x_4+x_5;u^{-2},u)=-2\frac{(1-u^4)(1-u^6)(1-u^5)}{u^6(1-u)^2(1-u^2)}x_1x_2x_3x_4x_5.$$

In conclusion, for each alphabet $\mathbb{X}$ of size n and each rectangular partition (k^r) with $n\geq 2r$, there is an explicit specialization of (q,t) such that $P_{k^r}(\mathbb{X};q,t)$ belongs to the kernel of $L_+^{q,t}$.

C. Staircase partitions

We examine here another family of highest weight Macdonald polynomials indexed by staircase partitions. By staircase partition, we mean a partition under the form $\lambda=[((\beta+1)s+r)^k,(\beta s+r)^l,\dots,(s+r)^l]$ where $\beta,s,r,k,l\in\mathbb{N}$ and $k\leq l$.

Theorem II.

Let $\beta,s,r,k,l\in\mathbb{N}$ with $k\leq l$. Consider the partition $\lambda=[((\beta+1)s+r)^k,(\beta s+r)^l,\dots,(s+r)^l]$. The polynomial $P_\lambda(x_1+\cdots+x_n;q,t)$ is a highest weight polynomial when

$$n=\frac{l+1}{s-1}r+l(\beta+1)+k \tag{69}$$

is an integer and

$$(t,q)=(u^{\frac{s-1}{g}},u^{-\frac{l+1}{g}}\omega_1) \tag{70}$$

where $g=\gcd(l+1,s-1)$ and $\omega_1=\exp\left\{\frac{2i\pi(1+dg)}{s-1}\right\}$ if d denotes an integer such that $w_1^r=1$.
Remark that the condition $\frac{l+1}{s-1}r\in\mathbb{N}$ implies that $\frac{s-1}{g}$ divides r and the condition $\omega_1^r=1$ implies g divides r. Hence, in all the cases, $s-1$ divides r.

Proof As for the rectangular partition, the starting point of our reasoning is the equality (63):

$$L_+^{q,t}.P_\lambda(\mathbb{X};q,t)=\sum_i \frac{1-t^{n-i}q^{\lambda_i}}{1-q}\prod_{j=i+1}^{n}\frac{1-q^{\lambda_i-\lambda_j-1}t^{j-i+1}}{1-q^{\lambda_i-\lambda_j}t^{j-i}}\frac{1-q^{\lambda_i-\lambda_j}t^{j-i-1}}{1-q^{\lambda_i-\lambda_j-1}t^{j-i}}P_{\lambda_{(i)}}(\mathbb{X};q,t). \tag{71}$$

Since $\lambda=[((\beta+1)s+r)^k,(\beta s+r)^l,\dots,(s+r)^l]$, there is only $\beta+1$ indices i_v $(0\leq v\leq\beta)$ such that $\lambda_{(i_v)}$ is a partition and hence has a non zero contribution in Eq. (71). These indices are characterized by $i_v:=k+lv$ and the corresponding node in the partition λ is

$$(i_v,j_v)=(k+lv,(\beta+1-v)s+r). \tag{72}$$

One verifies that λ is weakly (l,s,n)-admissible. Hence, Lemma 2 implies that $(t,q)=(u^{\frac{s-1}{g}},u^{-\frac{l+1}{g}}\omega_1)$ is a pole of neither $P_\lambda(\mathbb{X};q,t)$ nor $P_{\lambda_{(i_v)}}(\mathbb{X};q,t)$ for $v=0,\dots,\beta$. Hence, to prove the theorem it remains to show that the coefficient ϱ_v of $P_{\lambda_{(i_v)}}(\mathbb{X};q,t)$ in Eq.(71) vanishes under the specialization (70).
Let us examine first the denominator of ϱ_v. From Eq. (71) this denominator is a product of polynomials under the form $1-q^{\lambda_i-\lambda_j}t^{j-i}$, $1-q^{\lambda_i-\lambda_j-1}t^{j-i+1}$, $1-q^{\lambda_i-\lambda_j-1}t^{j-i}$ or $1-q$. Since λ is weakly admissible, Eq.(65) implies that the three first possibilities do not vanish for the specialization (70) whilst the fourth is straightforwardly not zero under this specialization. Hence, it suffices to prove that the numerator ς_v of ϱ_v vanishes for each v. One has to consider two cases. First consider that $v=l$. After simplification, one obtains

$$\varsigma_l=(1-q^{r+s})(1-q^{s+r-1}t^{n-k-\beta l+1}). \tag{73}$$

But since $n-k-l\beta+1=\frac{l+1}{s-1}r+l-1$, it follows

$$q^{s+r-1}t^{n-k-\beta l+1}=q^rt^{\frac{l+1}{s-1}r}=\omega_1^r=1, \tag{74}$$

from the hypothesis. Hence, $\varsigma_l=\varrho_l=0$. Suppose now that $v<l$. One has

$$\varsigma_v=(1-t^{n-i_v}q^{\lambda_{i_v}})\prod_{j=i_v+1}^{n}(1-q^{\lambda_{i_v}-\lambda_j-1}t^{j-i+1})(1-q^{\lambda_{i_v}-\lambda_j}t^{j-i-1}). \tag{75}$$

If we set $j=i_v+l$ then $\lambda_{i_v}-\lambda_j=s$ and the factor $1-q^{\lambda_{i_v}-\lambda_j-1}t^{j-i_v+1}$ in ς vanishes under the specialization (70). It follows that $\varsigma_v=\varrho_v=0$. This implies our theorem. □

Let us give some examples. The following polynomials are highest weight Macdonald polynomials:

1. $P_{53}(x_1+x_2+x_3+x_4+x_5;q=u^{-2},t=u)$ $(\beta=2,\ s=2,\ k=0,\ r=1,\ l=1)$.
2. $P_{63}(x_1+x_2+x_3;q=-u^{-1},t=u)$ $(\beta=2,\ s=3,\ k=0,\ r=0,\ l=1)$.
3. $P_{422}(x_1+x_2+x_3+x_4+x_5;q=u^{-3},t=u)$
4. $P_{533}(x_1+\cdots+x_8;q=u^{-3},t=u)$
5. $P_{633}(x_1+\cdots+x_5;q=u^{-3},t=u^2)$

Note that the converse of Theorem (2) is false as shown by the counter-example :

$$L_+^{q,t}P_{42}(x_1+\cdots+x_2;q,t)=0 \tag{76}$$

for $n\geq 2$ and $q=-1$. For the moment, the problem of the characterization of the highest weight Macdonald polynomials is still open.

VI. HIGHEST WEIGHT JACK POLYNOMIALS

A. Some necessary conditions

As it is shown in Ref.(22), the action of L_+ on the polynomials $P_\lambda^{(\alpha)}(\mathbb{X})$ can be recovered from Eq. (63) by setting $q=t^\alpha$ and sending t to 1.

$$\begin{aligned}L_+.P_\lambda^{(\alpha)}(\mathbb{X}) &= \sum_i\frac{n-i+\lambda_i\alpha}{\alpha}\times\\ &\times\prod_{j=i+1}^{n}\frac{(\alpha(\lambda_i-\lambda_j-1)+j-i+1)(\alpha(\lambda_i-\lambda_j)+j-i-1)}{(\alpha(\lambda_i-\lambda_j)+j-i)(\alpha(\lambda_i-\lambda_j-1)+j-i)}P_{\lambda_{(i)}}^{(\alpha)}(\mathbb{X})\end{aligned} \tag{77}$$

If one asks the highest weight condition $L_+.J_\lambda^{(\alpha)}=0$ in terms of $J_\lambda^{(\alpha)}$ as in Ref. (15), then one has the necessary condition $n-l(\lambda)+1+\alpha(\lambda_{l(\lambda)}-1)=0$. However for some specializations we may have $J_\lambda^{(\alpha)}=0$ (for instance $J_{53}^{(-1)}(x_1+x_2+x_3+x_4)=0$). To make sense, the property must be stated in terms of P: If $L_+.P_\lambda^{(\alpha)}(x_1+\cdots+x_n)=0$ then

$$n-l(\lambda)+1+\alpha(\lambda_{l(\lambda)}-1)=0. \tag{78}$$

Indeed, we need that the coefficient of each $P^{(\alpha)}_{\lambda_{(i)}}(\mathbb{X})$ in Eq. (77) vanishes. In particular, the coefficient of $P^{(\alpha)}_{\lambda_{(l(\lambda))}}(\mathbb{X})$,

$$\frac{\lambda_{l(\lambda)}(n-l(\lambda)+\alpha(\lambda_{l(\lambda)}-1)+1)}{1+\alpha(\lambda_{l(\lambda)}-1)}$$

must equal 0. Since $P^{(\alpha)}_{\lambda_{(l(\lambda))}}\neq 0$ occurs in Eq. (77) ($\lambda_{(l(\lambda))}$ being always a partition and $P^{(\alpha)}_{\lambda_{(l(\lambda))}}$ being dominated by $m_{\lambda_{(l(\lambda))}}$), it follows that one has necessarily $n-l(\lambda)+1+\alpha(\lambda_{l(\lambda)}-1)=0$.

Set $\lambda=(r_m^{l_m},\dots,r_1^{l_1})$ with $r_m>\cdots>r_1$. Eq. (78) provides a necessary condition relying α and n. It follows that if $L_+.P_\lambda^{(\alpha)}(x_1+\cdots+x_n)=0$, then α is a negative rational number and that the last part of λ is strictly greater than 1. Other parts of λ gives also further information, which fixes the two values. Suppose $n>l(\lambda)-1+l_1\frac{r_1}{r_2-r_1}$ and $\alpha\neq 0$. We examine the coefficient of $P^{(\alpha)}_{\lambda_{l(\lambda)-l_1}}$ in Eq. (77) after simplification :

$$\frac{\alpha(\alpha(r_2-r_1-1)+l_1+1)(r_2-r_1)(\alpha(r_2-1)+n+1-l(\lambda)+l_1)(\alpha r_2+l_1)}{(\alpha(r_2-r_1)+l_1)(\alpha(r_2-r_1-1)+1)(\alpha r_2+n+l_1-l(\lambda))(\alpha(r_2-1)+1+l_1).} \tag{79}$$

If $L_+.P_\lambda^{(\alpha)}(\mathbb{X})=0$, at least one of the five factors α, $(\alpha(r_2-r_1-1)+l_1+1)$, (r_2-r_1), $(\alpha(r_2-1)+n+1-l(\lambda)+l_1)$ or (αr_2+l_1) vanishes. From the hypothesis $\alpha>0$ and $r_2>r_1$. Hence, it remains three factors: $(\alpha(r_2-r_1-1)+l_1+1)$, $(\alpha(r_2-1)+n+1-l(\lambda)+l_1)$ and (αr_2+l_1). Suppose $\alpha r_2+l_1=0$, Eq.(78) implies

$$n=l(\lambda)-1+l_1\frac{r_1-1}{r_2}<l(\lambda)-1+l_1\frac{r_1-1}{r_2-r_1}. \tag{80}$$

But this contradicts the hypothesis $n>l(\lambda)-1+l_1\frac{r_1}{r_2-r_1}$. In the same way, $\alpha(r_2-1)+n+1-l(\lambda)+l_1=0$ implies $n=l(\lambda)-1+l_1\frac{r_1}{r_2-r_1}$ which also contradicts the same hypothesis. It remains $\alpha(r_2-r_1-1)+l_1+1$, that is $\alpha=\frac{l_1+1}{1-(r_2-r_1)}$.
Straightforwardly, this implies $1-(r_2-r_1)\neq 0$ or equivalently $r_2>r_1+1$. Substituting $\alpha=\frac{l_1+1}{1-(r_2-r_1)}$ in $n-l(\lambda)+1+\alpha(\lambda_{l(\lambda)}-1)=0$, one obtains :

$$n=l(\lambda)-1+\frac{l_1+1}{r_2-r_1-1}(r_1-1). \tag{81}$$

Since n is an integer, this implies that $\frac{r_2-r_1-1}{\gcd(r_2-r_1-1,l_1+1)}$ divides r_1-1.

In conclusion, under the condition $n>l(\lambda)-1+l_1\frac{r_1}{r_2-r_1}$ and $\alpha\neq 0$, if $L_+.P_\lambda^{(\alpha)}(\mathbb{X})=0$ then

$$\alpha=\frac{l_1+1}{1-(r_2-r_1)}. \tag{82}$$

It follows that $r_2>r_1+1$ and $\frac{r_2-r_1-1}{\gcd(r_2-r_1-1,l_1+1)}$ divides r_1-1.
In others words, when n is big enough, $\alpha\neq 0$ and for a fixed partition λ with at least two distinct parts, the polynomial $L_+.P_\lambda^{(\alpha)}(\mathbb{X})$ vanishes for at most one value of (n,α).

B. Rectangular partitions

In this paragraph, we are interested in characterizing highest weight Jack polynomials indexed by rectangular partitions. One has, as a straightforward consequence of Eq.(78),

$$L_+.P_{k^r}^{(\alpha)}(\mathbb{X})=0 \text{ implies } \alpha=\frac{r+1-n}{k-1}. \tag{83}$$

Furthermore, if in addition $n \geq 2r$ and $\gcd(n-r+1, k-1) = 1$, a special case of Theorem (1) by sending u to 1 gives the equivalence of the two equalities. Let us give some examples of such highest weight Jack polynomials by computing their expansion over the basis $(\tilde{c}_\lambda)_\lambda$. The simplest examples are provided by partitions whose all the parts equals 2. In this case, Eq. (83) implies $\alpha = l(\lambda) - 1 - n$. The coefficients seem easy to obtain and the first computations suggest the general equality

$$P_{2^l}^{(1-l-n)}(x_1 + \cdots + x_n) = (-1)^l \frac{n!^2}{(n-l)!^2} \sum_\mu (-2)^{l(\mu)} \tilde{c}_\mu, \tag{84}$$

summed over the partitions μ of $2l$ having only even parts. Note that when $\gcd(n-r+1, k-1) \neq 1$, $P_{k^r}^{(\alpha)}$ can have a pole at $\alpha = \frac{r-1-n}{k+1}$, as shown by the example

$$\begin{aligned} P_{33}^{(\alpha)}(x_1 + \cdots + x_5) \ &= \ m_{3,3} + 3\,\tfrac{m_{3,2,1}}{2\,\alpha+1} + 6\,\tfrac{m_{3,1,1,1}}{(2\,\alpha+1)(\alpha+1)} + 6\,\tfrac{m_{2,2,2}}{(2\,\alpha+1)(\alpha+1)} \\ &\quad +3\,\tfrac{(3\,\alpha+5)m_{2,2,1,1}}{(2\,\alpha+1)(\alpha+1)^2} + 36\,\tfrac{m_{2,1,1,1,1}}{(2\,\alpha+1)(\alpha+1)^2}, \end{aligned}$$

$P_{33}^{(\alpha)}$ having a pole at $\alpha = -1$. Furthermore, even correctly defined normalizations of $P_{33}^{(-1)}$ are not annulled by L_+,

$$L_+.J_{33}^{(-1)}(x_1 + \cdots + x_5) = 48m_{221} + 288m_{2111}. \tag{85}$$

We conjecture the following property : Suppose $n \geq r$, then the assertions :

$$P_{k^r}^{(\alpha)}(\mathbb{X}) \text{ is well defined and } L_+.P_{k^r}^{(\alpha)}(\mathbb{X}) = 0 \tag{86}$$

and

$$\alpha = \frac{r-1-n}{k-1},\ n \geq 2r \text{ and } n-r+1 \text{ is not a divisor of } k-1 \tag{87}$$

are equivalent.

C. Staircase partitions

Let β, s, r, k and l be five integers such that $1 \leq k \leq l$, $0 < \beta, s$ and $\gcd(l+1, s-1) = 1$. Let $\lambda = [((\beta+1)s + r)^k, (\beta s + r)^l, \ldots, (s+r)^l]$ be a staircase partition and $\mathbb{X}$ an alphabet of size n verifying $n > \beta l + k - 1 + l_1 \frac{s+r}{s}$. Let $\alpha \neq 0$ be a non zero complex number. From Theorem (2) and Eq. (78), the two following assertions are equivalent :

1. $\{L_+.P_\lambda^{(\alpha)}(\mathbb{X}) = 0.\}$
2. $\{n = \frac{l+1}{s-1} r + l(\beta+1) + k \quad \text{and} \quad \alpha = \frac{1+l}{1-s}.\}$

Again, the implication $1 \Rightarrow 2$ is a direct consequence of Eq.(82). The implication $2 \Rightarrow 1$ comes from a special cases of Theorem (2) sending u to 1. The enumeration of the first cases allows us to propose the following conjecture : If $L_+.P_\lambda^{(\alpha)}(\mathbb{X}) = 0$ then λ is a staircase partition. If we assume the two previous conjectures, we can propose a complete characterization of highest weight Jack polynomials : Let $\mathbb{X}$ be an alphabet of size n, $\alpha \neq 0$ and λ be a partition such that $l(\lambda) \leq n$. The polynomial $P_\lambda^{(\alpha)}(\mathbb{X})$ is annulled by L_+ if and only if λ is a staircase partition and one of the two following assertions is verified :

1. {If $\lambda = k^r$ is a rectangular partitions then $\alpha = \frac{r-1-n}{k-1}$, $n \geq 2r$ and $n-r+1$ is not a divisor of $k-1$.}
2. {If $\lambda = [((\beta+1)s+r)^k, (\beta s+r)^l, \ldots, (s+r)^l]$ is not rectangular, then $\gcd(l+1, s-1) = 1$, $n = \frac{l+1}{s-1} r + l(\beta+1) + k$ and $\alpha = \frac{1+l}{1-s}$.}

Remark that $\lambda = [((\beta+1)s + r)^k, (\beta s + r)^l, \ldots, (s+r)^l]$ can be written in terms of occupation numbers as :

$$\lambda \equiv [n_0, 0^{s+r-1}, l, 0^{s-1}, l, 0^{s-1}, \ldots, l, 0^{s-1}, k, 0^\infty]. \tag{88}$$

This includes the Jack polynomials indexed by partitions $[n_0, 0^{s+r-1}, l, 0^{s-1}, l, 0^{s-1}, \ldots, l, 0^{s-1}, l, 0^\infty]$ of Ref.(15), but the last part may be different. For instance, for $n = 5$ and $\alpha = -3$, one has :

$$L_+.P_{422}^{(-3)}(x_1 + x_2 + x_3 + x_4 + x_5) = 0. \tag{89}$$

VII. CONCLUSIONS

We have characterized highest weight Macdonald and Jack polynomials for special partitions : rectangular and staircase. We have also formulated conjectures concerning a possible generalization. To summarize, these conjectures should be deduced from that a necessary condition for a Macdonald polynomial to have a highest weight is that its partition is a staircase, which is suggested by numerical evidences. The underlying mechanism seems to be related to the vanishing properties of staircase Macdonald polynomials under the specialization $q^at^b = 1$. These vanishing properties could be translated in terms of factorizations under specializations of the variables x_i. For example, one has the identity :

$$\begin{aligned} &P_{44}((1+t+t^2+t^3)x_1 + (1+t+t^2)x_2; q = \omega t^{-2}, t) = \\ &(*)_{q,t}(x_1+\omega x_2)(x_1+\omega t^3x_1)(x_1-t^6x_2)(x_2+\omega t^6x_1)(x_1-t^9x_2)(x_2-t^9x_1)(x_2-t^{12}x_1) \end{aligned} \tag{90}$$

where $(*)_{q,t}$ is a scalar depending only on q and t, and $\omega = \exp\left\{\frac{2i\pi}{3}\right\}$. The link between highest weight and factorizations, generalizing the results of Ref. 26, is a promising study that will be explored in a future paper. These wavefunctions may eventually prove useful for the construction of candidate quasiparticle/quasihole states and their manipulation by analytical or numerical[27] means.

As a final remark, we note that similar results are known on non symmetric Jack polynomials (called "singular") which are in the kernel of the Dunkl operators[28,29]. In this context, the study of singular non symmetric Macdonald polynomials seems to be relevant.

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Appendix A: About some eigenvalue problems

The eigenvalue problem for a delta function interaction restricted to the lowest landau level on symmetric functions admit only a handful of explicit eigenstates[30–33]. When the angular momentum in the planar geometry is equal to the number of particles the lowest energy "yrast" state is given by :

$$\Phi_n = \prod_{i=1}^{n}(x_1 + \cdots + x_n - nx_i). \tag{A1}$$

This quantity is a highest weight polynomial with a nice expression when written in terms of the functions $\tilde{c}_\lambda$:

$$\Phi_n = n^n n! \sum_{\lambda}(-1)^{l(\lambda)+n}\tilde{c}_\lambda \tag{A2}$$

where the sum is over the partition λ of n having no part equal to 1. For example if $n = 7$, one has

$$\Phi_7 = \prod_{i=1}^{7}(x_1 + \cdots + x_7 - 7x_i) = 4150656720(\tilde{c}_7 - \tilde{c}_{5,2} - \tilde{c}_{4,3} + \tilde{c}_{3,2,2}) \tag{A3}$$

There also other exact states[30–33] that are known :

$$\Phi_L^n = \sum_{i_1<\cdots<i_L}\prod_{k=1}^{L}(x_1 + \cdots + x_n - nx_{i_k}). \tag{A4}$$

Surprisingly, this polynomial has the same expression (up to a multiplicative coefficient) in terms of $\tilde{c}_\lambda$ as Φ_n:

$$\Phi_L^n = n^L L! \binom{n}{L} \sum_{\lambda}(-1)^{l(\lambda)+n}\tilde{c}_\lambda \tag{A5}$$

(again the sum is over the partition λ of n having no part equal to 1). Note that the two polynomials are *not* equal since they are evaluated on different alphabets. For example, we have :

$$\begin{aligned}\Phi_7^9 &= \sum_{1\le i_1<i_2<\cdots<i_7\le 9}\prod_{k=1}^{7}(x_1 + \cdots + x_9 - 9x_{i_k}) \\ &= 867821895360(\tilde{c}_7 - \tilde{c}_{5,2} - \tilde{c}_{4,3} + \tilde{c}_{3,2,2})\end{aligned} \tag{A6}$$

Note also that the Φ_L^n are not the symmetrized of the Φ_n functions since their expansions in terms of $\tilde{c}_\lambda$ are quite different. For example, the symmetrized of Φ_7 on the alphabet $\{x_1, \ldots, x_9\}$ is :

$$\begin{aligned}&\sum_{1\le i_1<i_2<\cdots<i_7\le 9}\prod_{k=1}^{7}((x_{i_1} + \cdots + x_{i_7}) - 7x_{i_k}) = \\ &486777211200\,\tilde{c}_7 - 462909867840\,\tilde{c}_{5,2} - 492922366272\,\tilde{c}_{4,3} + 447162907968\,\tilde{c}_{3,2,2}.\end{aligned} \tag{A7}$$

Appendix B: Highest weight polynomials and the eigenvalues of the Read-Rezayi Hamiltonian

The k-type Read-Rezayi state[8,9] is the exact zero energy ground state of smallest degree of :

$$\mathcal{H}^{(k)} := \sum_{i_1<\cdots<i_{k+1}} \delta^{(2)}(x_{i_1}-x_{i_2})\dots\delta^{(2)}(x_{i_k}-x_{i_{k+1}}). \tag{B1}$$

In the lowest Landau level we are only interested by the description of the spectral properties of the operator :

$$\mathbf{h}_k.f(x_1,\dots,x_n) = \sum_{i_1<\cdots<i_k} f(x_1,\dots,x_{i_1-1},X,x_{i_1+1},\dots,,x_{i_{k+1}-1},X,x_{i_{k+1}+1},\dots,x_n), \tag{B2}$$

where X denotes $X=\frac{x_{i_1}+\cdots+x_{i_{k+1}}}{k+1}$, acting on symmetric functions. The computation of the eigenspaces of the operator $\mathbf{h}_k$ is highly non trivial since its characteristic polynomial generally does not factorize in the field of rational numbers. One eigenfunction can be easily shown for any k : the sum of the variables $c_1=x_1+\cdots+x_n$. More precisely, a straightforward computation gives :

$$\mathbf{h}_k.c_1 = \binom{n}{k+1} c_1. \tag{B3}$$

Furthermore, $\mathbf{h}_k$ commutes with the multiplication by c_1:

$$\mathbf{h}_k c_1 f(x_1+\cdots+x_n) = c_1\mathbf{h}_k f(x_1+\cdots+x_n). \tag{B4}$$

The equality $\tilde{c}_1=c_1$ combined to the fact that $\mathbf{h}_k$ is diagonalisable implies that it suffices to understand the eigenspaces of the restriction of $\mathbf{h}_k$ to the space generated by the $\tilde{c}_\lambda$ where λ is a partition without 1, that is the algebra of highest weight symmetric polynomials. Note that in the special case $k=1$, the polynomials Φ_L^n are eigenfunctions of $\mathbf{h}_1$ with eigenvalues $\frac{1}{2}L(L-\frac{n+2}{2})$.

In fact it is enough to find the highest weight eigenfunctions of $\mathbf{h}_k$. Let us illustrate this principle with the simplest example $n=3$ and $k=1$. This is a particularly simple case, since the characteristic polynomial factorizes. One has to find as many eigenfunctions as the numbers of partitions with parts only equal to 2 or 3 which is given by the generating function :

$$\begin{aligned}\tfrac{1}{(1-t^2)(1-t^3)} &= 1+t^2+t^3+t^4+t^5+2t^6+t^7+2t^8+2t^9+2t^{10}+2t^{11}+\\ &\quad 3t^{12}+2t^{13}+3t^{14}+3t^{15}+3t^{16}+3t^{17}+4t^{18}+3t^{19}+\cdots.\end{aligned} \tag{B5}$$

The square of the Vandermonde determinant $(x_1-x_2)^2(x_1-x_3)^2(x_2-x_3)^2$ belonging to the kernel, it remains to compute as many functions as the numbers described by the generating series

$$\frac{1}{(1-t^2)(1-t^3)} - \frac{t^6}{(1-t^2)(1-t^3)} = 1+\frac{t^2}{(1-t)} = 1+t^2+t^3+t^4+t^5+t^6+t^7+t^8+\dots, \tag{B6}$$

i.e. only one by degree. We conjecture that the following polynomials are eigenfunctions of $\mathbf{h}_1$:

$$\Psi_L^{(2)} := \tilde{c}_{3^p2^\ell} + 6(p+\ell)\tilde{c}_{3^{p-2}2^{\ell+3}} + 36(p+\ell)(p+\ell+1)\tilde{c}_{3^{p-4}2^{\ell+6}} + \dots + 6^q(p+\ell)\dots(p+\ell+q)\tilde{c}_{3^\epsilon 2^{\ell+3q}}, \tag{B7}$$

where p is the maximal integer such that $L=3p+2\ell$ with $0\leq\ell$ and $p=2q+\epsilon$ with $\epsilon=0$ or 1.

Some examples are :

$$L=21:\ \tilde{c}_{3333333} + 42\tilde{c}_{33333222} + 2016\tilde{c}_{333222222} + 108864\tilde{c}_{3222222222}, \tag{B8}$$

$$L=22:\ \tilde{c}_{33333322} + 48\tilde{c}_{333322222} + 2592\tilde{c}_{3322222222} + 155520\tilde{c}_{22222222222}, \tag{B9}$$

$$L=23:\ \tilde{c}_{33333332} + 48\tilde{c}_{333332222} + 2592\tilde{c}_{3332222222} + 155520\tilde{c}_{32222222222}. \tag{B10}$$

Unfortunately, the general case is not so simple. But, we hope that the subproblem of the description of the kernels can be solved more easily by means of a similar reasoning. The difficulty consists in finding a “good” family of symmetric functions such that a basis of the kernel can be “nicely” described. Numerical evidences suggest that the only highest weight Jack polynomials belonging to the kernel of $\mathbf{h}_k$ are rectangular. Furthermore, the first computations suggest that these polynomials play an important role in the description of the kernel. For example, when $n=4$, the restriction of the kernel of $\mathbf{h}_2$ to the space of highest weight polynomials is generated by two algebraically independent polynomials $P_{22}^{(-3)}$ and $P_{33}^{(-\frac{3}{2})}$. But the construction is not understood in the general case, for example when $n=5$ the kernel of $\mathbf{h}_4$ (restricted to highest weight polynomials) is generated by $P_{22}^{(-3)}$, $\mathcal{S}P_3^{(-\frac{3}{2})}(x_1+x_2+x_3)P_2^{(-2)}(x_4+x_5)$ and $\mathcal{S}P_{33}^{(-\frac{3}{2})}(x_1+x_2+x_3+x_4)$, where $\mathcal{S}$ means symmetrization *w.r.t.* the alphabet $x_1+x_2+x_3+x_4+x_5$.